ELEMENTARY TREATISE

ON

ELECTRICITY AND MAGNETISM

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BY

G. CAREY FOSTER, F.R.S.

FELLOW OF AND EMERITUS PROFESSOR OF PHYSICS IN
UNIVERSITY COLLEGE, LONDON

AND

ALFRED W. PORTER, B.Sc.

FELLOW OF AND ASSISTANT-PROFESSOR OF PHYSICS IN
UNIVERSITY COLLEGE, LONDON

FOUNDED ON

JOUBERT'S

"TRAITÉ ÉLÉMENTAIRE D'ÉLECTRICITÉ"

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PREFACE

THE scheme of this book was thus stated, in the Preface to the First Edition, by the late Dr. E. Atkinson and one of the undersigned: "In undertaking the present work our first idea was simply to translate M. Joubert's *Traité élémentaire d'Électricité*, perhaps introducing such small modifications as might seem to make it better adapted to English requirements. But, on further consideration, we decided, while adhering to the general plan and scope of the original, to endeavour to introduce into it that view of the nature of electrical phenomena which was originated by Faraday, and developed by Maxwell in his classical treatise. Although these views have now for some years formed the starting-point of almost all the greatest forward steps in electrical science, they have not yet, so far as we are aware, been adopted as the basis of a systematic exposition of so elementary a kind as the present.

"The endeavour to carry out this idea implied the keeping in view from the beginning of the dual character of electrification and the emphasising of the essential part played in the most familiar electrical phenomena by the dielectric medium in which they occur. This led to the early introduction of the idea of lines and tubes of force, and involved the rewriting of several chapters or parts of chapters."

In preparing a new edition, while still following the

general plan of the original work, we have carefully revised the whole book and have rewritten or rearranged a large part. The parts which have been least changed are those relating to the theory of dynamo-electric machines, namely, Chapters XXXIII., XXXIV., and XXXV. These are to a great extent translated from the second edition of M. Joubert's book. In the preface to the first English edition it was stated that M. Joubert could not be held responsible for any errors or defects that might be found in it; the further changes that have been introduced into this edition make M. Joubert's responsibility even less than before.

Among the principal additional matters introduced into this edition we may mention:—

The theory of electric images, as applied to the mutual electric influence of spherical conductors; the idea of ionisation of electrolytes, and its bearing on electrolysis and allied phenomena; the propagation of electric waves along a "concentric cable," and of a plane wave in an unlimited isotropic medium; the properties of the cathode stream, of Röntgen and Becquerel rays, &c.; the Zeeman effect, and a brief account of recent investigation bearing on the theory of electrons.

The introductory chapters on magnetism have been re-written, and Gauss's law of the inverse-cube of the distance, for the field due to a small magnet, has been taken as the experimental starting-point for the quantitative discussion of magnetic action, in place of Coulomb's law of the inverse square as applied to magnetic poles. A great part of the chapters relating to electromagnetism has been re-modelled.

Lastly, we should perhaps refer to some points of terminology. We have used the term influence instead of

induction in reference to the condition of a conductor in an electrostatic field, reserving induction to express what Maxwell calls electric displacement. Similarly, in magnetism, we use influence to express the general fact of the magnetisation of a magnetic body in a magnetic field and induction to denote the property whose rate of change determines the magnitude of induced electromotive force. Seeing that we have avoided reference to magnetic poles, except incidentally, it seemed more consistent to avoid the correlative expression, magnetic force: we speak, instead, of magnetic field and lines of field.

G. C. F.
A. W. P.

May 1903.

PREFACE TO THE THIRD EDITION

In the preparation of this Edition the whole book has been carefully revised and a good many additions have been made, not always for the sake of introducing new matter so much as in order to make more complete the discussion of subjects already included. The final chapter has been rewritten, and it is hoped it will be found a useful summary of recent progress in electrical science.

We desire to acknowledge our indebtedness to Dr. C. Chree for his kindness in looking through the chapter on Terrestrial Magnetism, and giving us information as to the magnetic elements.

G. C. F.
A. W. P.

November 1909.

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ERRATA

Page 242, line 14, *for* " M (M being)" *read* " Mh (M being)."

Page 242, line 15, *delete* " h."

Page 602, line 5, *for* "S. J. Allen" *read* " H. S. Allen."

Page 475, line 13, *for* " $2\pi a\ 2\pi r \sin\theta$ " *read* $2\pi a = 2\pi r \sin\theta$."

Page 475, line 15, *for* " $H = ev \sin\theta/r$ " *read* " $H = ev \sin\theta/r^2$."

ELECTRICITY

CHAPTER I

FUNDAMENTAL PHENOMENA

1. Electrification by Friction.—If a plate of glass and a plate of vulcanised india-rubber, both thoroughly dry, are laid on each other and rubbed together, it is found that some force is required to separate them, and that, after having been separated, they attract each other if not too far apart. If a second piece of glass and a second piece of india-rubber are treated in the same way, it is found that not only does either piece of glass attract either piece of india-rubber, but that the two pieces of glass repel each other, and that the two pieces of india-rubber do the same. Bodies which are in such a condition as to exhibit these properties are said to be *electrified.*

Attraction also occurs when an electrified body is brought near bodies that have not been subjected to friction, and if these are light enough (bits of pith, feathers, shreds of paper, wool, &c.), they may be lifted. Effects of this kind often afford an easier test of the existence of electrification than such experiments as those previously referred to. It is accordingly to be noted that although attraction occurring between two bodies may be evidence of previous electrification, it does not prove that *both* the mutually attracting bodies were previously electrified; and if only one was thus electrified, we cannot say, without other evidence, which it was. On the other hand, repulsion, as between two pieces of glass or two pieces of india-rubber, does not occur unless both the repelling bodies have been previously electrified.

2. Electrification by Contact.—If a pith-ball (A) hung by a fine fibre of raw silk (Fig. 1) is allowed to touch a piece of glass (D), that has been electrified by friction, it is repelled by the glass after contact (Fig. 2). Again, a second pith-ball, similarly hung by a

silk fibre, if allowed to touch a piece of electrified india-rubber, is repelled by the india-rubber after contact. The two pith-balls, however, if brought near together, are found to attract each other;

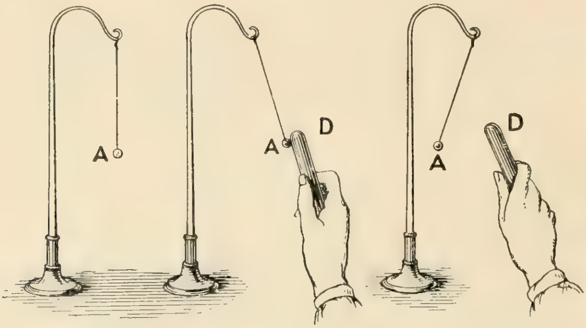


Fig. 1. Fig. 2.

in fact, they respectively behave towards the glass and india-rubber, and towards each other, just as a second pair of pieces of glass and india-rubber which had been rubbed together would do. That is, the two pith-balls have acquired properties exactly like those of the glass and india-rubber, and are consequently said to be electrified by contact.

A metal rod or cylinder supported by a dry glass stem or suspended by dry silk threads, can be electrified by contact with a piece

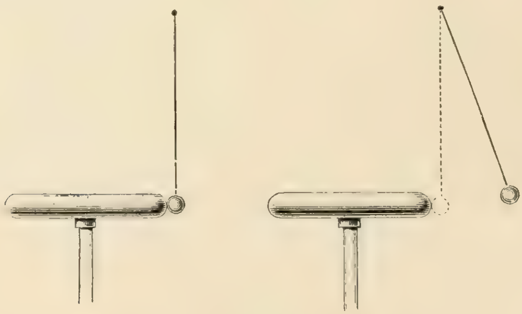


Fig. 3.

of rubbed glass or of rubbed india-rubber. A pith-ball hanging by a silk fibre so as just to touch the metal before electrification may be repelled in this way to a considerable distance (Fig. 3).

A piece of metal thus electrified exhibits signs of electrification at all parts of its surface, but the evidence of electrification is less marked the larger the piece of metal. At the same time, the electrification of the primarily electrified body is found to be lessened by having imparted its own properties to the metal. In fact, the whole process takes place as though the communication of electrification depended on the sharing of some thing or condition, existing at the surface of the originally electrified body, between it and the body with which it is brought into contact, so that what is gained by one is lost by the other. Facts of this kind have led to the idea of an unknown something capable of being developed on bodies by friction, when two different substances are rubbed together, capable also of spreading from one body to another by contact, and giving rise to the phenomena characteristic of electrification. This supposed cause of electrical phenomena is spoken of as *electricity*.

3. Good and Bad Electrical Conductors.—Bodies which, when electrified at one part, are immediately found to be electrified all over, are called good *conductors* of electricity, and the process is commonly described as depending on the freedom with which electricity can move over them. There are, however, many substances—such as glass, shellac, india-rubber, silk, wool, sulphur, amber—which, when electrified at one part, allow the electrical state to spread only very slowly and gradually to other parts. Such bodies are conceived of as not permitting the passage of electricity through them or along their surface: they are hence called *non-conductors* of electricity. More properly, bodies of this kind are called bad or slow conductors, for they do not absolutely prevent the communication of electrification, and hence their properties differ rather in degree than in kind from those of good conductors. The differences in degree, however, are often enormous, the spread of electrification being in some cases apparently instantaneous, while in others it is so slow as to require very careful observation to detect it at all.

All the metals belong to the class of electrical conductors, as well as graphite and various metallic ores. Water in its ordinary condition is a conductor, and so are aqueous solutions of soluble salts; but when water is very carefully purified its conducting power is greatly diminished, so that it seems likely that absolutely pure water, and, probably, every non-metallic liquid in a pure unmixed state, is a non-conductor. Living plants and animals, being for the most part pervaded throughout by watery fluids, are conductors, though many animal and vegetable tissues, when

thoroughly dry, are non-conductors. The air, again, and other gases, as well as vapours, are very perfect non-conductors.

4. Insulators.—The facts stated in the last two paragraphs serve to explain certain points connected with the electrification of bodies by friction. If two bodies, both belonging to the class of electrical non-conductors, are rubbed together, they are both found to be electrified, the degree of electrification varying with the nature of the bodies. But if a piece of metal or other conducting material held in the hand is rubbed against a non-conductor—say a piece of dry flannel—only the non-conductor appears afterwards to be electrified. The reason is that the electrification produced on the metal spreads over the hand, arm, and body of the experimenter to the floor and walls of the room, and is thereby so much diminished in intensity as not to be recognisable without special means. In order that a piece of metal may be electrified either by friction or by contact, it is needful that it should be cut off from electrical communication with the room and surrounding objects by being supported by non-conducting materials. The conductor is then said to be *insulated*, and so long as its insulation remains perfect, the degree of its electrification cannot increase or decrease.

In consequence of their use for preventing the communication of electrification, non-conducting bodies are often spoken of as *insulators.* No more perfect insulator is obtainable than air, if there is a sufficient thickness of it. Of solid insulators, clean and dry

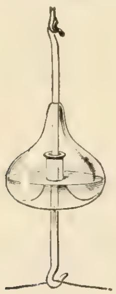


FIG. 4.

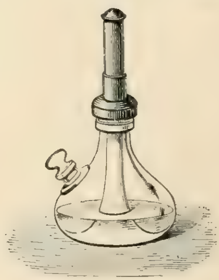


FIG. 5.

flint-glass and ebonite are among the most efficient and convenient for practical purposes. Glass, however, is apt to become covered with a conducting film of moisture deposited from the atmosphere. This can sometimes be prevented by keeping it slightly warm —not *hot*, as it then conducts appreciably—or by coating the

surface with shellac-varnish, or, better still, by keeping the air in its neighbourhood dry by means of strong sulphuric acid (see Figs. 4, 5).

5. Two Correlative Electrical States.—Let us return to a fact mentioned in (**1**). Suppose a piece of glass and a piece of vulcanised india-rubber to have been electrified by being rubbed together, and then that a second piece of each material is treated in the same way. As already stated, the two pieces of glass and the two pieces of india-rubber both repel each other, but either piece of glass attracts either piece of india-rubber. Hence we see that although both the glass and the india-rubber have acquired special properties as the result of the friction, that is, although both are electrified, their properties are not identical. As compared with each other, no difference is apparent: the glass attracts the india-rubber and the india-rubber attracts the glass, and either of them can lift small bits of pith. But towards a third electrified body their action is not only different, it is directly opposed: a body which is repelled by one is attracted by the other. Moreover, if two bodies are rubbed together without being separated, the parts that were subject to mutual friction remaining in contact, and are then presented simultaneously to a third electrified body, no sign of electrification can be detected, since the repulsion that would be caused by one of the rubbed bodies if it were alone is exactly neutralised by the attraction due to the other. Experiments of this kind, which in their details may be varied to an almost endless extent without altering the general result, prove that *the process of electrification consists in the simultaneous development of two electrical states so related that the effects produced by a body in one of these states are opposed to those produced under similar conditions by a body in the other state*, and also that *the opposite electrical states are always produced simultaneously to such an extent that they can mutually neutralise each other's action.*

The relation between these two correlative electrical conditions is analogous to that between quantities of opposite sign in algebra; that is, when oppositely electrified bodies are superposed, their combined effect is equal only to the difference of the effects they are capable of producing separately, just as the result of adding quantities of opposite algebraic sign is a quantity of the same sign as the greater of the two, and equal in magnitude to their arithmetical difference. This relation is conveniently expressed by calling one of the two electrical states, that assumed by the glass, *positive*, and the other, that assumed by the india-rubber, *negative*. Again, just as two quantities of opposite sign are said to be equal

when their sum = 0, so the electrification of two oppositely electrified bodies is said to be equal in amount, or the bodies are said to be charged with equal opposite quantities of electricity, when the result of their superposition is to do away with all evidence of electrification. Using this mode of expression, we may repeat in a more concise form what was said above, and assert that *the process of electrification consists in the simultaneous and correlative production of equal quantities of positive and negative electricity.*

6. Electrification of Various Substances.—Hitherto, in speaking of electrification by friction, we have almost confined ourselves to a single typical case, that of glass and india-rubber. But in general, any two bodies, after being rubbed together, they having been held, if needful, by insulating supports, are found, if examined with sufficient care, to be more or less electrified. One of them repels a piece of glass that has been rubbed with vulcanised india-rubber and attracts the india-rubber; the other repels the india-rubber and attracts the glass. Accordingly the first is said to be positively electrified, and the second negatively. If a number of different substances are taken and rubbed together two and two, it is possible to arrange them in consecutive order, so that each one becomes positive when rubbed against one of those that follow it in the series, and negative when rubbed against one of those that precede it. For example, the substances in the following list are arranged in this way:—

Fur of a live cat.	Silk.
Polished glass.	Shellac.
Flannel.	Resin.
Feathers.	Ground glass.
Woody fibre (pure cotton-wool).	Metals.

Other things equal, more marked effects are obtained with two substances which occur far apart in such a list than with two that occur nearer together; the effect appears, however, in many cases to depend on small changes of condition in a way that is not understood. An example is afforded by the difference between polished and ground glass.

7. The Electric Field.—When two bodies have been rubbed together so as to electrify them, the result cannot be detected until the surfaces have been separated (**5**). In separating them, work has to be done in order to overcome the force with which they mutually attract each other (**1**). When the rubbed bodies have been separated, any other electrified body brought into their neighbourhood, and especially if put directly between them, is

subject to force tending to make it move from one and towards the other (**1**). If, therefore, the mechanical resistances are so small that motion can actually take place, work is done by the electrical forces. It may be said, then, that the space near the electrified bodies possesses special properties depending on the relative positions of these bodies, and on the degree of their electrification. The existence of the forces already referred to constitutes the most obvious and easily observed of these properties, but, as we shall see later, this space possesses other properties not less characteristic (**23, 68**). In order to connote collectively these properties without reference to the special way in which the electrification may have been produced, any space where they exist is called an *electric field.*

An electric field is always bounded by surfaces possessing equal opposite correlative electrifications (**5**). It may, or may not, have definite geometrical limits. Consider, for example, the field produced by rubbing together and then separating two bodies which are at a very great distance from all other objects as compared with their distance from each other. The boundaries of the field are the surfaces of the electrified bodies; but as these are entirely external to each other, they do not enclose any definite space, and the field extends on all sides. The only sense in which it can be said to be limited is that, at a greater or less distance, electrical effects can no longer be detected; but this distance depends essentially on the means of detection employed, and as these are rendered more and more sensitive the apparent extent of the electric field becomes greater and greater. If, on the other hand, one of the bounding surfaces entirely encloses the other, the electric field is definitely limited to the space between them. For example, a case that frequently occurs is when one of the rubbed bodies is a conductor in electrical connection with the inside of the room where the experiment is made. The floor, walls, and ceiling are always composed of materials possessing sufficient conducting power to enable the electrification of the body connected with them to spread over them. In this case the electric field extends between the surface of the rubbed body that remains insulated and the inner surface of the room.

8. Electric Discharge.—If two oppositely electrified conductors are brought into contact, there is a sudden decrease in the degree of electrification. Sometimes both conductors are found, after contact, to be entirely unelectrified, and in such cases we infer that their previous electrifications were equal (**5**). In cases where any electrification remains, the electrification of both conductors is

found to be of the same sign, and we then infer that the conductor whose original electrification was of this sign was electrified in a greater degree, or to a greater amount, than the other. The disappearance of electrification under the above conditions is called *electric discharge*, and the conductors are said to be discharged or to have lost their electric charges. Discharge occurs also when oppositely electrified conductors are connected by a third conductor instead of being put into direct contact.

If the degree of electrification is considerable, discharge can take place through an appreciable thickness of non-conducting matter, such as air, or oil, or glass. In such cases it is accompanied by a sudden disruption of the non-conductor, as is rendered evident in the case of air by a sudden sound—which may vary in loudness from a barely audible snap to a bang like that of a gun—and by a spark, which may be just visible in the dark or may be a dazzling flash. Liquid non-conductors are thrown violently about by the discharge and solids are permanently pierced.

Complete discharge takes place, with or without an appreciable spark and sound, when the two conducting boundaries of an electric field come into contact or are connected by a conductor. For example, suppose a metal ball supported by a dry glass stem, or hung by a silk cord, to be flapped with a piece of warm dry flannel: an electric field is set up, the negative boundary of which is the surface of the ball, while the positive boundary is, at the first instant, on the flannel; but whether the experimenter keeps this in his hand, or lays it on the table or on the floor, the positive electrification gradually spreads over the floor, walls, and ceiling of the room, and the field then extends between all parts of the inner surface of the room and the ball (**7**, end). If now the experimenter touch the ball with his finger, his body forms a conductor connecting the boundaries of the field, and all signs of electrification cease. A body thus touched, or connected with the room by a wire, is often spoken of as being *connected to earth*, or earth-connected. This mode of securing absence of electrification is of continual use in experimenting. We shall deal with this subject more fully in Chapter IX.

9. Energy of an Electric Field.—In whatever way an electric field may be produced, a certain amount of energy must be expended in the process. In such a case as that referred to in (**7**) this is evidently equal to the work done against electric force in separating the electrified surfaces, and may be represented by the product fl, if f denotes the average force tending to prevent separation, while two points originally in contact are separated to the distance l.

Again, when an electric field exists, the forces that act therein can do work on electrified bodies moving in the field. The total amount of work that can thus be done may be called the *energy of the electric field:* it is equal to the energy expended in establishing the field. How a definite power of doing work, in other words, a definite amount of energy, can be associated with a given electric field may be understood by a special case. For example, let A and B (Fig. 6) be two insulated metal plates, equally and oppositely electrified, forming the boundaries of an electric field, and let C be a gilt pith-ball hanging between them by a fine silk fibre. If C comes into contact with A, which we will suppose positively electrified, it will become electrified, and will then be urged to B by the electric forces of the field, an amount of work being done equal to the product of the distance traversed into the average electric force exerted. When C arrives at B, there will be mutual neutralisation of the positive electricity conveyed by it, and of an equal quantity of the negative electricity of B. This will be accompanied by a proportionate diminution of the energy of the field. Next, C will acquire negative electricity from B, and will be urged back to A, the forces of the field again doing work upon it. On its arriving there, the negative electricity of C and an equal amount of the positive electricity of A will mutually neutralise each other, and so a further diminution of electric energy will take place. If this process is repeated over and over again, it is evident that the final result will be the entire disappearance of electrification, and that simultaneously the power of doing work by motion of the pith-ball will come to an end.

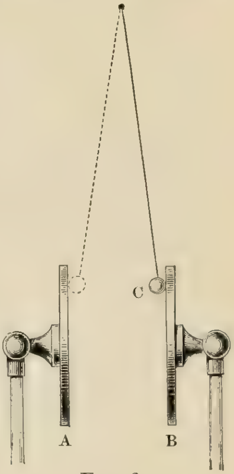


FIG. 6.

10. Seat of Electric Energy.—It has been pointed out already that the space between two correlative oppositely electrified surfaces, a so-called electric field, may be said to possess energy. It is a matter of great importance, in connection with any inquiry into the ultimate nature of electrification, to ascertain the seat of the condition to which this energy is due,—whether it is a property of the electrified surfaces by which the field is bounded, or whether it is to be sought in the non-conducting medium existing throughout the field itself.

The facts that the operations whereby an electric field is established are carried on with the bodies forming the bounding surfaces of the field, and that the question whether electric force acts at any given point or not depends upon the position of the point relatively to these bodies, long caused attention to be fixed almost exclusively upon them, and the accepted terminology of electric science has been framed almost entirely from this point of view. On the other hand, such facts as those mentioned in (**15**), that the conducting boundaries of a field affect it only by virtue of the geometrical relations of their surfaces, whereas the forces acting anywhere in the field, and therefore its total energy, depend essentially on the physical properties of the intervening non-conducting medium,—seem to indicate the latter as that in which the electrical state is really inherent. There is, in fact, evidence, to be discussed later, tending to show that the medium occupying an electric field is in a state of stress, consisting in a tension at every point, in a direction parallel to that in which an electrified particle at that point would be urged, and of a pressure in all directions at right angles thereto. The forces tending to make the oppositely electrified conductors that bound the field approach each other are supposed to result from this stress, which, however, is not made evident by direct mechanical effects produced in the homogeneous field itself. The case is to some extent comparable with that of an ordinary fluid under pressure: the stress existing in the fluid tends to make the bounding surfaces of the containing vessel move away from each other, but any portion of the fluid itself is in equilibrium under the action of balancing forces.

On this view, what is commonly called the quantity of electricity on a conductor, or the amount of its charge, is to be looked on as an indication of what total force is applied to its surface,[1] and the sharing of the charge of a conductor with another previously unelectrified, as the transfer of the surface of application of part of the total force from the first conductor to the second.

Although it may not be possible as yet to give a complete account of the part played by non-conducting media in the production of electrical phenomena, there can be no doubt that any satisfactory explanation of these phenomena must take account of the functions of non-conductors as well as of conductors.

[1] In exact language, what is here meant is the surface integral of the outward normal component of force.

CHAPTER II

LAW OF INVERSE SQUARES AND APPLICATIONS

11. Gold-leaf Electroscope—Proof-plane.—The experimental study of electric states requires the use of apparatus which we have not yet described. It can be carried out to a considerable extent by means of the two instruments named at the head of this paragraph.

1. *The gold-leaf electroscope* serves for the detection, and to some extent for the comparison, of smaller degrees of electrification than could be recognised by any of the means yet referred to. It consists essentially (Fig. 7) of two strips of gold-leaf suspended close together from an insulated metal stem. To protect the gold-leaves from air-currents and from mechanical injury, they are surrounded by a glass bell-jar, through the top of which the metal stem projects, being terminated by a ball or sometimes by a metal plate. The bell-jar is closed below by a piece of wood, covered at least on the upper side with tinfoil and uninsulated. Two vertical uninsulated strips of tinfoil, or else two metal rods, are carried up inside the bell-jar so that their upper ends are a little above the bottom of the strips of gold-leaf.

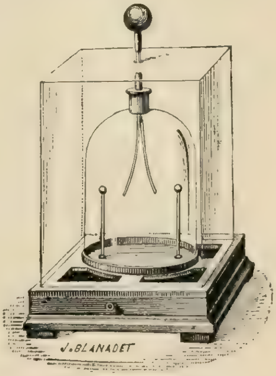


FIG. 7.

When the electroscope is entirely unelectrified, the two gold-leaves hang vertically face to face; but a very small degree of electrification causes them to stand apart at the bottom, the amount of divergence increasing with the electrification. When the electroscope is charged, it becomes *pro tanto* one of the boundaries of an electric field, and we may say that the divergence

of the gold-leaves results from the tendency of the boundaries of the field to approach each other.

2. The *proof-plane* consists of a small disc of copper-foil or of gilt cardboard fastened at the end of an insulating rod of shellac or glass. If the disc is applied tangentially to any part of the surface of a conductor, it coincides sensibly with the surface at that part and (as will be proved in the next section) becomes charged with the quantity of electricity previously existing thereon. If it is then removed at right angles to the surface, it carries this charge with it. The relative amounts of the charges acquired by the proof-plane in different experiments can be judged of by bringing it into contact with the knob of the gold-leaf electroscope and observing the divergence. This requires that the electroscope shall have been previously graduated, which can be done in the following way, depending upon the experiments next to be described. A hollow insulated conductor is connected to the knob, and a proof-plane charged from a constant source is brought into contact with the inside of the conductor. The proof-plane gives up the whole of its charge and the leaves diverge. If this is done repeatedly, always charging the proof-plane from the same source and taking care not to discharge the electroscope, the charge on the latter will each time increase by the same amount; and thus the divergences corresponding to charges whose relative values are 1, 2, 3, 4, . . . &c., can be ascertained. A *proof-ball* may be used instead of a proof-plane, but the relation between the charge received by a ball when put in contact with a conductor is related to the charge per unit area at the part touched in a much more complicated way than is the case with a proof-plane, for which the two are simply proportional.

12. The Charge of a Conductor in Electrical Equilibrium is purely Superficial: in other words, **An Electric Field does not penetrate the Surfaces of the Conductors which form its Boundaries.**— The following are some of the most striking experiments whereby this important principle may be verified.

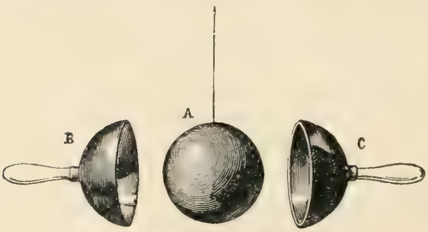


Fig. 8.

1. An insulated metal sphere, A (Fig. 8), is electrified and then enclosed in two metal hemispheres, B and C, of somewhat larger diameter than itself, held by insulating

handles. The hemispheres while kept closed are raised or lowered so as to touch the inner sphere for a moment and are then separated and removed simultaneously. On now testing the ball with the electroscope, it is found entirely unelectrified, while the hemispherical cups are found to be charged.

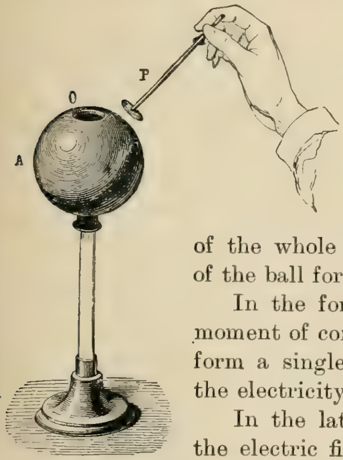


FIG. 9.

This experiment, which is due to Cavendish, will be interpreted somewhat differently according as we fix our attention only on the charge of the ball, or take account of the whole electric field of which the surface of the ball forms one boundary.

In the former case, we may say that at the moment of contact the ball and the enclosing cups form a single conductor, and that the whole of the electricity passes to the outer surface.

In the latter case, we have to consider that the electric field extends from the surface of the ball to the surface (usually the inner surface of the room where the experiment is made) on which the correlative opposite charge to that of the ball is situated, and that the charge passes from the ball to the outside of the cups in accordance with the general tendency to longitudinal contraction of the field.

2. A hollow insulated conductor of any shape (Fig. 9) is electrified. If now a proof-plane is made to touch any part of the outer surface, it is found when removed to be electrified. But if the proof-plane is inserted through a hole, so as to touch any part of the inner surface, it is found on removal to be entirely unelectrified.

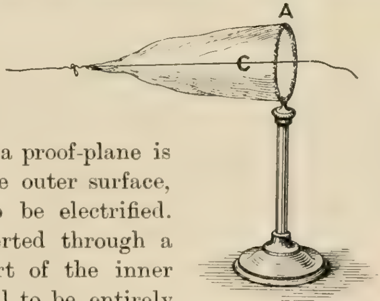


FIG. 10.

The first part of this experiment may be regarded as a partial repetition of that just described. The result of the second part is just what might be looked for when it is considered that the whole electric field is outside the conductor.

3. Faraday made an experiment analogous to the last by means of a sort of muslin butterfly-net, mounted, as shown in the figure (Fig. 10), on an insulating stand. When the net is electrified it

can be shown by the proof-plane that the charge is entirely on the outside. But if the net is turned inside out by means of a silk thread fastened to the point, the charge passes to the other side of the muslin, so as again to be on the outer surface, and this happens every time the net is reversed.

4. Faraday had a chamber constructed with conducting walls, and supported on insulating feet, large enough for him to get inside it and make experiments. The chamber was connected with a powerful electrical machine outside, and charged so that strong sparks could be drawn from any part of the outer surface; but nevertheless not the smallest trace of electricity could be detected inside, nor the least sign of electric force, even by the most delicate instruments.

There is no need in this experiment that the surface of the conductor should be perfectly continuous. It may, in fact, be easily performed by means of a parrot-cage formed of a simple network of wire. A gold-leaf electroscope placed anywhere inside the cage remains unaffected even when the cage is strongly electrified.

5. An important experiment correlative to that of Faraday is due to Mr. F. C. Webb. Taking an electrical machine into an insulated chamber, and connecting the rubbers of the machine with the floor of the chamber, he was able to electrify insulated conductors inside the chamber, exactly as he might have done in an ordinary room. When this was done, all parts of the inner surface of the chamber were found to be electrified, but no sign of electrification could be detected outside.

In this form of the experiment the electric field extends from the surface of the enclosed conductors to the inner surface of the chamber, and is complete inside the chamber. In Faraday's form of the experiment the field extends from the outer surface of the chamber to the inner surface of the room in which it is placed, and is complete outside the chamber.

All these experiments go to establish a fundamental principle of electric distribution, that there is *neither electricity nor electric force* within the substance of a conductor: the charge is entirely superficial, being, in the case of a hollow conductor, on the outside or the inside surface according as the correlative opposite charge is external or internal respectively.

13. Law of Force—Bertrand's Proof.—We have no means of judging of the degree of electrification of bodies except by the mechanical forces they exert by reason of it; we shall therefore assume the possibility of taking one as a measure of the other. So

that, if in any case the force on a small charged body due to surrounding electrification is double that which it is in another case, the surrounding electrification remaining the same, then we assume that the charge on the small body in the first case is also double that in the second. This being granted, the experiments we have just described enable us to show that there is one and only one possible law of action between elementary charges. The most superficial experiments suffice to show that the force between two bodies becomes less when the distance is increased, and we shall now prove that the law of variation is that of the inverse square of the distance, provided that the charges are localised sensibly at points.

Consider an electrified conducting sphere at a great distance from other objects, and therefore at a great distance from the opposite boundary of the corresponding electric field. In such a case the only forces of appreciable magnitude tending to produce a definite distribution of the charge upon the sphere are those arising from the mutual repulsion between the several elementary portions of the charge itself. Considerations of geometrical symmetry require us to assume that, in such a case, the charge distributes itself uniformly over the surface of the sphere, and consequently, if at any point within it we place a very small charge, the force upon it due to a given small element of the surface will be proportional to the area of the element, but it will also depend in some definite way on the distance of the element from the point, that is, it will be what is called in mathematics a *function* of this distance. What we have here said may be expressed by saying that the force due to a small element of surface of area ds, at a point distant r centimetres from it, is represented by $ds.f(r)$, where the symbol $f(r)$ stands for the function of the distance r by which the force is determined. We have to find out what this function is.

If the force decreases in proportion as the square of the distance increases, that is, if the force varies inversely as the square of the distance, we may write

$$ds.f(r) = ds\frac{A}{r^2},$$

where A is a constant quantity for the electrified sphere, and may be interpreted physically as standing for the force at unit distance due to a very small portion of the surface taken as unit of area. The same formula may also be written as

$$r^2 f(r) = A,$$

which we may read as a statement that, *if* the electric force at a point varies inversely as the square of the distance of that point

from the acting electricity, the product of the square of the distance into the function that expresses the way in which the force varies with distance is the same for all distances. On the other hand, if the law of inverse squares does not hold good, the product $r^2 f(r)$ must either increase or decrease as r increases, or must vary in the same direction as r within a certain range of values, and in the opposite direction for another range. If the product is not constant, it will always be possible to find two values of r, say r_1 and r_2, such that, as r increases from one to the other, the product in question varies always in the same direction. For the sake of argument, we will suppose it to increase, and will consider the consequences of this supposition.

Construct a sphere (Fig. 11) of which the diameter is $r_1 + r_2$, and consider the point P which divides the diameter into the two segments r_1 and r_2. Let ds and ds' be the two small areas intercepted in the surface of the sphere by a double cone of infinitesimal solid angle with its vertex at P. These small areas will lie on opposite sides of a plane through P at right angles to the given diameter; let r and r' be their respective distances from P. Then the force at P due to ds is $ds.f(r)$ and that due to ds' is $ds'.f(r')$. But since ds and ds' are sections of the small cone of equal obliquity (as is immediately seen on drawing radii to them) their areas are proportional to the squares of their respective distances from the vertex. Consequently, for the ratio of the forces at P due to these areas, we have

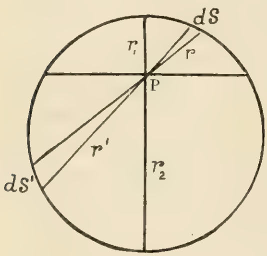


FIG. 11.

$$\frac{ds.f(r)}{ds'.f(r')} = \frac{r^2 f(r)}{r'^2 f(r')}$$

and, by the supposition from which we started, if r is less than r', the numerator of this ratio is less than the denominator, and therefore the force $ds.f(r)$ is less than the force $ds'.f(r')$, and the same will be true for every element of surface on one side of the plane through P as compared with the corresponding element on the other side. Now, by the symmetry of the arrangement, the resultant force due to all the elements of surface taken together which lie on one side of this plane must act along the diameter through P, and the resultant force due to all the elements on the other side must also act along the same diameter in the opposite direction. But if the law of inverse squares does not apply, in

other words, if the product $r^2 f(r)$ is not constant whatever the value of r, these two resultants are not equal, and there must be a final resultant force at P equal to their difference. This, however, is in contradiction to the experimental evidence, for P is an internal point. We therefore conclude that the law of inverse squares is true, *i.e.* that

$$r^2 f(r) = \text{constant} = A \text{ (say)}$$

or

$$f(r) = \frac{A}{r^2}.$$

14. Coulomb's Torsion-Balance.—The law connecting force and distance was discovered experimentally by Coulomb, who compared the forces between two small electrified spheres at different distances by means of the torsion-balance. A thin glass or shellac rod, $b\ d$ (Fig. 12), suspended horizontally by a very fine vertical wire, carries a small ball, b, of gilt elder-pith at one end, and a second similar ball, a, also carried by a glass rod, is supported in a fixed position. The fine wire is fastened above to a torsion-micrometer, shown separately to the right of the figure, whereby the upper end can be turned about a vertical axis through any required angle. The position of the movable ball can be read off on a graduated scale put round the cage of the instrument, A, B, C, D. The position of equilibrium of the arm, $b\ d$, is that in which the vertical wire is without twist: if it is displaced from this position through an angle α, the moment of the couple which tends to restore it is proportional to this angle. If now the two balls are electrified in the same way, all other electrified bodies being removed from the neighbourhood or discharged by being connected with the inside of the room, the movable ball is in a field of force extending between the fixed ball and the floor or table, walls, and ceiling. The ball b consequently moves away from a, and the suspending wire is thereby twisted. By turning the upper end of the wire in the opposite direction, b can be brought back to within any required distance of a. Let α be the angular distance between

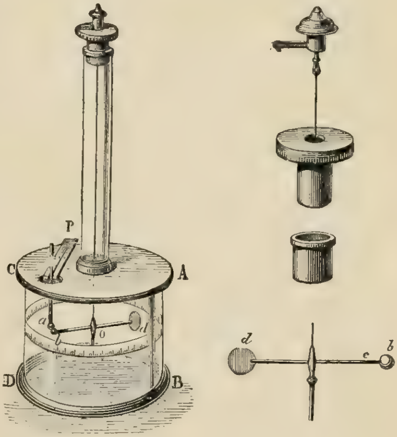


FIG. 12.

the balls when the top of the wire has been turned through the angle A; then the total twist on the wire is $A + \alpha$, and the moment of torsion, proportional to the angle of twist, balances the repulsion between the balls and may be taken as a measure of it. If the equilibrium position of the movable ball (corresponding to no torsion) would coincide with the position actually occupied by the fixed ball, then the lower end of the wire must have been turned through the same angle, α, as that which measures the distance between the balls. If $A, A_1, A_2, \ldots$ be the angles through which the top of the wire has to be turned in order that the distance between the balls may be $\alpha, \alpha_1, \alpha_2, \ldots$ respectively, the following relation is found to hold good, namely—

$$(A+\alpha)\,\alpha^2 = (A_1+\alpha_1)\,\alpha_1{}^2 = (A_2+\alpha_2)\,\alpha_2{}^2 \ldots = \text{constant},$$

provided that the angles $\alpha, \alpha_1, \ldots$ are small, and that the balls are small compared with the distance between them.[1]

Since, as we have seen, the angle of twist $A+\alpha$ measures the electric force between the balls, and the distance between them is proportional to α, this result shows that the electric force is obtained by dividing a constant (whose value depends on the degree of electrification and the construction of the instrument) by the square of the distance between the balls. In other words, the *electric force between two small electrified bodies*, of which the electri-

[1] Even when the balls are very small, this result is not quite accurate. Strictly speaking, it is the product fd^2 which is constant, f being the force between the balls at a distance d. If l be the common distance of the balls A and B from the centre of the circle (Fig. 13), their distance apart $d = \text{AB} = 2\,l \sin\frac{\alpha}{2}$; the moment of the force f tending to increase the twist of the wire is $f.\,\text{OC} = f\,l\cos\frac{\alpha}{2}$. Hence putting C for the couple required to keep the wire twisted through unit angle, $f\,l\cos\frac{\alpha}{2} = C\,(A+\alpha)$, and therefore

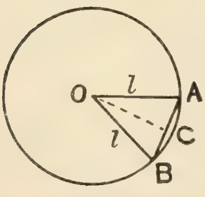


FIG. 13.

$$fd^2 = 4Cl\,(A+\alpha)\sin\frac{\alpha}{2}\tan\frac{\alpha}{2}.$$

As above stated, this quantity is constant for all values of f and d, and for a given instrument C and l are both constant; hence we get $4\,(A+\alpha)\sin\frac{\alpha}{2}\tan\frac{\alpha}{2} = \text{constant}$. If α is a small angle, $2\sin\frac{\alpha}{2}$ and $2\tan\frac{\alpha}{2}$ are both nearly equal to α, and so we have, in this case, the result given in the text. For example, if $\alpha = 30°$, α^2 differs by little more than 1 per cent. from $4\sin\frac{\alpha}{2}\tan\frac{\alpha}{2}$. As the errors of experiment are commonly as great as this, it is not generally worth while to use the more complicated formula.

fication remains the same, *is inversely proportional to the square of the distance between them.* In symbols—

$$\frac{f}{f'} = \frac{A + a}{A' + a'} = \frac{a'^2}{a^2}.$$

This is known as the "law of inverse squares," and is often referred to as "Coulomb's law." It applies equally to the attraction between oppositely electrified bodies and the repulsion between bodies electrified in the same way, and is found experimentally to be more and more accurately verified the smaller the bodies experimented on, so long as the force between them is great enough to admit of accurate measurement. Consequently we are justified in supposing that the law of inverse squares is the fundamental law of electric force, and that it expresses accurately the mutual action between the smallest parts of two electrified bodies. This being admitted, it follows that the experimentally measurable forces between electrified bodies must be the resultants of the forces acting according to the law of inverse squares between their smallest parts.

Experiments with the torsion-balance, however, only prove the approximate truth of the law, owing to unavoidable errors of measurement; the belief in its sensible exactness rests on inference from the absence, inside a closed conductor, of any force that can be detected in the most delicate experiments (**13**).

15. Quantity of Electricity — Unit of Electricity.—We measure the degree of electrification of bodies by the mechanical forces they exert by reason of it. If, while the distance between the balls in a Coulomb's balance and the electrification of one of them are kept constant, the electrification of the other is varied in such a way as to make the force 2, 3, 4 . . . times what it was before, we say that the *quantity of electricity*, or the *electric charge* of this ball, has become 2, 3, 4 . . . times as great. And if two equal balls placed successively in the same position relatively to a third electrified ball, all other conditions remaining the same, exert the same force, we say that they have equal charges, or quantities, of electricity.

We shall adopt as the *unit quantity of electricity*, or *unit electric charge*, the quantity that must be imparted to each of two small spheres, so that, when placed unit distance apart in air, they may repel each other with unit force.

The unit of force is the force that produces unit acceleration in unit mass, or, using, as we shall do throughout this work, 1 *centimetre* as unit of length, 1 *gramme* as unit of mass, and 1 *second* as

unit of time,—that is, the so-called C.G.S. (centimetre-gramme-second) system of units,— the unit of force is 1 *dyne*. We may accordingly define the unit of electricity in concrete terms as follows:—*The C.G.S. electrostatic unit of electric quantity is that quantity which repels an equal quantity at a distance of* 1 *centimetre in air with a force of* 1 *dyne*. To give some idea of the magnitude of this unit, we may say that if two pith-balls, each weighing 0·1 gramme, are hung from the same point by silk fibres 1 metre long without appreciable weight, and are equally electrified so as to make their centres separate 10 cm. from each other, the charge of each ball must be about 22 units.

This unit being very small in comparison with the quantities that come into account in many electrical phenomena, another unit, known as the *coulomb*, equal to three thousand million, 3×10^9, times the unit just defined, is often employed, especially in connection with practical applications. The reasons why it is convenient to adopt this particular quantity as the unit for practical purposes will appear afterwards.

It is found by experiment that, if an electrified conducting sphere is allowed to share its electrification with another equal sphere previously unelectrified, the charges of the two spheres, which are necessarily equal by reason of symmetry, are each equal to one-half of that which the first sphere had to begin with. Thus, as stated in (**2**), electricity is shared between the two spheres as though it were a material substance, no increase or decrease taking place on the whole.

A point of considerable importance in connection with the sharing of electrification between two or more conductors is, that the ratio in which a given quantity divides itself among them does not depend upon any physical characters of the conductors, as whether they are solid or hollow, or composed of one substance or another, but only on their geometrical properties, size, shape, and relative position. The attraction or repulsion, however, between two electrified bodies depends, not only on their electrification and geometry, but also on the nature of the insulating material between them; so that two balls, electrified to a given degree and placed at a given distance apart, attract or repel each other with a different force in air from what they do if immersed in a non-conducting liquid.

It follows from Coulomb's law that the force between two small bodies possessing electric charges q and q' respectively, separated by the distance r, may be represented by the formula—

$$f = \frac{1}{K} \cdot \frac{qq'}{r^2}.$$

In this formula K is a coefficient depending on the properties of the insulating medium in which the bodies are placed, and known as its "specific inductive capacity," or "dielectric coefficient." Its numerical value depends on the units adopted for the measurement of q and f. With the units adopted above, if q and q' are each unity, and r is one centimetre, we have

$$f = \frac{1}{K};$$

but, under these conditions, if the electrified bodies are in air, the force f is one dyne. Hence it follows that the coefficient K must have the value 1 for air. It should be noted that this value is not obtained as the result of an investigation of the electrical properties of air, but simply as a consequence of an arbitrary definition which includes a reference to these properties. For other substances it is recognised as having a value different from unity. The quantity K, however, is not a mere number: it has a physical meaning as well, and expresses the fact that the force due to a given quantity of electricity is influenced in some way, not yet understood, by the nature of the medium within which it is exerted. But in the electrostatic system of electrical measurements, the physical significance of K is not taken into account in the case of air; it is treated as having the merely numerical value unity, and therefore (as not affecting the value of any expression into which it enters as a factor) it is very commonly omitted altogether. It must be understood, however, that both the numerical value of K and the ignoration of its physical significance are peculiar to the electrostatic system to which the unit above defined belongs. In the later part of this book we shall find it convenient to measure electric charges in terms of another unit (the electromagnetic), to which neither of the above statements applies.

If q and q' in the above formula are both of the same sign, their product is positive. Consequently the force acting between two similarly electrified bodies, that is to say, repulsion, is to be considered positive, while a negative value of f is to be interpreted as indicating attraction.

16. Electric Density.—When the boundaries of an electric field are formed by conductors, the electric forces may cause motion of electricity on or in these without causing motion of the conductors themselves, and it is found that the resulting distribution of electricity is in conformity with the same general law, the law of inverse squares, as that which determines the motion of electrified bodies.

If a quantity, q, of electricity exist within a space of volume, v, the quotient $\rho = q/v$ gives the average quantity in each unit of volume, or the mean *electric density*, of the space. If the space is very small, the density at any point within it cannot differ appreciably from the mean density. Hence we may define the electric density at a given point as being that to which the value of the above quotient approximates when the volume v is taken smaller and smaller, but always so as to contain the point in question. This is expressed by writing for the density at a point $\rho = dq/dv$.

In the same way, if a quantity q of electricity exists on a surface of area s, the average quantity per unit of area is $\sigma = q/s$, and the surface-density of the charge at a given point on an electrified surface is expressed by $\sigma = dq/ds$, the limiting value of the quotient q/s when s diminishes without limit, but so as always to include the given point. When it is necessary to distinguish, ρ is called volume-density, and σ surface-density.

The electric density at each point of a conductor is determined by the force of the electric field in accordance with the general tendency of the field to contract longitudinally, that is, along the direction of force, and to spread transversely,—sometimes one of these tendencies and sometimes the other producing the predominating effect.

17. Determination of the Electric Density at Different Parts of an Electrified Conductor.—It is proved by experiments such as those described in (**12**) that there is no electricity within an electrified conductor, or in other words, that the volume-density of electrification is zero. The surface-densities at different parts may be compared by help of a proof-plane (**11**) (Fig. 14), which, when applied tangentially at any part and then removed normally, carries with it the charge existing on the part of the surface to which it was applied. It only remains to compare the charges of the proof-plane in different cases. This may be done by means of the torsion-balance if a constant charge is given to the movable ball and the proof-plane is introduced in place of the fixed ball. In this case the torsion required to bring the movable ball to a definite position is evidently a measure of the charge of the proof-plane.

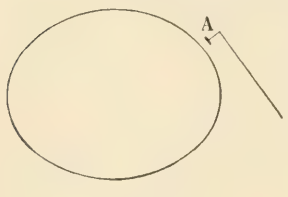


FIG. 14.

But a similar comparison can be made much more easily and quickly by putting the proof-plane inside a metal jar connected with an electrometer graduated as indicated in (**11**). If the proof-

plane is allowed to touch the jar, the jar and electroscope must be discharged before making a new experiment.

Whatever method is adopted, it is needful to guard against errors that may be caused by the gradual loss of electrification due to imperfect insulation. This may be done by the method of *alternate trial.* To compare the densities at two points, A and B, three experiments are made, first at A, then at B, then at A again, equal intervals of time being allowed to elapse between the first and second, and between the second and third. The charge obtained from B is compared with the mean of the two charges obtained from A.

18. General Conditions determining Surface-Density.—For purposes of calculation, as we have already seen (**14, 15**), we may treat electricity as though it were an actual substance, each smallest portion of which exerts force upon every other according to Coulomb's law, and may regard the force exerted upon a quantity, q, of electricity by any electrified bodies whatever as the resultant of the forces exerted by all the elementary portions of the electric charges, each of them being considered as acting independently of all the rest, whether they belong to different bodies or make up the charge of a single body. We shall speak of the value of this resultant for the case of a unit quantity of positive electricity, supposed to be placed at a given point without modifying the distribution of electricity in the neighbourhood, as the *intensity of electric force at the point*, or sometimes, more shortly, as the *electric intensity at the point*, or again simply as the electric force at the point.

Starting from this point of view, it is merely a question of calculation to determine *à priori* the law of the distribution of electricity on a specified system of conducting surfaces, though it is only in a comparatively small number of cases that the required calculations can be put into a sufficiently simple form to admit of being carried out accurately. From what has been said (**12**), it will be seen that the problem to be solved may be stated as follows:—What must be the electric density at each point of the conducting surface, or system of surfaces considered, in order that the electric force may be zero at every point within the conductor or conductors?

Another mode of stating the same problem, which, though different in form, is equivalent in effect, is this: What must be the electric density at each point of the surface, or surfaces, in order that the direction of electric force at every such point may be along the normal to the surface? It follows in fact from the

properties of conductors that the resultant electric force must be everywhere normal to the surface; otherwise, there would be a component tangential to the surface which would cause movement of electricity along the conducting surface, and consequently there could not be electrical equilibrium.

19. Power of Points.—The observed facts as to the electric density in conductors, as has been already stated (**18**), are such as might be deduced mathematically from the idea of two *electricities*, conceived of as capable of moving easily through conducting substances, but not able to pass through non-conductors, and such that each smallest portion of one repels every other portion of the same kind and attracts every portion of the second kind, according to the law of the inverse square of the distance. The same facts, however, might also be deduced from the conception of stresses in the dielectric field of the nature of a tension between the opposite boundaries and a pressure in directions transverse to the lines of tension. When the opposite boundaries of the field are near together,—as, for example, in the case of the field between two metal plates facing each other at a short distance,—the transverse force tending to widen the field cannot produce much effect, because of the small length of the field, and the resulting form of the field (or, as it is expressed in the other order of ideas, the electric distribution) is mainly determined by the tension. On the other hand, if the boundaries of the field are very far apart, the effect of the transverse pressure becomes so great that that of the tension is inappreciable in comparison.

If an electric field is bounded on the one hand by a flat conducting surface of considerable size, say the wall of a room, and on the other by a metal point presented towards the flat surface at a moderate distance, then both the tension and the pressure in the electric field tend to cause a concentration of it about the point, so that the electric force in the air just outside the point becomes very great. But, as will be shown more fully later, the air cannot sustain more than a limited electric force. When this limit is exceeded, the air gives way to the electric stresses, and all signs of electrification cease, almost or quite as completely as if the bounding conductors of the field had been brought into contact.

When an electric field is formed between a sharp point and an opposing plate, the stress in the field close to the point may be sufficient to cause a disruptive discharge although at other parts of the field the stress may be far less. On this account the arrangement mentioned is often used in the construction of electrical apparatus when it is desired to facilitate discharge through a

greater or less thickness of air. On the other hand, points or projections of any kind are avoided as far as possible when it is wished to prevent such discharge. By connecting a point and a plate with the two sides of an electric machine, the electric field between them is constantly reproduced, and discharge takes place continuously. The air close to the point having given way, the stress in the air farther off has nothing to counterbalance it, and the air consequently moves in a continuous stream away from the point, constituting what is known as the *electric wind.* This effect is easily made evident by means of a candle-flame (Fig. 15). The breaking down of tension in front of the point may also be illustrated by the *electric wind-mill* (Fig. 16), the

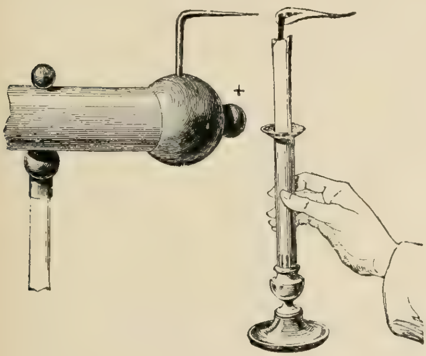


FIG. 15.

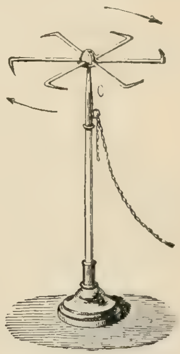


FIG. 16.

points of which move backwards for the same kind of reason that a man pulling hard at a rope tumbles backwards if the rope breaks.

If the air near the point is charged with solid or liquid particles, such as exist in smoke, these are rapidly deposited on the opposing surface and the smoke disappears. This process has been practically employed to hasten the deposition of dust or fumes from the air.

The production of the electric wind is accompanied by a sort of hissing sound, and in the dark the point exhibits a projecting tuft of violet light if it is positive, and a small brilliant star if it is negative.

20. Electric Force outside a Uniformly Charged Sphere.—We have seen that the force inside a uniformly charged sphere is everywhere zero.

To determine the effect of such a sphere at an external point P, join P and the centre O (Fig. 17), and take P′ on OP so that

$$\text{OP}' \,.\, \text{OP} = R^2,$$

R being the radius, and let P′ be the vertex of a cone of very small solid angle, $d\omega$, which intercepts infinitesimal areas, A and A', at the surface of the sphere.

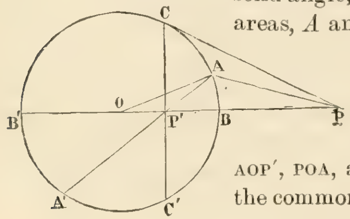


FIG. 17.

Putting r for P′A and r' for P′A′, and also α for the angle OAP′ = OA′P′ = OPA = OPA′ (from the similarity, each to each, of the triangles AOP′, POA, and A′OP′, POA′ which follows from the common angle at O in each pair of triangles respectively, and from the relation $\left.\frac{\text{OP}'}{R} = \frac{R}{\text{OP}}\right)$, we get, for the areas A and A',

$$\frac{r^2 d\omega}{\cos\alpha} \text{ and } \frac{r'^2 d\omega}{\cos\alpha}.$$

Put $s = \text{PA}$ and $s' = \text{PA}'$: then the electric forces at P due to the charges on A and A' are

$$\sigma\frac{r^2 d\omega}{s^2\cos\alpha}\frac{1}{K} \text{ and } \sigma\frac{r'^2 d\omega}{s'^2\cos\alpha}\frac{1}{K},$$

σ being the uniform surface-density of the charge; but, by the similarity of triangles above referred to, we have both $\frac{r}{s}$ and $\frac{r'}{s'} = \frac{R}{\text{OP}}$, and therefore the forces exerted by the elementary areas at A and A′ are equal, and their resultant bisects the angle between AP and A′P, that is, it acts along OP. Obviously the whole surface may be divided into similar pairs of elements, one lying on one side and the other on the other of the plane, CP′C′, perpendicular to OP: the resultant effect of each such pair, and therefore of all the pairs, or of the whole spherical surface, must be along OP, as might have been assumed without other proof than the consideration of geometrical symmetry.

We shall get the effective component of the force due to an element of the surface, that is, the component along OP, by multiplying the total force due to the element by cos α. Hence the effective component due to any element may be written

$$\sigma\frac{R^2}{\text{OP}^2}\frac{1}{K}.\,d\omega,$$

and, to get the effect of the whole sphere, we have only to multiply the constant factor $\sigma \dfrac{R^2}{\text{OP}^2} \dfrac{1}{K}$ by the sum of all the values of $d\omega$. This evidently is 4π.

Hence the force at the point P due to the sphere charged to the uniform surface-density σ is

$$f = \frac{4\pi R^2\sigma}{\text{OP}^2} \frac{1}{K} = \frac{Q}{D^2} \frac{1}{K},$$

if Q be put for $4\pi R^2\sigma$, the whole charge of the sphere, and D for the distance OP. Hence it appears that the force at an external point due to a charge distributed uniformly over the surface of a sphere is the same as if the whole charge were concentrated in one point at the centre of the sphere. This result is of very great importance on account of its frequent application.

If the point P is close to the surface of the sphere, we have $D = R$, and therefore, in this case

$$f = 4\pi\sigma \frac{1}{K},$$

and the force acts along the normal to the surface, outwards if σ is positive, inwards if σ is negative.

It may be noted that, for a given value of the charge, the force at a given distance is independent of the radius of the sphere.

21. Solid Sphere with Volume Charge.—It follows that a homogeneous sphere, or a sphere that might be subdivided into concentric homogeneous layers, acts at any external point as though its whole mass were concentrated at the centre. To find the action of a homogeneous sphere at an internal point distant r from the centre, imagine a spherical surface, concentric with the sphere, drawn through the point: then, the force due to the part outside this surface vanishes, and the part inside acts as though it were concentrated at the centre. The action of the whole sphere is therefore represented by

$$f = \frac{\frac{4}{3}\pi r^3 \rho}{r^2} \frac{1}{K} = \tfrac{4}{3}\pi\rho \,.\, r \frac{1}{K},$$

if ρ is the volume-density of the sphere. The force is consequently proportional to r.

The case here considered could only occur electrically if the sphere were composed of some non-conducting material, for with a conductor we should have $\rho = 0$. K in the last formula must be taken as the dielectric coefficient of the material of the sphere.

CHAPTER III

ELECTRIC INFLUENCE

22. Lines of Force.—We have already (**18**) defined the *intensity of electric force at a given point :* it is the resultant of the forces exerted by every portion of electricity upon a *unit* of positive electricity imagined as being at that point.[1]

The intensity of electric force has a definite magnitude and direction at every point in an electric field. If a line is drawn in the field so that it is, at every point in its course, tangential to the direction of electric force, it is called a *line of force.* It represents the path which would be followed by an electrified particle without mass, entirely free and unacted on by any other force. It is reckoned as running in the direction in which a positively electrified particle could move. Lines of force, therefore, are to be thought of as traversing the whole electric field from the positive to the negative boundary.

One line, and one only, passes through any given point of an electric field. If it were possible for two lines of force to pass through the same point, the resultant electric force at this point would have two directions at the same time, which is absurd.

As we have seen (**12, 18**), there is no electric force inside a conductor, and, just outside, the direction of force is everywhere normal to the surface. Consequently, lines of force terminate at conducting surfaces, and always meet them normally. A surface from which lines of force start is a positively electrified surface: one at which they end is negatively electrified.

If it were possible to draw all the lines of force of an electric field, the resulting diagram, being completely full of lines, would be unintelligible; but by drawing representative lines selected according to some easily recognisable system, the general course of the whole set may be indicated, and so the character of all parts

[1] As it is only in imagination that a unit of electricity is put at the point in question, there is no need to be concerned about possible disturbances of equilibrium which it might cause if put there in reality.

of the electric field, as to direction and intensity, can be represented. The rule generally followed is to make the number of lines such that the number traversing the unit area of a surface, supposed perpendicular to their direction at any part of the field, shall represent the electric intensity at that part. Hence, if the electric intensity diminishes as we follow a line of force, this is represented by divergence of the lines, and increase of intensity by convergence. A uniform field, that is, one in which the intensity and direction are everywhere the same, is represented by equidistant parallel straight lines, and reciprocally. In representing the space distribution of these lines on a flat surface special devices are adopted in individual problems, which will be described as they occur.

In many cases it facilitates thinking and reasoning about the properties of the electric field to refer the stresses of which it is the seat to the lines of force, and to think of these as having not merely a geometrical significance, but as possessing certain physical properties, namely, a tendency for each line to shorten, and a tendency for separate lines to repel each other.

23. Electrification by Influence.—There being no electric force within a conductor, there can be no lines of force within it. Suppose, then, an unelectrified piece of metal placed in an electric field: the space outside the surface of the metal is to be thought of as full of lines of force, and the space inside the surface as free from such lines. The conductor thus represents a gap in the field, or an interruption of the lines of force. The lines which previously traversed the space occupied by it are each cut into two branches, one branch ending on one part of the surface, and the other, originally the continuation of the same line, starting from another part. But (**22**) a surface at which lines of force end is negatively electrified, and one from which such lines start is positively electrified. Hence we should expect an insulated unelectrified conductor immersed in an electric field to acquire equal opposite electrifications at different parts of its surface, that part becoming positive which is presented towards the negative boundary of the field, and *vice versâ*.

Such electrification is found experimentally to occur, and is known as *electrification by influence.*

24. Laws of Electrostatic Influence.—In order to give a more complete account of electrostatic influence we will consider a special typical case of a kind that frequently occurs in actual experiments.

a. Consider an insulated metal ball or other conductor that has been positively electrified by an ordinary electrical machine, the

rubber of which is in electrical connection with the floor of the room. We have, then, an electric field extending from the ball on all sides to the floor, walls, and ceiling, which together constitute the second boundary. To assist the mind in picturing the distribution of force in the field, we may think of the lines of force as if they were stretched elastic threads extending between the two boundaries of the field, each one repelling every other, and their extremities capable of moving freely about on the conducting surfaces to which they are attached, but not able to leave these surfaces, except when, under extreme conditions, by the crowding together of lines, the force in any part of the field becomes so great that the medium breaks down and disruptive discharge takes place.

Mechanical consequences of the suppositions made as to the properties of the lines of force are that they would all meet the boundaries of the field at right angles, and that there would be the greatest concentration of lines at the parts of the field where the opposite boundaries are nearest together, and where, therefore, the lines are shortest. These results correspond to the facts already pointed out (**18**, **19**), that the direction of electric force close to the surface of a conductor is normal to the surface, and that the electric force is relatively great at any part of an electric field where the opposite boundaries come near together. Again, since the beginning of a line of force corresponds to a positive charge, and the termination of a line to a negative charge, and since the number of beginnings and endings are necessarily equal, the conception of lines of force includes the essential equality of the opposite charges on the two boundaries of an electric field. It was, moreover, pointed out in (**20**) that the electric force, just outside a uniformly electrified sphere, is equal to the surface-density of the charge multiplied by 4π and divided by the dielectric coefficient of the surrounding medium; and we shall see subsequently (**44**) that this relation between electric force and surface-density of electrification is quite general. Hence, a statement of the number of lines of force starting from or ending upon a given area, and a statement of the surface-density of the charge of the same area, are only different modes of expressing the same facts. The above statements may be verified experimentally by suspending a good-sized metal ball by a silk cord near one of the walls of a room, and electrifying it. The surface-density of the charge on different parts of the ball and of the wall can be easily compared by means of a proof-plane and electroscope.

b. Let the conductor, A, be completely surrounded by a hollow

conductor, B, of any shape whatever. This conductor divides the electric field into two parts, that between A and the inner surface of B, and that between the outer surface of B and the inside of the room. All the lines of force issuing from A are cut by B, so that the number starting from A and ending on the inner surface of B is identical, and so that also the continuation of every line ending inside B is represented by a line issuing from its outer surface and ending on the surface of the room. In other words, the surface of the charged body A, the inner and outer surfaces of the surrounding conductor B, and the surface of the room, are all charged with equal quantities of electricity, the charges of the two external surfaces (that of A and the outer surface of B) being of one kind, and those of the two internal surfaces (the inner surface of B and that of the room) being of the opposite kind.

Since no force acts inside the conductor B, no force can be transmitted through it, and the electric fields inside it and outside it are entirely independent of each other so far as concerns the distribution of force in each, or (what comes to the same thing) the density of charge at any part of the boundary of each. In fact, surface-density and distribution of lines of force in each field are entirely determined by the forces of that field.

c. The two fields—for convenience we may call them *internal* and *external*—are indeed so entirely independent of each other, that either of them may be destroyed without in any way affecting the other. For example, if the electrified body A be displaced so as to come into contact with the inside of B, the internal field ceases to exist, but no electrical effect of any kind is thereby produced outside B. Again, if the conductor B be put into contact with the inside of the room, either directly or by means of a wire, the external field is destroyed, but nothing whatever is thereby altered inside B. To say that an electric field anywhere ceases to exist is the same thing as to say that there all electric force ceases to act; hence, from the point of view which refers electric force to the action at a distance of electric charges, we may express the facts last stated by saying that the two internal charges mutually neutralise each other so far as regards all action outside the conductor B, and that the two external charges mutually neutralise each other in regard to action inside B. This experiment is often described as depending on the tendency of an electric charge placed inside a hollow conductor to pass to the outside surface. But the fact that this outside surface appears charged as long as the charged body A is inside, and that this external charge is in no way altered when A comes in contact with the outer

conductor, shows that such a statement of the case is at least incomplete.

d. If, after the external field has been destroyed, the conductor B is opened and the electrified body A is withdrawn, what was previously the internal field remains still unchanged as to the number of lines of force in it—that is, as to the quantity of electricity on its bounding surfaces,—but the form of the field is altered by the changed relative positions of its boundaries. In accordance with their tendency to shorten and to separate from each other, the lines of force which previously all ended on the inner surface of B come, more and more of them, to end on the outer surface as A is removed to a greater distance. If A is put in contact with the inside of the room, it becomes electrically part of a conducting enclosure surrounding B, and the result is that we have now an electric field whose positive boundary is the surface of the room, and whose negative boundary is the outside of the conductor B. We have then, in a sense, reversed the original field extending between A and the room: the charges remain throughout of the same magnitude as at first (always assuming perfect insulation), but the charge of the room is positive instead of negative, and that of the enclosed conductor is negative instead of positive.

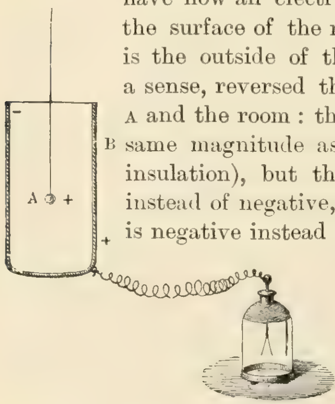


FIG. 18.

In cases *b*, *c*, *d*, the various results stated can be verified by means of a metal jar (Faraday used a pewter ice-pail) connected with a gold-leaf electroscope (Fig. 18), together with a second electroscope and a proof-plane. The conductor A may be a small metal ball at the end of a silk thread.

e. If an insulated conductor is placed in the electric field existing between any charged body and an enclosing surface, as that between the electrified body A and the room, it becomes charged, as pointed out in (**23**), differently at different parts of its surface, the parts presented towards each boundary of the field acquiring a charge opposite to that of the corresponding boundary. If the conductor is withdrawn, it still remaining insulated, or if the electric field is destroyed, all signs of the electrification of the conductor cease, thus proving that the positive and negative charges possessed by different parts were equal in amount. This might also be inferred *à priori* from the consideration that the conductor, having remained insulated, cannot have received positive or nega-

tive electricity from without, and therefore the production of one kind in or upon it implies an equal production of the other. The same conclusion also follows very simply from the consideration of lines of force. The negative charge of one part of the surface corresponds to the termination on that part of a certain number of interrupted lines of force, and the continuation of each of these lines must start from some other part of the surface.

For this experiment an insulated metal cylinder with rounded ends is convenient. It should be put lengthwise between a good-sized electrified body and the wall of the room, not very far from either. For this purpose any large, flat, conducting surface, electrically connected with the room, may represent the wall. The electrification of the cylinder can be tested with a proof-plane and electroscope, and the increase of influence with decrease of distance can be similarly proved. Another way of making the experiment is to hang at intervals along the cylinder pairs of pith-balls attached to very fine wires (Fig. 19): the greater or less divergence of the pith-balls indicates the greater or less density of charge at the different parts of the cylinder.

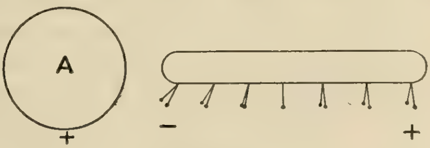


FIG. 19.

f. If the insulated conductor be brought nearer and nearer to the charged body, which for distinctness we will still call A and suppose to be positively electrified, it will interrupt more and more lines of force—that is, its influence-charges will become greater. When it gets very near to A, the parts of the lines of force extending between A and it, being very short, cannot repel each other much (the total repulsion between two lines being supposed proportional to their lengths); consequently they crowd more and more about the parts of the two conductors which are nearest together, until at last disruptive discharge takes place between A and the conductor brought near it. This is accompanied by the disappearance of all the lines in question, but the continuations of all these lines still extend between the insulated conductor and the surface of the room or other enclosure. This is the process of sharing the charge of one conductor with another, previously unelectrified, which is brought into contact with it. The total charge is not altered thereby, but merely its distribution.

g. If the insulated conductor, instead of being moved into contact with the conductor A, is made to approach and finally touch the surface of the enclosure, the resulting changes are essentially

the same as in the last case, with only such differences as arise from the fact that the conductor is in general passing from a stronger to a weaker part of the field. When it comes sufficiently near to the surface of the enclosure, mutual discharge takes place between the positive influence-charge on the conductor and an equal quantity of the negative charge on the enclosure, with the result that the conductor retains only a negative charge, which, together with what remains on the enclosure, is equal to the total original charge of the enclosure. In effect, the enclosure shares its charge with the insulated conductor, just as in the previous case the body A shared its charge. In the one case, the insulated conductor virtually becomes part of the positive boundary of the field; in the other, it virtually becomes part of the negative boundary.

h. If the insulated conductor, after coming into contact with the surface of the enclosure, be separated from it again, the lines of force from A end, some of them on the insulated conductor, and the rest on the surface of the enclosure. Let now A be brought into contact with the enclosure: the lines of force between it and the enclosure shorten up to nothing and the electrification represented by them disappears; however, the lines of force which just before ran from A to the insulated conductor remain, but their positive ends, instead of being confined to A, are now free to spread over the whole surface of the enclosure. We thus get, as in case *d* above, a reversed field, which, however, contains a smaller number of lines of force than the original field, instead of the same number, as in the previous case.

In cases *f*, *g*, *h*, an electrified ball hung inside a metal jar may represent the conductor A, and a proof-plane, also placed inside the jar, may represent the insulated conductor. Or, what is better, the inside of the room may represent the enclosure, and the apparatus employed in the verification of case *e* may be used.

It is easy to see that the results we have mentioned as being produced by displacing a conductor in an electric field may also be produced by connecting the conductor with one or other boundary of the field by means of a wire. In some cases this method would be experimentally more effectual than moving the conductor.

25. Conducting Sphere in a Uniform Field.—In the case of a conducting sphere in a uniform electric field, it is not difficult to calculate the surface-density of electrification required to satisfy the general condition, pointed out in (**18**), that the electric force at any point within the conductor shall be zero. The force at any such point must evidently be the resultant of the forces which

would be exerted there by the field and by the induced electrification of the sphere if these acted separately. And as the force of the field is, by supposition, everywhere the same in magnitude and direction, the problem comes to be to find such a surface-distribution of electricity on the sphere as would, if it existed independently, produce at all internal points a uniform force equal and opposite to that of the influencing field. We get this by supposing two spheres, at first coincident, of equal opposite electric densities, $+\rho$ and $-\rho$, uniform throughout, to undergo a small relative displacement AA′ (Fig. 20), the positive sphere moving in the direction of the force of the field, and the negative one in the opposite direction. The result of this is to give, as shown in the figure, a central portion, throughout which the electric densities $+\rho$ and $-\rho$ are superposed, and where therefore the resultant density is zero; while there is a layer of density $+\rho$ at B symmetrically distributed with respect to the diameter parallel to the force of the field, and a similar layer of density $-\rho$ on the opposite side of the sphere. The diameter just mentioned may be called the axis of electrification of the sphere, or simply the axis; the points where it cuts the surface may be called the poles; and a great circle perpendicular to the axis and half-way between the poles (passing through C and D in the figure), the equator of the sphere. The two electric layers, called *displacement-layers*, which respectively cover the two hemispheres, have a maximum thickness equal to AA′ at the poles. The equator separates the positive from the negative layer, and the thickness there is nothing.

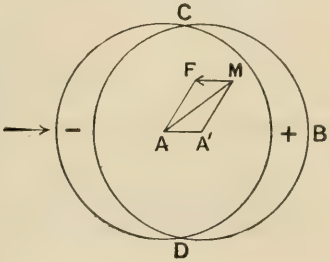


FIG. 20.

The required electrification of the conducting sphere in the case we are considering is given by supposing the two halves of its surface oppositely electrified to a surface-density proportional to the thickness of these displacement-layers.

To prove this, let us investigate the electric force at a point, M, within the two nearly coincident spheres. The force due to the positive sphere with centre A′, is $\frac{4}{3}\pi\rho \,.\, \text{A}'\text{M} \,.\, \frac{1}{K}$ (21), acting in the direction A′M, and that of the negative sphere, with centre A is $\frac{4}{3}\pi\rho \,.\, \text{MA} \,.\, \frac{1}{K}$, in the direction MA. In the triangle A′MA, the

sides A′M and MA are respectively parallel and proportional to the two forces acting at M, and consequently the third side, A′A, is parallel and proportional to their resultant. The resultant is therefore $\frac{4}{3}\pi\rho$. A′A. $\frac{1}{K}$, and is independent of the position of the point M—that is, the force due to the given distribution is uniform in magnitude and direction at all internal points. Further, it is evident that the force at any point, internal or external, would be unaltered by removing the coincident portions of the two spheres and leaving only the displacement-layers; for as the density throughout the region of coincidence = 0, the force must be what the displacement-layers would exert if they existed alone.

If we put $\sigma_\circ$ for the maximum surface-density, that at the pole B, we have

$$\sigma_\circ = \rho.\ \mathrm{AA}',$$

and for the force at an internal point due to the electrification of the sphere $-\frac{4}{3}\pi\sigma_\circ\frac{1}{K}$ the negative sign indicating that it acts in the direction A′A opposite to the force of the field. Putting f for the force due to the field alone, and considering that the resultant force inside the sphere is nothing, we get

$$f - \tfrac{4}{3}\pi\sigma_\circ\frac{1}{K} = 0,$$

which determines $\sigma_\circ$ when f is given.

To get the density elsewhere than at the poles, we must remember that the thickness of the displacement-layers is everywhere AA′ if measured parallel to the axis; hence as is easily seen on drawing a figure, the thickness measured along the radius at any point is AA′ cos α, if α is the angle which the radius through the point makes with the axis. The surface-density being proportional to the radial thickness of the displacement-layer is then given by

$$\sigma = \sigma_\circ \cos\alpha = \frac{3fK}{4\pi}\cos\alpha.$$

The value $\alpha = 90^\circ$ corresponds to points on the equator, where, as we have already seen, the density is nothing.

To get the action of the displacement-layers at external points, we may replace them by the two overlapping spheres with centres at A and A′. Each of these would act as though its whole mass were concentrated at the centre **(21)**; hence, the effect is the same as would be produced by two equal masses of opposite signs concentrated in the two points A and A′. The corresponding

magnetic problem is discussed in (**188**), and the result there arrived at applies equally to the present case.

The lines of force for a conducting sphere in a uniform field are shown in Fig. 21, which must be considered as a section of the field of force by a plane through the axis. The lines are so drawn that if they are imagined to rotate about the horizontal line of symmetry they will trace out surfaces of revolution, and these surfaces will mark out zones upon the sphere, each of which will contain the same charge.

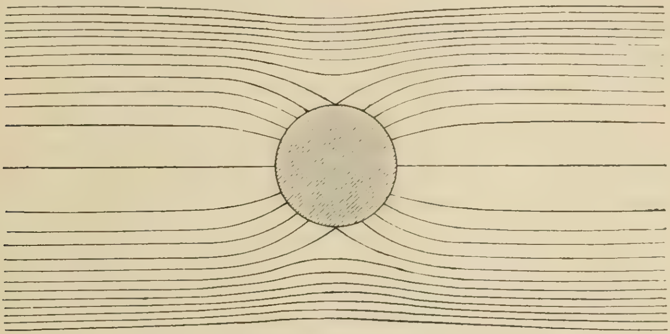


FIG. 21.

26. Electric Influence on Bad Conductors.—The effects of influence are less distinct with badly conducting substances. Such a body placed in an electric field seems to behave as though it were composed of a multitude of minute conductors, separated by an insulating medium, each of which is acted on by influence in the way already described, the side nearest the positive boundary of the field acquiring a negative charge and *vice versâ*. The perceptible effects are as though the body became electrified in the same manner as a conductor occupying the same space would do, but to a smaller extent. A non-conductor in an electric field is often said to be *polarised*.

If the action last for only a very short time, as soon as it ceases the polarisation disappears, and the body returns to its natural state. But if it last for a good while, the body may exhibit for some time afterwards two oppositely electrified regions. This effect may be supposed to result from a more or less complete mutual neutralisation of the opposite charges of neighbouring conducting particles taking place through the intervening imperfectly insulating medium.

27. Attraction of Light Bodies.—Let us return to the

experiment mentioned in (**1**), of the attraction of bodies that have not been previously electrified, when a body that has been electrified by friction or otherwise is brought near. It is evident that in such a case electrification by influence precedes attraction, and that what is really observed is attraction between opposite electric charges. It may indeed be taken as a fact proved experimentally that ordinary matter as such is not acted upon by electric force, which acts only between different quantities of electricity.

If a pith-ball pendulum is used to illustrate electric attraction, the effect is somewhat different according as the suspending thread is of linen or cotton, which are pretty good conductors, or of silk, which is a non-conductor. In the former case, the pith-ball is in electrical connection through the thread with the inside of the room, which commonly forms one boundary of the electric field; the pith-ball, therefore, forms a part of this boundary, and is naturally strongly attracted when the electrified body which forms the other boundary is brought near. When the suspending thread is an insulator, we have the pith-ball merely subjected to the electric field extending between the electrified body and the room. It is electrified by influence, as described above (**23**, **24** *e*), and it moves in obedience to the resultant force, which acts towards the stronger part of the field. As the means of testing whether a body brought near to it is electrified or not, a pith-ball suspended by a conducting thread is accordingly more sensitive; but if we want to find out *how* a body is electrified, we must use a pith-ball hung by a non-conducting thread: we electrify the ball in a known manner, positively or negatively, and observe whether repulsion or attraction takes place when the body to be tested is brought near.

28. Electrification of a Gold-Leaf Electroscope by Influence.—A very common example of electric influence, and one sufficiently important to make it desirable to refer to it specially, is presented by the gold-leaf electroscope (Fig. 7). Suppose the instrument, as usual, to be standing on the table, and that an electrified body is held over it at a distance of several inches, or even a foot or two. An electric field extends between the electrified body and the table, and envelops the electroscope. Lines of force run between the electrified body and the cap, and the continuation of the same lines between the leaves and the table, and the opening of the leaves is to be attributed to the tension in these latter lines, together with the repulsion between the lines proceeding from one leaf and those proceeding from the other.

If the cap is now touched with the finger, and thereby put

electrically into connection with the table, the lines of force between the latter and the gold-leaves disappear and the divergence ceases, but at the same time the field between the electrified body and the cap of the electroscope becomes stronger than before. After removing the finger, let the electrified body be removed; the electric field is thereby distorted, and we may think of the body as dragging after it the lines of force that are attached to it and stretching them out. As they become longer, their mutual repulsion asserts itself more strongly; they spread from the cap in greater or less numbers over the rest of the electroscope, some of them pass to the gold-leaves, and these again open. This action is carried to its limit if the electrified body is put into electrical contact with the room, but it comes to sensibly the same thing if it is removed in any direction to a much greater distance from the electroscope than the surface of the table or other parts of the second boundary of the electric field. It is important to note that the electrification which is now indicated by the electroscope is *opposite* to that of the electrified body brought near it at the beginning of the experiment. For definiteness, assume the body to have been positive, and therefore the table negative: in the first stage of the experiment, the divergence of the gold-leaves indicated lines of force running from the leaves to the table; the final divergence indicated lines running from the table to the leaves.

When an electric field of a known kind has been established between the electroscope and the table—that is, when the electroscope has been charged with a known kind of electricity—it serves as a delicate test to distinguish the nature of the electrification of any object brought near it. If a body, slowly moved up towards the cap of the electroscope from a considerable distance, causes the leaves to diverge continuously more and more, we may conclude that it is electrified in the same way as the electroscope. If, on the other hand, it makes the leaves gradually fall together, collapse completely, and then open again as it approaches nearer, we may conclude that it is electrified oppositely to the electroscope. In the first case, the electric field due to the body to be tested is superposed upon the field due to the electroscope and strengthens it. In the second case, contrary fields are superposed: as the second field is gradually strengthened, there is first weakening, then neutralisation, and finally inversion of the original field of the electroscope. If an oppositely electrified body is quickly brought near the electroscope, the inversion of the field may occur before the leaves have had time, by falling together,

to indicate the previous weakening of the original field. In such a case we may mistake the sign of the electrification. It is important, therefore, to bring up the body slowly towards the electroscope and to note the first visible effect. A mere diminution of divergence when a body is brought near the cap of the electroscope is not trustworthy evidence of contrary electrification. If the hand, or any uninsulated conductor, be held a little way above the cap, it alters the electric field, and so causes a diminution of the divergence of the leaves, in a way that is not difficult to follow out.

29. The Electrophorus.—The action of the electrophorus (Fig. 22) depends essentially on electrical influence. The instrument consists of a flat, smooth plate of resin or ebonite (called the *cake*), resting on a metal plate (the *sole*), and of a second metal plate (the *cover*), of rather less diameter than the cake, provided with an insulating handle.

To use the electrophorus, the cake is electrified by beating it with a catskin, the cover is put on, connected with the sole, insulated from it again, and lifted by the insulating handle. The cover is now found positively electrified (if the cake is negative as usual), and its charge may be transferred to any other conductor as desired. To renew the charge, it is only necessary to replace the cover, connect it for an instant with the sole, and raise it again by the insulating handle, and this can be repeated a great number of times in succession without its being needful to renew the electrification of the cake, which only undergoes a very gradual loss due to imperfect insulation.

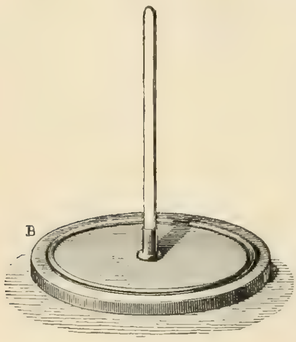


FIG. 22.

The electrical changes that take place may be described as follows:—By friction with the catskin, an electric field is set up, originally between the catskin (positive), and the resin (negative). When the catskin has been removed to a distance, the lines of force no longer extend all the way from it to the cake, but are interrupted by neighbouring conducting objects, usually by the table on which the electrophorus stands. They then shorten, and the electric field contracts so as to be confined almost entirely to the space occupied by the cake; the upper surface of the sole becomes the positive boundary, and the upper layer of the resin the

negative boundary. The lines of force are thus short lines, practically straight, running up through the cake. When now the cover is laid on and connected with the sole, the positive ends of the lines are to a great extent transferred from the sole to the cover, as they thereby become still shorter than before. A great part of the field is thus transferred to the upper side of the cake, where it occupies the thin stratum of air between the face of the cover and the cake, and also probably a thin layer of resin itself near the upper surface. When the cover is raised, the field is drawn out and soon divides into two, one extending from the cover to the surface of surrounding objects, and the other, as before, from the sole to the upper part of the resin.

CHAPTER IV

ELECTRICAL POTENTIAL

30. Conditions of Electrical Equilibrium.—If two electrified conductors are brought into contact, or, remaining at a distance, are connected by a third conductor, as, for instance, by a long fine wire, there is in general a change in the electrical state of both; whatever signs of electrification were previously shown by one of them become weaker, while those exhibited by the other become stronger. The effect is what is expressed by saying that electricity passes from the first to the second. In special cases, however, it may happen that no electrical change is produced—that no electricity passes either way. We then say that the connected conductors were in *electric equilibrium*. In any case, electric equilibrium is established in a very short time after connection is made.

Electric equilibrium, when attained, evidently consists in such a distribution of electricity that the resultant electric force at every point of the conductor, or system of conductors, in question is equal to nothing; for, if there were electric force at a point of a conductor, there would be displacement of electricity, and consequently equilibrium would not exist.

The question then arises, On what does electric equilibrium depend? What is the condition which, when possessed in the same degree by two conductors, makes it possible to connect them together without any alteration of their electric state? It is not equality of total charge, nor yet equality of surface-density, for two unequal spheres are not in equilibrium when they have either equal charges or the same surface-density.

31. Electric Potential.—The required condition, which is distinct from any that we have so far considered, is what is known as *electric potential*. This term expresses a condition in relation to electrified bodies which is analogous to that denoted by temperature in the case of heat. Just as bodies are said to have the same temperature if no transfer of heat from one to the other occurs when

they are put into contact, so electric conductors between which no transfer of electricity takes place when they are connected are said to have (or to be at) the same potential. On the other hand, if two conductors which have different potentials are connected, a passage of (positive) electricity takes place from the one which has the higher to that which has the lower potential, until equality of potential, and therewith electric equilibrium, is reached.

To speak of any system of bodies as being in a state of electric equilibrium, is equivalent to saying that there is no electric force tending to cause movement of electricity in any way among them. But all the sensible phenomena by which we become aware of the electrical state of bodies—such as the motion of an electroscope, or the passage of an electric spark—are evidences of electric force. Hence it follows, from what we have just said, that if all the objects about us were at the same potential, they would not exhibit any electrical phenomena, for, being at the same potential, they would all be in electrical equilibrium. This would still be the case, however high, or however low, the common potential might be. It consequently appears that we have no means of recognising the absolute potential of bodies, but only the *differences* between the potentials of different bodies.

The strict definition of electrical potential is founded upon the fact, just alluded to, that the existence of electric force implies a difference of electric potentials. We may define electric potential as a property of space whose value at any given point, above or below that of some point or body chosen as an arbitrary zero of potential, is susceptible of numerical statement, and such that the line through that point along which it decreases most rapidly is the direction of electric force, and that the rate of its decrease in any direction is equal to the intensity of force in that direction.

This definition, being necessarily expressed in general terms, may perhaps fail to convey any definite meaning on a first reading. If so, the following comparisons may help to make the matter clearer. Adopting a similar form, we might say that temperature is a property of space whose value at any given point above or below an arbitrary zero (say the temperature of melting ice) is susceptible of numerical statement, and such that the direction through that point in which heat would flow (if the point were within an isotropic conducting material) is the direction in which temperature decreases most rapidly, and that its rate of decrease in this direction is proportional to the intensity of the flow of heat. [The fact that we are not in the habit of speaking of *thermal force* as determining the transfer of heat between bodies of different

temperatures makes it impossible to keep the wording of the definition of temperature parallel throughout to the definition of potential.]

Potential may also be usefully compared with surface-level in the case of liquids; for the passage of electricity, from a conductor of higher to one of lower electrical potential when they are connected by a wire, is very closely analogous to the flow of water from a cistern at a high level to one at a lower level when a connecting pipe is opened between them. In fact, the analogy is so complete, that the expression "electrical level" is not infrequently used instead of electric potential.

In connection with the comparison between potential and temperature, it is to be noted that whereas the temperature of a body is not altered by mere change of position, so long as it does not lose or gain heat, the electric potential of a body is in general altered whenever it is moved in an electric field. Another point of difference is that, whereas important changes in the physical properties of matter are produced by changes of temperature, nothing of the same kind results from rise or fall of potential. Great differences of potential between neighbouring parts of a dielectric medium produce various physical effects, such as the mechanical shattering that accompanies discharge; but any portion of matter throughout which potential is uniform remains unaffected, however high or low the potential may be.

32. Electric Force and Potential.—In order to see what is involved in the definition of potential given in (**31**), we will apply it to a special case where the electric force is known, that of an electric field bounded by two concentric spherical conducting surfaces. Let P (Fig. 23) be a point in the field at a distance, r, from the common centre, O, of the spherical boundaries. If Q be the charge on the inner surface, and therefore $-Q$ the charge on the outer surface, the electric force at P will be $\frac{1}{K}\cdot\frac{Q}{r^2}$ acting in the direction OP (because the charge of the inner sphere will act at P as though it were concentrated at O, and the charge $-Q$ on the outer surface will exert no force at P). Hence the potential

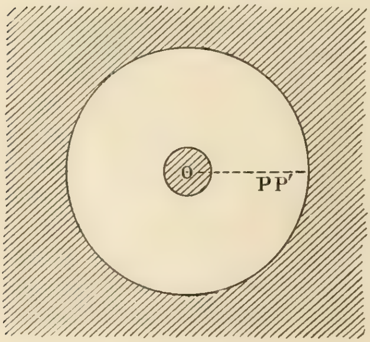


FIG. 23.

at P must be a quantity such that its rate of decrease is most rapid in the direction OP (or r), and is equal to $\frac{1}{K} \cdot \frac{Q}{r^2}$. Putting V for the potential at P, and V' for the potential at a point P′, distant r' from the centre, the change of potential as we pass from P to P′ is $V' - V$, and the average rate of change is this divided by the distance PP′, or $\frac{V' - V}{r' - r}$. If the point P′ be taken nearer and nearer to P, the distance PP′, or $r' - r$, becomes smaller and smaller, and may finally be denoted by dr, the difference of potentials, $V' - V$, simultaneously becoming infinitely small so as to be represented by dV. Hence we see that the rate of variation of the potential *at* the point P is expressible by the ratio dV/dr. The two factors of this ratio, however, represent increments of the corresponding quantities; hence, to express the rate of *decrease* of potential, which by the definition is the electric force at P, we must prefix a negative sign. We thus get finally for the electric force at P

$$f = -\frac{dV}{dr} = \frac{1}{K} \cdot \frac{Q}{r^2}.$$

By the definition of (**31**), the rate of decrease of potential in any direction must give the electric force in that direction. Therefore, if dV be again the change of potential that takes place in the distance ds, measured from P in any direction, s, making an angle θ with r, and if f_s be the component of electric force in this direction we must have

$$f_s = -\frac{dV}{ds}.$$

But resolving the resultant force f along s and at right angles thereto, we have also

$$f_s = f \cos \theta.$$

Consequently,

$$\frac{dV}{ds} = \frac{dV}{dr} \cos \theta,$$

and since $\cos \theta$ is always less than unity when θ is greater than 0, dV/ds is less than dV/dr, or the rate of change of potential along s is less rapid than along r.

33. Equipotential Surfaces.—Consider the case where θ is a right angle, that is, the case where ds is perpendicular to r, and therefore lies along the surface of a sphere passing through P, and with centre at O. We know that, in this case, the component

force $f_s=0$, and consistently with this, the last equation gives $\frac{dV}{ds}=0$ when θ is a right angle.

Put into words, this gives the result that there is no variation of potential along the surface of a sphere with centre O, or that the potential has the same value at all points of such a surface; or again, more shortly, that a spherical surface with centre O is an *equipotential* surface. And similar reasoning would show that in all cases a surface that is everywhere at right angles to the direction of electric force is an equipotential surface.

Conversely, an equipotential surface must be such that no electric force acts along it from one point to another of the surface. Hence electric force (if any) acts everywhere normally to an equipotential surface.

It follows that the surface of any conductor or system of conductors in electrical equilibrium is an equipotential surface, and that electric force (if any) is everywhere normal to such a surface.

34. Work Done by Electric Force—The formula $f_s = -dV/ds$ may be said to embody the definition of potential. If we multiply both sides by ds and write it

$$f_s\,ds = -dV,$$

it admits of a simple physical interpretation. The product of the force f_s into the infinitesimal displacement ds in the line of action of the force, would evidently be the work done by electric force on a unit of positive electricity moved from P through the distance ds. We may accordingly write

$$dW = f_s\,ds = -dV;$$

whence it appears that the rate of decrease of potential in any direction is identical with the rate at which work is done by electric force upon a unit of positive electricity displaced in that direction. It follows that the total work done upon a unit of positive electricity transferred from a given point, A, to another point, B (it being understood that the electric force is not altered in any way by the transfer), is equal to the whole decrease of potential from A to B, or, in symbols

$$W_{AB} = \int_A^B dW = \int_A^B -dV = V_A - V_B.$$

If the quantity of electricity transferred is any quantity Q, the work done is, of course, equal to Q times that which corresponds to the transfer of a positive unit. Thus, if we put W for the electric work done, when any quantity, Q, of electricity is trans-

ferred from a point where the potential is V to a point where the potential is V', we have

$$W = Q\,(V - V').$$

From this it appears that the electrical work done when electricity passes from any point of one equipotential surface to any point of another, depends only on the difference of potential of the surfaces, and not on the particular points between which the passage takes place, nor upon the path by which it occurs.

The proof of this very important proposition is contained in what has been said already, but it may be well to consider it from another point of view. Let M and N (Fig. 24) be portions of two equipotential surfaces having respectively the potentials V and V', then what we have to prove is that, if a quantity of electricity be transferred from any point, A, of the surface M to any point, B, on the surface N, the work done by electric forces is the same as if an identical quantity of electricity were carried from any other point, C, of the first surface to any point, D, of the second. That this is necessarily the case follows easily from the principle of the conservation of energy, combined with the conception of an equipotential surface. For if it be possible for these two amounts of work to differ, let the electric work along AB, per unit of electricity, exceed that along CD by any finite amount, say, 10 ergs: then it will require an expenditure of energy less by 10 ergs to carry a unit of electricity back from N to M by the path DC than to carry it by the path BA. Consequently we may carry a unit of electricity from M to N along AB, during which process an amount of work, say W ergs, is done by electric force, then transfer it from B to D along the equipotential surface N—during this process, since no force acts between any points on the same equipotential surface, no work will be done—next carry it to the surface M along the path DC, doing $(W - 10)$ ergs of work against electric force, and lastly restore it to the starting-point A, by carrying it along the equipotential surface M. As the result of this series of operations, no permanent electrical change whatever will have been produced, since all the electricity taken from A has been brought back to it, but 10 ergs more work will have been done by electric force than has been done against it, or there has been a gain of energy to the extent of 10 ergs, without any corresponding expenditure; and by repeating the same operation

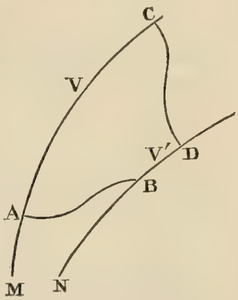


FIG. 24.

again and again, it would be possible to gain any amount of energy, a result inconsistent with the principle of conservation of energy. Hence we conclude that it is not possible for the amount of electric work done during the passage of electricity from one equipotential surface to another to vary with the path followed.

35. Difference of Potentials defined in terms of Electrical Work.—The proposition arrived at in (**34**) furnishes a new mode of defining difference of potentials, which may be put in the following form :—*The electric potential at any point,* A, *exceeds the potential at any other point,* B, *by an amount equal to the work that would be done by electric force upon a unit of positive electricity transferred from* A *to* B, it being understood that no change of force is caused by the transfer. If we put W for the work done when a quantity of electricity, Q, is transferred, the above statement is equivalent to the equation

$$V_A - V_B = \frac{W}{Q}$$

36. Difference of Potentials as depending on Electric Charge.—Returning to the consideration of an electric field contained between two concentric conducting surfaces (Fig. 25), such as that referred to in (**32**), it is easy now to calculate the difference of their potentials when the charge Q and the radii of the surfaces, say a and b, are given. As we have seen, the force at any point, P, distant r from the centre, O, is $\frac{1}{K}\cdot\frac{Q}{r^2}$, and is directed along OP. If we take a point, P′, very near P in OP produced, the force there is $\frac{1}{K}\cdot\frac{Q}{r'^2}$, and is nearly equal to the force at P. The average electric force along PP′ is therefore very near to either of these values, and still more nearly equal to the intermediate value $\frac{1}{K}\cdot\frac{Q}{rr'}$. Adopting this expression, we have for the electrical work corresponding to the passage of a unit of positive electricity from P to P′

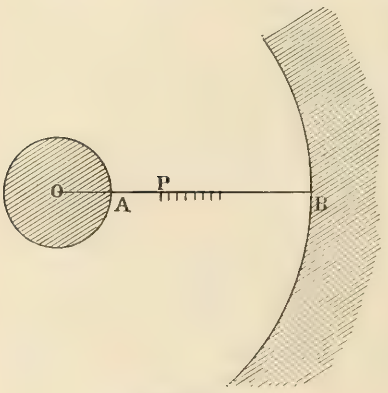


FIG. 25.

$$\frac{1}{K}\cdot\frac{Q}{rr'}(r'-r) = \frac{Q}{K}\left(\frac{1}{r}-\frac{1}{r'}\right).$$

Similarly, for the work done between P′ and P″, a point still farther from O at the distance r'', we have

$$\frac{Q}{K}\left(\frac{1}{r'}-\frac{1}{r''}\right),$$

with precisely similar expressions for each of the very small segments into which the line AB, extending from one boundary of the field to the other, may be subdivided. The total work done by electric force when the unit of electricity is carried from A to B, or the amount by which the potential V at A exceeds the potential V' at B, would be found by adding together all such expressions as the above formed by giving to r in succession every value from $r=a$ to $r=b$. Accordingly we get

$$V-V'=\frac{Q}{K}\left[\left(\frac{1}{a}-\frac{1}{r_1}\right)+\left(\frac{1}{r_1}-\frac{1}{r_2}\right)+\;.\;.\;.\;.\;+\left(\frac{1}{r_n}-\frac{1}{b}\right)\right].$$

In this sum, every term, except the first and last, enters once negatively and once positively, and therefore disappears from the result, which becomes

$$V-V'=\frac{Q}{K}\left(\frac{1}{a}-\frac{1}{b}\right).$$

Similarly, for the difference of potentials between a point at an intermediate distance, r, from the centre O, and the outer surface, we should have

$$V_r-V'=\frac{Q}{K}\left(\frac{1}{r}-\frac{1}{b}\right).$$

If the radius b of the outer boundary of the field is very great, its reciprocal $1/b$ is very small, and may be neglected in comparison with $1/r$; hence, in such a case, the last expression becomes

$$V_r-V'=\frac{Q}{Kr}.$$

And if the outer boundary of the field, without being spherical, is of great extent and everywhere very distant from the inner spherical surface,—as, for example, the walls, floor, and ceiling of a large room near the middle of which a small electrified ball is suspended,—it is evident that the force within a moderate distance of the sphere will be sensibly the same as in the last case, and that the force at great distances from it will be very small; consequently the difference of potentials between a point at distance, r, and the outer boundary of the field will still be $\dfrac{Q}{Kr}$.

The potential of the earth or of any conductor connected with

it is usually taken as the conventional zero of potential. Accordingly if V' is the potential of the walls of a large room, as in the case supposed above, we may put $V'=0$, and the potential at any point whose distance, r, from the centre of an isolated electrified sphere is small compared with its distance from the nearest point of the surrounding surface becomes

$$V_r = \frac{Q}{Kr}.$$

At the surface of the sphere this becomes $V_a = \frac{Q}{Ka}$; and, since there is no force within the sphere, this must also be the value of the potential at all internal points.

The equipotential surfaces outside the sphere are spheres concentric with it, the difference of potential between any two of them being proportional to the difference of the reciprocals of the radii.

37. Potential due to Several Charges.—Suppose now that, in the neighbourhood of a given point, P, there are several electrified bodies, each so small compared with its distance from P that its charge acts as though it were concentrated at a single point, and suppose also that the point P is very distant from any other electrified bodies; then the potential at P will be the algebraic sum of the potentials due to the several electrified bodies, each supposed to act by itself. Thus the expression for the potential at the point P becomes

$$V = V_1 + V_2 + \ldots = \frac{1}{K}\left(\frac{Q_1}{r_1} + \frac{Q_2}{r_2} + \ldots\right) = \frac{1}{K}\Sigma\left(\frac{Q}{r}\right),$$

where the symbol $\Sigma\left(\frac{Q}{r}\right)$ means that each charge is to be divided by its distance from the given point, and that the sum of the quotients is to be taken.

In general, the distribution of electricity in the neighbourhood of a given point cannot be taken account of by treating it as being concentrated at a finite number of separate points. The process of calculating the potential at a point is then equivalent to considering the whole quantity of electricity as subdivided into an infinite number of infinitesimal portions, dQ, each at its own distance, r, from the point, and determining the value of the integral $\int\frac{dQ}{r}$ for the whole.

As a special case, suppose a quantity of positive electricity, Q, concentrated at a point, A (Fig. 26), and a quantity Q' at A′; the

potential at any point, P, whose distances from A and A′ are r and r' respectively, is then given by

$$V=\frac{1}{K}\left(\frac{Q}{r}+\frac{Q'}{r'}\right).$$

If we make V constant, this equation characterises an equipotential surface. All such surfaces, in the case supposed, are symmetrical

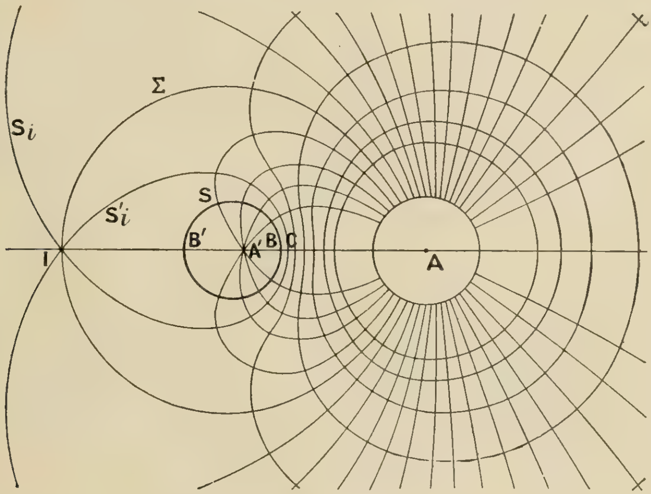


FIG. 26.

about the line AA′ as an axis, and Fig. 26 represents a section by a plane through the axis of a system of equipotential surfaces corresponding to the values $Q=+20$ and $Q'=-5$, the potentials of consecutive surfaces differing by unity from each other. The surface given by the equation

$$\frac{Q}{r}+\frac{Q'}{r'}=0,$$

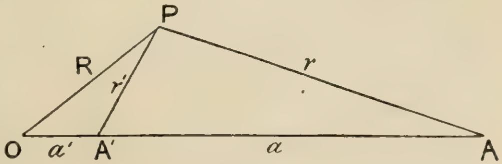


FIG. 27.

or the surface of zero-potential, is a sphere S surrounding the point A′. For if from any point P on this surface (Fig. 27) a line PO be drawn making the same angle with PA′ that PA makes with AA′, and cutting AA′ produced in O; and if we put

$$AP=r,\quad A'P=r'$$
$$OP=R,\quad OA'=a',\quad OA=a,$$

we have

$$\frac{r'}{r}=\frac{R}{a}=\frac{a'}{R}$$

whence

$$\frac{r'^2}{r^2}=\frac{a'}{a} \text{ and } 1-\frac{r'^2}{r^2}=\frac{a-a'}{a}=\frac{\text{AA}'}{a}.$$

But the left-hand side of this equation is constant, and since AA′ is also constant, a must be constant; in other words, O is a fixed point for all positions of P. Again, the first equation above shows that R is $a \times$ a constant, *i.e.* OP is a fixed length and therefore P must be on a sphere with centre O. At all points inside this surface the potential is negative; and at all points outside it is positive.

One of the equipotential surfaces, marked S_i in Fig. 26, consists of two sheets which cut each other at the point I, where the electric force is nothing, that is to say, the point for which

$$\frac{Q}{r^2}+\frac{Q'}{r'^2}=0, \text{ or } \frac{r^2}{r'^2}=-\frac{Q}{Q'}.$$

For the point I, we have $r = r' + \text{AA}'$, and therefore, with the values adopted for Q and Q', $(r' + \text{AA}')^2/r'^2 = 4$, whence $r' = \text{AA}'$ and $r = 2\text{AA}'$. This gives, for the potential at I, and therefore for the whole of the equipotential surface we are considering

$$V=\frac{20}{2\text{AA}'}-\frac{5}{\text{AA}'}=\frac{5}{\text{AA}'},$$

and since in the figure AA′ is taken equal to 2·5 cm., the numerical value of V is 2. The surfaces with potentials greater than 2 are closed surfaces enclosing the point A only. The surfaces having positive potentials less than 2 consist of two separate parts, one being a closed surface lying between the inner loop of the surface S_i and the sphere BB′ of potential zero, and the other, also a closed surface, lying outside the outer loop of the surface S_i and enclosing both the points A and A′. These are not shown in the figure.

The curves which are shown cutting the equipotential surfaces at right angles represent lines of force. These are represented on the same principle as those in Fig. 21. It will be seen that 40 lines diverge from A, and that 10 converge to A′, thus indicating the relative magnitude of the charges supposed to exist at these points respectively. The remaining 30 lines from A go off to infinity.

38. Electrical Images.—If a conductor is introduced into an electric field, it causes in general a change of potentials throughout, which may be expressed as a distortion of the equipotential surfaces.

For example, if a metal cylinder is introduced into the field referred to at the end of (**36**), where the equipotential surfaces are concentric spheres, the spheres will be distorted. The surface of the cylinder will form one sheet of an equipotential surface of some definite value, say V_1, and will be intersected at right angles along the line of zero surface-density by a deformed surface of the same potential derived from the original field. The other equipotential surfaces will be bent so as to follow the form of the cylinder more and more closely as their potentials are nearer to the special value V_1, those of higher value than V_1 passing between the cylinder and the sphere and enclosing the latter only, while those of lower value pass outside the cylinder and enclose both it and the sphere.

The only case in which no such distortion is produced is when the surface of the conductor coincides with one of the pre-existing equipotential surfaces. For example, if an earth-connected spherical conductor be placed so as to coincide with the spherical surface of zero potential (Fig. 26) the field both inside and outside will remain unchanged. The inside field may now be destroyed without affecting that outside (**24** *c*). We therefore conclude that the field surrounding a point-charge in the neighbourhood of an earth-connected conducting sphere is identical at all points external to the sphere with that due to the same point-charge, together with a second one of magnitude and position determined by the equations in the previous section. Hence, if we can determine the field corresponding to the latter case, we have in so doing determined the field due to the given point-charge and an earth-connected sphere in its neighbourhood,—a case which it would be very difficult to deal with by a direct method. The second point-charge is called the Electrical Image of the real one, and this method of solution, which was devised by Lord Kelvin, is called the Method of Images. The data necessary would be the distance a from the point-charge Q at A to the centre of the sphere O and also the radius of the sphere. The image must be placed at a point A′ on the line OA such that OA′ . OA $= R^2$; its magnitude must be such that $Q'/Q = -R/a$.

The real charges present are the point-charge and the electrification upon the surface of the sphere. We can calculate the surface-density of the latter by first finding the resultant force just outside the sphere. Referring to Fig. 27, the force at P is the resultant of a force $Q/(\text{AP})^2$ acting along AP, and a force $Q'/(\text{A}'\text{P})^2$ acting along PA′. Resolve each of these into two components, one along OP (and therefore normal to the sphere), the

other parallel to OA. Since the resultant force must be normal to the sphere the algebraic sum of the latter components must be zero. If in the triangle APO the side AP be taken as representing the force due to Q, the required components are represented by the lengths of the sides AO and OP; similarly, if PA′ represent the force along PA′, its required components are PO and OA′. Hence the components along OP are

$$\frac{Q}{(\text{AP})^2}\,\frac{\text{OP}}{\text{AP}} \quad \text{and} \quad \frac{Q'}{(\text{A}'\text{P})^2}\cdot\frac{\text{OP}}{\text{A}'\text{P}}$$

whence, remembering that $Q' = -QR/a$, the resultant force at P is

$$\begin{aligned} Q\,.\,\text{OP}&\left(\frac{1}{(\text{AP})^3} - \frac{R}{a}\cdot\frac{1}{(\text{A}'\text{P})^3}\right) \\ &= \frac{QR}{(\text{AP})^3}\left(1 - \frac{R}{a}\left(\frac{\text{AP}}{\text{A}'\text{P}}\right)^3\right) \\ &= \frac{Q}{(\text{AP})^3}\cdot\frac{R^2-a^2}{R} \end{aligned}$$

since AP/A′P is constant and equal to a/R. The surface-density at P is this quantity divided by 4π; it is always negative (for a is necessarily greater than R); and it varies inversely as the cube of the distance of P from the charge at A. If R is infinite all portions of the sphere at a finite distance lie on a plane; if d is the distance of the point-charge in front of this plane $a = R + d$ and the electric force at a point P on the plane is

$$\frac{Q}{(\text{AP})^3}\cdot\frac{R^2-a^2}{R} = \frac{Q}{(\text{AP})^3}\cdot\frac{R+a}{R}\,(R-a)$$

$$= -\frac{Q}{(\text{AP})^3}\cdot\frac{2R+d}{R}\,d = -\frac{Q}{(\text{AP})^3}\,2d, \text{ since } R \text{ is infinite.}$$

The image in this case is of the same magnitude as the point-charge and the same distance as it from the plane, but on the opposite side. These facts might be inferred either from the equations above which determine them, or from the obvious fact that in a field due to two equal opposite point-charges the potential will be zero at every point of the plane of symmetry normal to the line joining the charges.

If, instead of being earth-connected, the sphere is insulated and has an independent charge Q_1 the condition will be the same as if, having been first earth-connected, it had acquired, under the influence of the point-charge Q a charge $-Q\frac{R}{a}$, distributed in

accordance with the formula already arrived at, and then, being insulated, it had received a further charge $= Q_1 - \left(-Q\frac{R}{a}\right) = Q_1 + Q\frac{R}{a}$ distributed uniformly. The surface-density at any point P is therefore given by superposing the uniform density $\frac{Q_1 + Q \cdot R/a}{4\pi R^2}$ upon the density already arrived at. This gives the value

$$\frac{Q_1}{4\pi R^2} + \frac{Q}{4\pi R}\left(\frac{1}{a} - \frac{a^2 - R^2}{\mathrm{AP}^3}\right).$$

The first term of this expression evidently represents the uniform surface-density which would result from the charge Q_1 in the absence of the external charge Q, and the second term the surface-density which would result from the influence of the external charge upon the sphere if insulated and without charge ($Q_1 = 0$).

If the sphere is insulated and at potential V, it must have a charge Q_1 determined by the equation

$$V = \frac{Q_1}{R} + \frac{Q}{a}, \quad \text{or} \quad Q_1 = RV - Q\frac{R}{a}.$$

Putting this value for Q_1 in the expression for surface-density, we get

$$\frac{1}{4\pi R}\left(V - Q\,\frac{a^2 - R^2}{\mathrm{AP}^3}\right).$$

The mechanical force between the external charge and the sphere may be found by considering as before that its actual charge Q_1 is made up of the two parts, (i.) $-Q\frac{R}{a}$, acting externally as though it were concentrated at A′, the image of A, distant $\frac{R^2}{a}$ from the centre of the sphere and therefore at distance $\frac{a^2 - R^2}{a}$ from A; and (ii.) $Q_1 + Q\frac{R}{a}$ uniformly distributed over the surface, and therefore acting externally as though it were concentrated at the centre. Applying the law of inverse squares to these two parts of the charge separately, we get for the resultant force—

$$\begin{aligned} f &= \frac{Q\left(Q_1 + Q\frac{R}{a}\right)}{a^2} - Q^2\frac{R}{a} \cdot \frac{a^2}{(a^2 - R^2)^2} \\ &= \frac{QQ_1}{a^2} - Q^2\frac{R^3}{a^3} \cdot \frac{2a^2 - R^2}{(a^2 - R^2)^2}. \end{aligned}$$

The first term of the last form of the expression gives the force

between the charge Q at A and the charge Q_1 if distributed uniformly over the sphere. If Q_1 and Q are of opposite signs, it is clear that f must be in all cases negative, that is to say, that the force is an attraction. If they are of the same sign, the force may be either positive or negative. If, for shortness, we put r for a/R, the above expression gives $f=0$, when the ratio of the charges is

$$\frac{Q_1}{Q} = \frac{2\,r^2-1}{r\,(r^2-1)^2}.$$

If the ratio Q_1/Q has a greater value than this, the force is positive (repulsion); if it has a smaller value, the force is negative (attraction). Or, for a given ratio of charges the force may be positive or negative according to the value of r. Thus, for $Q_1=Q$, we have $f=0$ for $a=1{\cdot}618\ R$, while f is positive for greater values of a and negative for smaller ones. These results have an evident bearing on the familiar fact that a slightly charged pith-ball is repelled by a similarly electrified body held at some distance from it, whereas it may be attracted if the electrified body is brought nearer.

In any real case it is impossible for a finite charge to exist at a mathematical point: hence, in strictness, the above solutions only represent the limiting cases which are approached as the influencing body is made smaller and smaller; in real cases, however, they may be taken as giving a second approximation. Further approximations can be obtained by an extension of the method; as an illustration, we will take the case of two charged spheres of radii R and R', potentials V and V', and with their centres at O and O$'$ separated by the distance D. To determine the charges **Q** and **Q$'$** of the two spheres respectively, we may proceed as follows: First find the charges Q_1 and Q'_1 required to bring the first sphere to potential V while the second is at potential 0, and then the charges Q_2 and Q'_2 needed to bring the second sphere to potential V' while the first is at 0. Then, since each of these systems of electrification is separately in equilibrium, there will still be equilibrium if they are superposed. Accordingly, adding the charges Q_1 and Q_2 we get the total charge Q of the first sphere, and adding Q'_1 and Q'_2 we get the total charge Q' of the second sphere. Similarly, summing the resulting potentials, we get $V+0$ and $0+V'$, or V for the first and V' for the second, resulting from the coexistence of the total charge $Q=Q_1+Q_2$ on the first sphere and the charge $Q'=Q'_1+Q'_2$ on the second sphere.

As a first approximation to the charge Q_1, we may disregard the effect of the second sphere, and thus get $Q_1=RV$ uniformly

distributed, and therefore acting externally as though it were concentrated at O at distance D from the centre of the second sphere. But the presence of this charge on the first sphere would bring the potential of the second sphere to $\frac{RV}{D}$, and to restore it to zero the second sphere must receive a charge $= -\frac{RV}{D}R'$, and this would act externally as though it were concentrated at A′, a point on the line OO′ at a distance from the centre O′ $= d' = \frac{R'^2}{D}$. This charge, in its turn, would disturb the potential of the first sphere, which would now no longer be V, but $V - \frac{RR'V}{(D-d')D}$. To restore the required value, the first sphere must receive a further charge $\frac{R^2R'V}{(D-d')D}$, the external action of which would be given by treating it as concentrated at a distance from the centre O $= d = \frac{R^2}{D-d'}$. But this charge again would disturb the potential of the second sphere, and to maintain this at zero would require a still further charge to be given to the second sphere. Following out such considerations we see that they lead to a never-ending series of successive adjustments of the potential of each sphere, each of which disturbs the potential of the other. The successive disturbances and consequent readjustments, however, become progressively smaller and smaller, so that a fairly close approximation can usually be attained by a comparatively small number. To explain more definitely how the process can be carried out we will consider a concrete case. Suppose two spheres of radii 2 cm. and 3 cm. with their centres 6 cm. apart, and let it be required to find the charges which will make the potentials 120 and 10 respectively. We have to calculate separately the charges to produce potentials 120 and 0 and those to produce 0 and 10 at the two spheres. Considering first those needed to produce the potentials 120 and 0, let Q_n be the n^{th} charge given to the first sphere and d_n its distance from the centre of the sphere; also let Q'_n, d'_n denote similar quantities for the second sphere; then, putting in numerical values,

$$Q_n = -Q'_{n-1} \cdot \frac{R}{D-d'_{n-1}} = -Q'_{n-1} \cdot \frac{2}{6-d'_{n-1}}$$

$$Q'_n = -Q_n \cdot \frac{R'}{D-d_n} = -Q_n \cdot \frac{3}{6-d_n}$$

$$d_n = \frac{R^2}{D-d'_{n-1}} = \frac{4}{6-d'_{n-1}} \qquad d'_n = \frac{R'^2}{D-d_n} = \frac{9}{6-d_n}$$

Starting with $Q_1 = RV = 2 \times 120 = 240$, these formulæ enable successive terms to be calculated; the first six are recorded in the following table:—

n	Q_n	d_n	$6 - d_n$	Q'_n	d'_n	$6 - d'_n$
1	240·	0·	6·	− 120·	1·5	4·5
2	53·333	0·8889	5·1111	− 31·304	1·761	4·239
3	14·770	0·9436	5·0564	− 8·763	1·780	4·220
4	4·153	0·9478	5·0522	− 2·465	1·781	4·219
5	1·169	0·9481	5·0519	− 0·694	1·781	4·219
6	0·329	0·9481	5·0519	− 0·195	1·781	4·219
7 to ∞	·128	...	...	− 0766	...	...

The last row gives approximate values of the sum of all the remaining terms. These are obtained by observing that when d and d' become sensibly constant (already, at the sixth term, they are nearly so) each series of terms forms a geometric progression, the sum of which can be obtained. The sum of the values of Q_n, *viz.* 313·88, and the sum of the values of Q'_n, *viz.* − 163·50, are the two charges required to maintain the spheres at potentials 120 and 0. We require now to find the charges which would maintain them at 0 and 10. These are found in the same way, and the values of the first few terms are here tabulated:—

n	Q_n	d_n	$6 - d_n$	Q'_n	d'_n	$6 - d'_n$
1	...	...	...	+ 30·	0·	6·
2	− 10	·667	5·333	5·626	1·6875	4·3125
3	− 2·609	·9275	5·0725	1·544	1·7743	4·2257
4	− 0·731	·9466	5·0534	0·434	1·7810	4·2190
5	− 0·206	·9481	5·0519	0·123	1·7815	4·2185
6 to ∞	− 0·081	...	...	0·048	...	...

The sum of the values of Q_n is − 13·63,
and that of Q'_n is + 37·77,

and these are the charges that would maintain the spheres at potentials 0 and 10.

Hence the charges that would maintain them at 120 and 10 are

$$\mathbf{Q} = 313{\cdot}88 - 13{\cdot}63 = 300{\cdot}25$$
$$\mathbf{Q}' = -163{\cdot}50 + 37{\cdot}77 = -125{\cdot}73$$

39. Units of Measurement.—If we employ, in the preceding formulæ, the C.G.S. electrostatic system of units, work is expressed in *ergs*, and the unit difference of potentials exists between two

points when the transfer of an electrostatic unit of electricity from one point to the other requires work equal to one erg.

For practical purposes the unit of quantity generally adopted is the *coulomb*, which corresponds to 3×10^9 C.G.S. electrostatic units **(15)**, and differences of potential are expressed in terms of the *volt*, corresponding to $\frac{1}{300}$ of a C.G.S. electrostatic unit of difference of potential. The theoretic basis of these practical units will be explained later (Chapter XXXI.).

The work represented by the product of a volt into a coulomb is thus equivalent to $\frac{1}{300} \times 3 \times 10^9 = 10^7$ ergs. As this quantity recurs constantly in calculations, it has been adopted as the *practical unit of work*, and has been named a *joule.*

The joule being 10^7 ergs, and a kilogrammetre being $10^5 \times g$ ergs, it follows that a joule is equal to $100/g$ kilogrammetres, or to 1/10 kilogrammetre nearly, since g is about 981.

Thus in the fundamental formula—

$$W = Q\,(V - V'),$$

if Q and $V - V'$ are expressed in the C.G.S. *electrostatic units*, W is a number of *ergs;* whereas if Q is expressed in *coulombs* and $V - V'$ in *volts,* W is expressed in *joules.*

CHAPTER V

GENERAL THEOREMS

40. Electric Induction.—We have seen that the electric force in a field may be conveniently represented by lines whose direction is that of the force. In discussing the properties of a field, we shall find it convenient to define another quantity whose component in any direction at any point is everywhere equal to $K/4\pi$ times the component of the electric force in the same direction, where K is the dielectric constant of the medium. This quantity is called the *electric induction* (or simply, the *induction*) at the point; we shall denote it by the letter D. Its direction (at least in isotropic media, for which the value of K is independent of direction) is the same as that of the electric force, and consequently the lines which represent the force will in this respect serve to represent the induction also. In accordance with the above definition the value of the induction at a distance r from a point-charge Q will clearly be $\frac{Q}{Kr^2} \cdot \frac{K}{4\pi}$, *i.e.* $\frac{Q}{4\pi r^2}$, and is independent of the nature of the medium.

41. Tubes of Induction.—Let A be an equipotential surface, and dA an element of this surface at the point P (Fig. 28). If lines of induction be drawn through every point in the contour of this element they form a kind of tube which is everywhere perpendicular to the equipotential surfaces intercepted by it; such a tube we shall call a *tube of induction.*

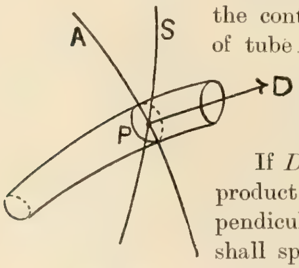


FIG. 28.

If D is the induction at the point P, the product DdA of the induction into the perpendicular section of the tube at P is what we shall speak of as the *flux of induction* within the tube at the point in question.

The tube cuts out an element dS from a surface S drawn in any manner through the point P. The flux of induction across any element of this surface is the product $D_n dS$ of the component of the induction normal to the element into the surface of the

element, and has the same numerical value as the flux through the right section dA of the tube in question; for if α is the angle between the normals to the two elements, we have

$$dA = dS \cos\alpha \qquad D_n = D\cos\alpha,$$

wherefore
$$D_n dS = DdA.$$

The total flux which traverses any surface is the sum of the fluxes across the various elements.

These expressions have an exact parallel in the case of the flow of an incompressible fluid, such as water is in relation to ordinary pressures. Referring again to the same figure (Fig. 28) let P be now a point within a quantity of water in a state of steady motion, and let dA be the area of an element at P of a surface A such that the direction of flow of the water particles at every point of this surface is normal thereto. Then if lines of flow are drawn through every point in the contour of the small area dA, these lines will all lie in a tubular surface which will be intercepted at right angles by the surface A. Such a tubular surface is called a *tube of flow*. If the cross-section dA is taken sufficiently small the velocity at every point of it is sensibly the same, say D; then the volume of the water which passes dA in unit of time is DdA. Again if dS is the area of an oblique section of the tube by any surface S and D_n the velocity normal to this section, the flux through dS, *i.e.* $D_n dS$, is at once seen to be equal to DdA.

It is on the basis of this analogy between the distribution of velocity in a mass of moving liquid and the distribution of induction in an electric field that the ideas of tubes of induction and fluxes of induction are founded. It may further be pointed out that the relation between electric force and induction is analogous to (though not identical with) that between mechanical force and velocity in the case of the resisted motion of a fluid. The reader, however, must be warned that mathematical equations are often identical in form although representing essentially different processes. Hence, it must be clearly understood that it is not intended to identify the quantity which we have called induction with a *velocity* in the medium, though many of the mathematical properties of the two things are similar to one another.

In establishing the following theorems (**42** to **49**) it is assumed that the dielectric is homogeneous; *i.e.* that the constant K has everywhere the same value.

42. Gauss's Theorem.—*Given a closed surface drawn in any way in an electric field, the total flux of induction across this surface is equal to the total quantity of electrification within it.*

We may observe, in the first place, that if a charge q is placed at the point O (Fig. 29), and if a cone of indefinitely small solid angle $d\omega$ be supposed to start from this point, the flux within this cone is the same at all parts.

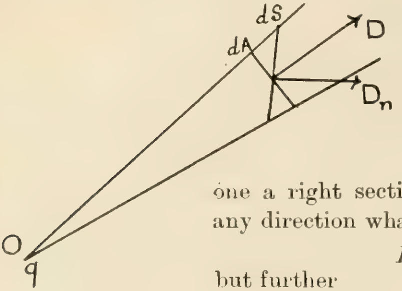


FIG. 29.

For consider two sections of the cone at the point A at the distance r from O, one a right section, dA, and the other, dS, in any direction whatever; we have

$$D_n dS = D dA,$$

but further

$$D = \frac{q}{4\pi r^2} \text{ and } dA = r^2 d\omega.$$

The flux is therefore

$$D_n dS = \frac{q\,d\omega}{4\pi},$$

and is independent of the distance r—that is to say, of the position of the point A.

Let us now draw in the field an imaginary, wholly convex surface (that is to say, a purely geometrical surface, and not the surface of a conductor), and consider an electric charge placed at the point O (Fig. 30).

The cone of infinitely small solid angle $d\omega$ intercepts two elements dS and dS', for which the fluxes are both equal; but as regards the surface they should be taken as of contrary signs, *because one enters the surface and the other leaves it;* we shall take outward flux—that is to say, that for which the normal component is directed to the outside—as positive. We have then, $D_n dS + D'_n dS' = 0$. All the surface may thus be cut into pairs of elements, the fluxes through which mutually compensate one another. If the surface has concavities, and if the cone in question cuts it more than twice, it cuts it an even number of times. The reader should verify this statement by trying to draw

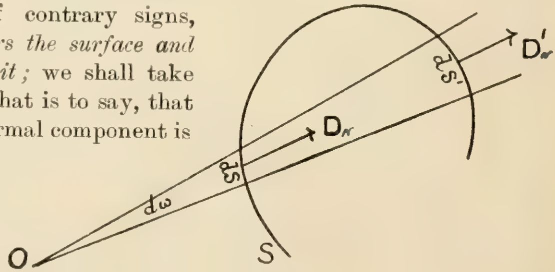


FIG. 30.

the outline of a surface for which this is not true; the sum of the elementary fluxes, which are always equal to each other and alternately of opposite signs, is still equal to nothing. Thus, the total flux which traverses a closed surface is zero when this flux proceeds from a charge outside the surface.

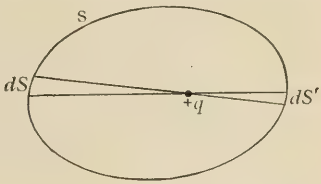


FIG. 31.

Consider now a charge q (Fig. 31) situated inside the surface. The two elements, dS and dS' intercepted by an infinitely small cone give as before

$$D_n dS = D'_n dS' = \frac{q\,d\omega}{4\pi}$$

but the two fluxes are of the same sign, positive if the charge is positive and negative if the charge is negative. To obtain the total flux through the entire surface, $d\omega$ must be replaced by 4π, that is, by the sum of the solid angles of all the cones into which the whole region round O may be divided. That is: The flux through a closed surface proceeding from a charge inside the surface is numerically equal to the charge itself.

Next consider any charges whatever, $q, q', q'', \ldots q_1, q_2, q_3, \ldots$ (Fig. 32), some inside and some outside the surface. On an element, dS, each charge, whether internal or external, has a normal component, and in order to get the total flux across the surface we must take the sum of the products obtained by multiplying each element by the normal component due to each of the charges. If in this sum all the terms corresponding to the same charge are added together the sum is zero if the charge is outside; and is equal to the charge if it is inside the surface. The total sum is therefore equal to the algebraical sum of all the internal charges.

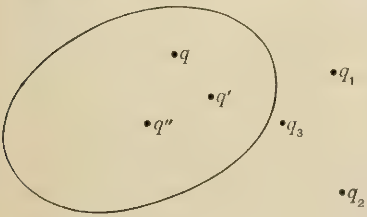


FIG. 32.

A theorem equivalent to this (within a constant) was given by Gauss for the case in which the dielectric constant is unity; although the theorem is stated above in a somewhat more general form, we shall refer to it as Gauss's theorem.

Conversely, in order to find the electric charge contained within a closed surface, it is sufficient to determine the outward normal

component of induction at each point, and to add all the products $D_n dS$.

43. Properties of Tubes of Induction.—Let us consider a tube of induction (Fig. 33) bounded by any two surfaces s and s′, and let us apply Gauss's theorem to the volume thus defined. If it contains no electricity, the total flux through its surface is nothing; but as the lateral surface has no influence on the sum since the normal component of induction is everywhere nothing, the sum reduces to the flux over the two ends, which must, therefore, be equal and of opposite signs. From this there results the following important consequence:—*At all points of the same tube of induction the value of the flux is constant.* If the tube is infinitely narrow and the terminal faces are perpendicular to the axis, we have $DdS = D'dS'$ and therefore, *the induction at each point of the tube is inversely as the cross-section.*

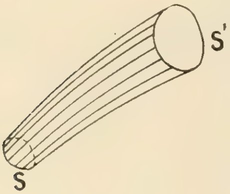


FIG. 33.

44. Coulomb's Theorem.—Let us take an element of surface, dS, on an electrified conductor in equilibrium in the presence of any given electric charges (Fig. 34); and let us suppose that the tube of induction corresponding to this element is terminated on the outside of the conductor by an equipotential surface s_1 infinitely near to dS, and on the inside by an arbitrary surface s_2. The two surfaces s_1 and s_2, taken along with the intervening portion of the tube of induction, form together a closed figure, to which, therefore, Gauss's theorem may be applied. The electric force is zero at every point of the surface s_2 and therefore the induction must also be zero, and the normal component is zero on the lateral surface of the tube; there is consequently no flux except on the outer surface dS_1. If D is the induction just outside the element, Gauss's theorem therefore gives

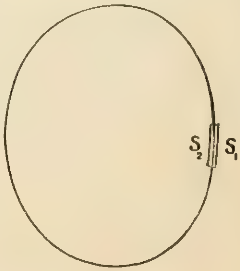


FIG. 34.

$$DdS_1 = \sigma dS$$

and as in the limit $dS_1 = dS$

$$D = \sigma$$

Thus the electric induction just outside a conducting surface in equilibrium charged to surface-density σ is numerically equal to

this surface-density. Its direction is, moreover, perpendicular to the surface. Since this is true for each unit area of the surface, it follows that the total flux starting from the whole surface is equal to the whole charge upon it. Thus, when we say that a surface is electrified, we may take it as implying that a definite flux originates there if the charge is positive, or comes abruptly to an end if the charge is negative. It may be pointed out that in the case of an actual flow of liquid the analogue of the charged surface would be a region where liquid suddenly sprang into existence or disappeared. It follows from the above that the electric force just outside the same surface is $4\pi\sigma/K$. This is known as Coulomb's theorem.

45. Electrostatic Pressure.—The total electric force at an external point M, infinitely near the element dS (Fig. 35) is made up of the force f, due to the element itself, and of the force f', due to all the other charges, and we have (**44**)

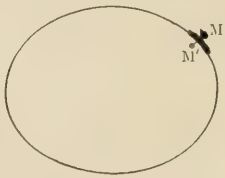


FIG. 35.

$$f + f' = F = 4\pi\sigma/K.$$

The total force is nothing at a point, M′, symmetrical with the first but just inside the conductor; further, the force f' due to all the electricity external to the element is the same at M′ as at M, and the action f of the element dS has only changed its sign; we have thus

$$f' - f = 0$$

and therefore

$$f' = 2\pi\sigma/K.$$

Hence the force due to all the electricity outside the element is equal to $2\pi\sigma/K$. The mechanical force on the quantity of electricity contained in the element dS, is $2\pi\sigma \, . \, \sigma ds/K$ and for unit surface is $2\pi\sigma^2/K$. This force is always directed outwards whatever be the sign of σ. It is called the *electrostatic pressure*.

46. Corresponding Elements.—Consider a tube of induction extending from one conducting boundary of an electric field to the other (Fig. 36), and terminated at its two ends by the areas dS and dS' respectively; these two areas are called *corresponding areas*. Since there is no induction inside the surface of either of the conductors by which the field is bounded, and

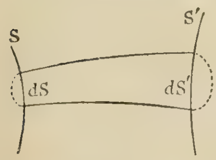


FIG. 36.

there is no induction normal to the walls of the tube, the total flux outwards from the tube is nothing. Consequently (**42**) the

algebraic sum of all the electrification within the tube must be nothing. But the only electrification within the tube is the quantity σdS upon one end, and $\sigma' dS'$ on the other end; whence it follows that

$$\sigma dS = -\sigma' dS',$$

and therefore that *corresponding elements of the bounding surfaces of an electric field possess equal and opposite charges.*

47. Complete Representation of the Field by Lines of Induction.—Lines of induction represent the direction of the induction at each point; we may further require them to be drawn so as to represent the magnitude also. Let the whole field be divided into tubes (Fig. 37), such that the flux through each one is the same and equal to unity: such tubes we shall call *unit tubes.* Each of these tubes may be represented in a diagram by means of a single line drawn centrally through it (its axis, in fact).

FIG. 37.

The number of these central lines thus drawn through a small unit area at right angles to them will be numerically equal to the induction at a point contained in the area. For the number of such lines is equal to the flux through the area, *i.e.* to DdA, and since dA is in this case unity, the number of lines equals D. In order that this may be so for the whole extent of the field, it is necessary and sufficient that the number of lines which start from each unit area of any conducting surface charged to surface-density σ, shall be equal to this surface-density. From an isolated charge q, a number of lines will start equal to q. Since the direction of induction and force is the same, these lines may be drawn, bit by bit, by the condition of being tangents at each point to the resultant of all the forces acting through that point. The lines of *force* of Fig. 26 were drawn on this principle. Since they have been drawn so that one line springs from each unit of charge, each line evidently represents one unit tube. They start from positive electrification. Some go off to infinity (or rather to very distant conductors); others terminate on the neighbouring negatively charged surface.

Since a line always has opposite electrification at its two ends, no line can stretch between two similarly charged points. Further, no line of induction can terminate at any point where there is no electricity. With these conventions, the expressions *flux of induction* and *number of lines of induction* are equivalent.

48. Application to Electric Influence.—Let q, q', q'' be quantities of electricity situated inside a closed hollow conductor, A (Fig. 38), whose external and internal surfaces are S and S′

respectively; and let M be the quantity developed by influence on the internal surface. By Gauss's theorem (**42**) the total induction through a closed surface is numerically equal to the algebraic sum of all the enclosed electrification; hence, considering an intermediate surface s_1 drawn *inside the mass of the conductor*, where the induction is nothing, we have

$$\Sigma q + M = 0.$$

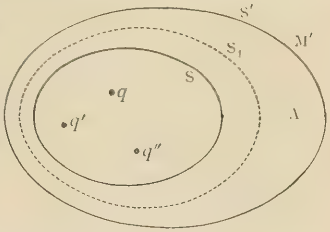


FIG. 38.

The electrification developed by influence on the inner surface of the conductor is thus equal and of opposite sign to the algebraic sum of all the charges inside the cavity (**24** b).

All the lines of induction proceeding from charges inside the cavity are then absorbed by the inner surface, S, of the conductor. The combined effect of all the internal electricity, and of the charge which covers the surface S, is thus nothing for all points outside this surface. In other words, so far as regards internal charges, the surface S forms an absolute *screen* for all points on the outside

Hence, any charge given to the insulated conductor, A, will be distributed on the outer surface in a manner independent of the internal charge, and only depending on the form of the surface S′, and on the distribution of the external charges. Reciprocally, the combined effect, at any point inside the conductor, of the electrification of the outer surface together with that of all electricity outside the conductor, is nothing.

This result agrees with the mutual independence of the electric fields inside and outside a hollow conductor that we have already pointed out (**24**).

In the case in which an insulated body, C, is in the presence of an electrified body, A, inside a closed conductor (**23**, **24** e), part of the flux from A terminates at the negative region of C; but an equal flux starts from the positive region of C, and ends finally at the closed conductor, so that there is always the same quantity of negative electricity on the interior surface, which follows, moreover, from the preceding equation.

49. Poisson's Theorem.—*The force which given electric charges contained within a closed surface exert at any point outside that surface, is the same as that of a layer of the same quantity distributed on this surface according to a certain law.*

In fact, let q, q', q'' or Σq be the internal charges, which we shall suppose fixed as if they belonged to non-conducting bodies, and let s be an ideal surface which encloses them. If this surface were a conductor, it would be covered on the inside with a quantity $-Q = -\Sigma q$, whose action on all outside points would be equal and opposite to that of the given charges. A quantity $+Q$ of electricity, distributed over the surface s according to the same law, will therefore produce at all external points an effect equal and similar to that of the enclosed charges Σq, and may therefore be substituted for them.

If the imaginary surface s were replaced by an actual conducting shell of the same size and shape, the inside of the shell would (as already pointed out in **24** *b*) have a charge $-Q$, distributed so as to counteract for all external points the effect of the enclosed charges, and the outside would have a charge $+Q$, but this would not be distributed according to the same law unless the surface s were an equipotential surface of the original system.

If, however, such a shell were coincident with a pre-existing equipotential surface, the electric conditions at all external points would be in no way altered by introducing it into the field. It follows that the force and induction at any point in an electric field would be unaltered if, the charge remaining constant, the shape and position of one or of both boundaries of the field were to change so that each of them should always coincide with one of the original equipotential surfaces. We have already (**20**) pointed out a special case of this kind: the force at any point due to an electrified sphere at a great distance from all other conductors is independent of the radius so long as the charge of the sphere and the position of the centre remain unchanged.

50. Heterogeneous Dielectrics.—In proving the foregoing theorems it has been assumed that the dielectric is the same throughout; we shall now inquire as to the laws which hold when more than one dielectric is in the field, and shall first take a case in which the boundary between the two dielectrics is at right angles to a line of force. This will be the case for all lines if a spherical mass of one dielectric (whose constant is K_1) surrounds a point-charge Q placed at its centre, while the remainder of the field is a dielectric of constant K_2; for geometric symmetry demands that the direction of the electric force shall then be everywhere radial from the charge. Since we may place an insulated conducting surface in coincidence with the surface of separation without disturbing the field; and since we thereby isolate the internal and external fields from each other's influence, it follows

that we may apply the theorems of the previous sections to each of these homogeneous fields in turn. Thus, for a point just inside the conducting surface the force will have the value $\frac{1}{K_1}\frac{Q}{r^2}$, where r is the radius of the surface, while for a point just outside the force will be the same as that due to a charge Q distributed uniformly over the surface, *i.e.* $\frac{1}{K_2}\frac{Q}{r^2}$. The values of the induction at the two points are these numbers multiplied respectively by $K_1/4\pi$ and $K_2/4\pi$, that is, each is equal to $\frac{Q}{4\pi r^2}$. Thus, in this case, the induction has the same value on both sides of the boundary, although the electric force is different on the two sides.

We shall now suppose the distribution to be such that the surface of separation between the two dielectrics is *parallel* to a line of force. Select two points in each medium (PQ and P′Q′, Fig. 39) such that the lines PQ and P′Q′ are very short, but that the lines PP′ and QQ′ are very short in comparison with them. Then if V_1 and V_2 are the potentials at P and Q they must also be the potentials at P′ and Q′; for the amounts of work done in carrying a charge from P to P′ or from Q to Q′ must each be infinitesimal. Now $\frac{V_1 - V_2}{\text{PQ}}$ and $\frac{V_1 - V_2}{\text{P}'\text{Q}'}$ become the values of the electric force on the two sides of the surface if Q and Q′ are allowed to approach indefinitely near to P and P′; and these values are equal. Thus when the electric force is parallel to the surface of separation of two dielectrics it has the same value close to the surface on both sides of it. It is obvious that the corresponding induction will not be equal on the two sides, but in the ratio of K_1 to K_2.

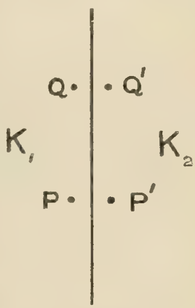


FIG. 39.

In the general case in which the electric force is inclined to the surface of separation we may imagine it to arise from two distributions of electrification—one of these producing the component whose direction is at right angles to the surface at the point under examination, while the second distribution gives rise to that component which is tangential to the surface. The theorems proved in this section will be applicable to these components, *i.e.* at the surface of separation between two dielectrics the normal components of induction are the same on both sides if the surface is uncharged, while the tangential components of the electric

force are the same on both sides. As a consequence of these relations a line of induction is bent or *refracted* on passing through the surface. For if θ_1 and θ_2 are the angles which it makes with the normal on the two sides, the above relations may be thus stated:

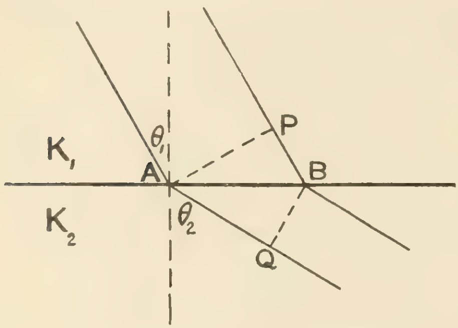


FIG. 40.

$$D_1 \cos\theta_1 = D_2 \cos\theta_2$$

and

$$F_1 \sin\theta_1 = F_2 \sin\theta_2$$

therefore

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{D_1}{F_1} \cdot \frac{F_2}{D_2} = \frac{K_1}{K_2}$$

where the suffixes refer to the two media.

Since the surface is uncharged, as many unit tubes must, by Gauss's theorem, be incident on one side of any portion of it, AB, as leave it on the other side. This also follows from the continuity in the normal component of induction. For since

$$D_1 \cos\theta_1 = D_2 \cos\theta_2$$

$$D_1.\text{AB}\cos\theta_1 = D_2.\text{AB}\cos\theta_2$$

or

$$D_1.\text{AP} = D_2.\text{QB}$$

where AP and QB are sections normal to the direction of the tubes; that is, the fluxes are continuous through the surface AB.

Thus the flux of induction can be represented throughout the field, whether the dielectric is everywhere the same or not, by continuous tubes or their representative axes *provided that there are no charges in the field*. If any surface between two dielectrics becomes charged (*e.g.* by friction), the lines due to this charge must be superposed upon those already existing in the field. If we take the case of a positive charge, lines originate on it, and since they extend in all directions outwards from the surface, their effect is to diminish the total number reaching the surface on one side, and to increase the total number on the other: the induction changes by the amount σ on passing through the surface.

CHAPTER VI

ELECTRICAL CAPACITY—CONDENSERS

51. Capacity of an Electric Field.—In nearly all cases in which we require to deal practically with electric fields, the bounding surfaces of the fields are formed of conducting materials, and we shall assume this to be the case throughout what follows. Under these circumstances the potential of each boundary is uniform, and the difference of potentials between the boundaries is, by definition **(35)**,

$$V - V' = \bar{F}e$$

if e is the distance from one surface to the other measured along a line of force, and $\bar{F}$ is the mean electric intensity along this line. The intensity at any point is proportional, other things remaining the same, to the charge on the field (meaning thereby the total quantity of positive electricity upon one boundary or of negative on the other), and therefore $\bar{F}$ must also be proportional to it. Consequently, for a given field, charge and difference of potentials are proportional to each other; and putting C to stand for a constant coefficient characteristic of the field, we may write

$$Q = C\,(V - V').$$

This coefficient, whose definition is implied in the above equation, is called the *capacity* of the field. In words we may define it as follows:—

The capacity of an electric field is the ratio of the charge on the positive boundary of the field to the difference of potentials between its boundaries, and is numerically equal to the charge required to establish unit *difference of potentials between them.*

The term "capacity" is borrowed from the theory of heat, where the thermal capacity of a body is defined as the ratio of a quantity of heat given to a body to the change of temperature produced by it, and as being numerically equal to the quantity of heat needed to produce unit change of temperature. Thermal capacity depends on mass and material: the capacity of an electric field,

on the size and shape—that is, on the geometry of the field—and on the material by which it is filled.

52. Capacity of a Tube of Induction.—An electric field may be thought of as made up of tubes of induction running from one boundary to the other, and each charged at its ends with a definite proportion of the whole charge on the field. The ratio of this charge to the difference of potentials between the ends of a tube may be called the capacity of the tube, just in the same sense as the ratio of corresponding quantities for the whole field is called the capacity of the field. For a given field, the difference of potentials is the same for all the tubes, because the boundaries of the field are equipotential surfaces. If, then, q be the terminal charge of a tube we may write for its capacity

$$c = \frac{q}{V - V'}$$

or for different tubes

$$c_1 = \frac{q_1}{V - V'}, \; c_2 = \frac{q_2}{V - V'}, \; c_3 = \cdot \quad \cdot \quad \cdot$$

$V - V'$ being a common denominator.

Adding these values we get

$$c_1 + c_2 + c_3 + \cdot \quad \cdot \quad \cdot = \frac{q_1 + q_2 + q_3 + \cdot \quad \cdot \quad \cdot}{V - V'} = \frac{Q}{V - V'} = C.$$

Whence we see that the capacity of an electric field is equal to the sum of the capacities of the tubes of induction into which it may be subdivided.

As for a complete field, so for a tube of induction, its capacity depends on its geometrical relations and on the material medium that occupies it. The difference of potential between the ends is $= Fe$, as above, or what comes to the same thing $\int_o^e F de$, where F is the intensity at any point in the tube, and de an element of length. If s is the area of the tube at the part where the intensity is F, and if s_o and F_o are corresponding values for the positive end, we have

$$Fs = F_o s_o$$

and therefore

$$V - V' = \int_o^e F de = F_o s_o \int_o^e \frac{de}{s} = \frac{4\pi\sigma s_o}{K} \int_o^e \frac{de}{s}$$

since, if K be the dielectric coefficient for the medium, $F_o = \frac{4\pi\sigma}{K}$.

The terminal charge of the tube is $q = \sigma s_0$, and therefore the capacity becomes

$$c = \frac{q}{V - V'} = \frac{K}{4\pi} \cdot \frac{1}{\int_0^e \frac{de}{s}}.$$

The value of the integral $\int \frac{de}{s}$ which occurs in this expression depends on the size and shape of the tube, and cannot be determined till these are known. It is therefore only in those cases in which the tube has a simple shape that its capacity can be readily calculated.

53. Capacities in Special Cases.—We shall now find expressions for the capacities of certain fields for which elementary methods are sufficient: in a later section (**127**) some more complicated cases will be solved.

If the conductors are *infinite parallel plates*, geometric similarity indicates that the lines of force must be straight, equidistant, and perpendicular to them; in other words, the intensity must be uniform from one boundary to the other; hence we have at once

$$V - V' = Fe = \frac{4\pi\sigma}{K} e$$

and therefore

$$C = \frac{QK}{4\pi\sigma e} = \frac{AK}{4\pi e}$$

where e is the distance between the plates and C is the capacity of area A.

In the case of *concentric spheres*, we have to remember that at any point in the field the intensity due to the charge of the outer surface is nothing, and therefore that the whole intensity is that due to the charge on the inner sphere, which acts as though it were concentrated at the centre. If b and a represent the radii of the outer and inner surfaces respectively then the intensity at the outer surface is $\frac{Q}{b^2K}$, at the inner surface $\frac{Q}{a^2K}$, and the mean intensity (comp. **36**) is $\frac{Q}{abK}$; consequently the difference of potentials is

$$V - V' = \frac{Q}{abK}(b - a)$$

and the capacity

$$C = K\frac{ab}{b - a}.$$

In the case of two *coaxal cylinders* of radii b and a and of great length, the tubes of force at a distance from the ends of the cylinders will be uniformly distributed and normal to the axis. They will be wedge-shaped spaces bounded by planes through the axis and by planes perpendicular to the axis, and the area of any cross-section will vary directly as its distance from the axis; the intensity will therefore vary inversely as this distance. Its value at the inner surface is $\frac{4\pi\sigma}{K}$, or if Q is the total charge on unit length of the surface, the intensity is $\frac{4\pi}{K}\cdot\frac{Q}{2\pi a}$, *i.e.* $\frac{2Q}{Ka}$ and at a distance e from the axis its value is $\frac{2Q}{Ke}\cdot$ Hence

$$V-V'=\int_a^b F de=\int_a^b \frac{2Q}{K}\cdot\frac{de}{e}=\frac{2Q}{K}\log_e\frac{b}{a}$$

and the capacity per unit length is

$$\frac{K}{2\log_e\frac{b}{a}}\cdot$$

For purposes of calculation it is important to bear in mind that the logarithm in the denominator is a so-called natural logarithm having the base 2·718 and is equal to the corresponding common logarithm (base 10) divided by 0·4343.

In a field of force of any shape, if the distance between the boundaries is uniform and small compared with their linear dimensions, it is evident that the electric intensity must be nearly uniform throughout the field, and therefore that the formula $\frac{AK}{4\pi e}$, found for two parallel planes, must apply to any case which satisfies the conditions named. It is easy to verify that the formulæ for cylinders and spheres reduce to this form when e is very small.

If the outer surface, in the case of a field bounded by concentric spheres, becomes very great, the corresponding radius, b, becomes very great, or, what comes to the same thing, the radius a becomes very small in comparison, and may be neglected in the difference $b-a$. Consequently the capacity of the field becomes

$$C=K\frac{ab}{b}=Ka$$

and depends only on the radius of the smaller (convex) sphere and on the nature of the dielectric medium.

When the outer boundary of the field is very distant, a change of its shape would not alter the potential at a point on the inner boundary. Consequently the expression

$$C = Ka$$

gives the capacity of any field having for one boundary a conducting sphere of radius a, and for the other boundary any very distant surface. For example, if a conducting sphere is suspended in a large room at a great distance from the walls or other conductors, the capacity of the electric field extending between the sphere and the walls, floor, and ceiling of the room is equal to the radius of the sphere multiplied by the dielectric coefficient of air.

54. Successive Electric Fields.—It happens not unfrequently that one electric field is itself completely surrounded by another, or, what comes to the same thing, that a field is divided into two parts by a closed conducting shell, and that it is required to express the charge in terms of the potentials of the first and last bounding surfaces. The arrangement indicated in Fig. 41 may be taken as a typical case of this kind. The shaded portions being understood to represent conducting material and the unshaded to represent dielectrics, we may consider that we have here a field of capacity, C_1, with bounding surfaces at potentials V and V_1 respectively, and that this is surrounded by a second field of capacity, C_2, with surfaces at potentials V_1 and V_2. Or we may look upon the figure as representing a single field, with surfaces at potentials V and V_2, divided into two parts by a conductor at potential V_1. It is obvious that the charge on each partial field is the same: representing it by Q we have

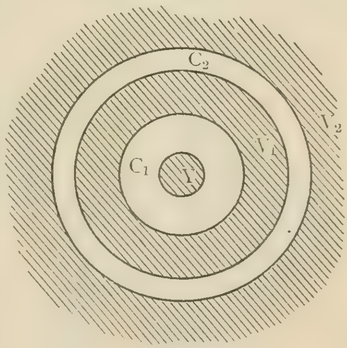


FIG. 41.

$$Q = C_1 (V - V_1) = C_2 (V_1 - V_2)$$

and eliminating V_1

$$Q = \frac{C_1 C_2}{C_1 + C_2} (V - V_2).$$

If we put C for the capacity of the simple field, which, with

surfaces at potentials V and V_2, would have the same charge as the two consecutive fields, we must write

$$C = \frac{C_1 C_2}{C_1 + C_2}.$$

By extending the same method of calculation, we may find the capacity of the single field equivalent to any number of separate fields arranged in succession. For example, if there are three fields of separate capacities C_1, C_2, C_3, the capacity of the single equivalent field is

$$C = \frac{C_1 C_2 C_3}{C_2 C_3 + C_1 C_3 + C_1 C_2}.$$

55. Condensers.—When two electric conductors are so placed that their surfaces form the boundaries of a field of relatively great capacity, the arrangement is very frequently spoken of as a *condenser*. For example, we have seen that a conducting sphere of radius a centimetres, at a great distance from other conductors, forms one boundary of a field of capacity Ka; whereas, if the sphere be surrounded by a concentric conducting shell of radius b, the resulting field has capacity $K\frac{ab}{b-a}$. Thus, if $a = 10$ cm., and $b = 11$ cm., the capacity in the second case is 11 times that in the first; or, a being still 10 cm., if $b = 10{\cdot}1$, the capacity of the field bounded by the two spheres becomes $1010\,K$, instead of $10\,K$.

56. Various Forms of Condensers.—One of the simplest forms of condenser is that known as *Æpinus's condenser* (Fig. 42), consisting of two equal insulated metal plates, supported so that they can be brought into contact with the opposite faces of a plate of glass larger than themselves, or simply placed so as to face each other at a short distance in air. In either case the capacity can be diminished by drawing the plates apart, or increased by moving them nearer together. To charge the condenser, it is only necessary to connect the two plates with opposite sides of an electric machine or other contrivance for producing a difference of potentials.

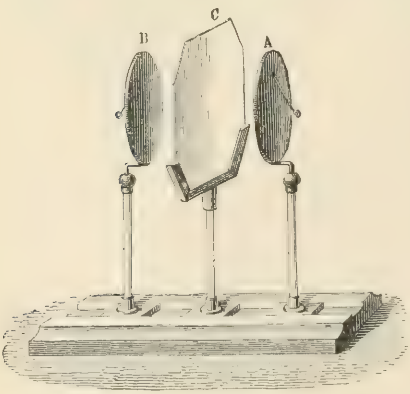


Fig. 42.

Another equivalent arrangement, except that the distance between the conductors cannot be varied, is that known as *Franklin's pane*, consisting of two equal pieces of tinfoil pasted opposite each other on the two faces of a sheet of glass large enough to project a good way beyond them on all sides. For better insulation, the uncoated parts of the glass are usually covered with shellac varnish.

When condensers of very great capacity are required, but are not to be exposed to great differences of potential, they are often made by superposing alternately sheets of tinfoil and thin plates of mica, or sheets of paraffined paper. The first, third, fifth . . . . sheets of tinfoil are connected together, and form one boundary of a field of great area but very small thickness, and the second, fourth, sixth . . . sheets are connected to form the other boundary.

Another form of condenser that is very frequently employed in connection with high differences of potential is the *Leyden jar* (Fig. 43). This consists of a glass jar or bottle, coated internally and externally with tinfoil to within a moderate distance of the opening, the rest of the surface being varnished with shellac. A brass rod, terminated by a ball, connected with the inner coating, and passing out through the neck without touching it, enables electrical connection to be made with the inner coating when required. Sometimes the jar is closed at the mouth by a non-conducting lid; in this case the rod is usually supported by being put tight through the lid, and makes contact with the inner coating by means of a flexible spring or a few inches of chain. Electrically, the Leyden jar is exactly equivalent to the coated pane, but it is more compact and convenient to handle. The tinfoil coatings may be replaced by conductors of any other material; the reason why tinfoil is commonly used being chiefly that its flexibility allows it to be readily adapted to the surface of the glass. Where very high insulation is required, it answers well to use strong sulphuric

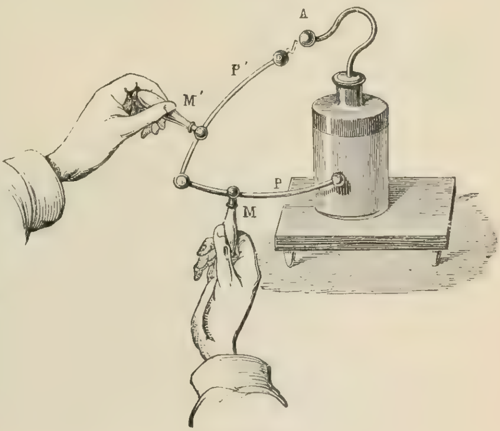


FIG. 43.

acid (oil of vitriol) to make contact with the inner surface of the glass (which, in this case, is not varnished), and to connect the acid with the other apparatus by a platinum wire dipping into it.

57. Quantity of Electricity in a Condenser.—The charge of a condenser is, of course, the charge on the electric field which is bounded by its conducting surfaces; it is therefore represented by

$$Q = C\,(V - V')$$

and depends partly on the condenser itself and partly on the difference of potentials established between its surfaces.

The formula $C = \frac{AK}{4\pi e}$ (53) applies with sufficient accuracy to the capacity of any of the forms of condenser described. From this we see that, with a given insulating material, the capacity can be increased by increasing the area A of the opposed conducting surfaces, for these determine the area of cross-section of the field, and also by decreasing the distance e between the surfaces, or, in other words, the length of the field. But if the charge is to be great, not only the capacity, but the difference of potentials, $V - V'$, must have a considerable value. This is usually determined by

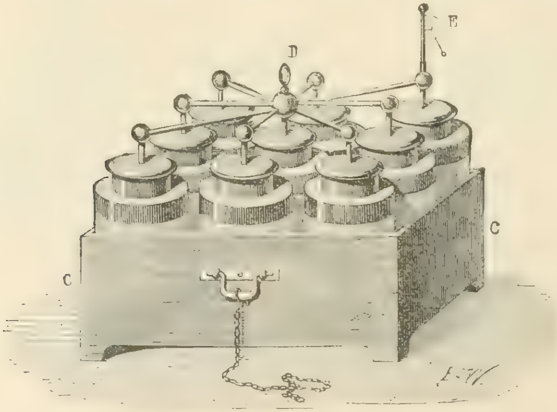


FIG. 44.

the electric machine or other instrument used for producing the charge, and under fixed conditions in this respect may be taken as constant. The value of the difference of potentials that is to be employed puts a practical limit to the extent to which capacity can be increased by diminishing the thickness, e, of the dielectric. A small thickness and great difference of potentials implies a high value of the ratio $\frac{V - V'}{e}$, which measures the mean electric inten-

sity in the field, and, as already mentioned (**19**), when the stress in the field reaches a limit depending on the nature of the material, the dielectric gives way and is burst through by a disruptive discharge. The maximum stress that a given material can support without allowing discharge to take place through it is called its *electric strength.* Little is known about the exact value of this property for different substances, but it is certainly many times greater for glass and various other solid dielectrics than it is for air. For this reason, in condensers intended to support great differences of potential, as well as to have a large capacity, glass is generally used as the dielectric, and the commonest form is that of the Leyden jar. To obtain sufficient surface without using jars of unwieldy size, several jars are often combined, as shown in Fig. 44, all the inner coatings being electrically connected by insulated metal rods, and the outer coatings by placing the jars on a tray, or in a box lined with tinfoil. The capacity of such a combination of jars, sometimes called a *Leyden battery*, is equal to the sum of the capacities of the separate jars (comp. **52**).

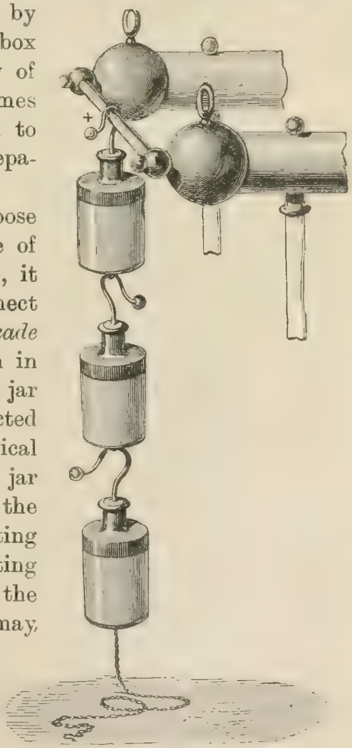


FIG. 45.

When it is not desirable to expose a condenser to the full difference of potentials that may be available, it is sometimes advantageous to connect two or more jars in series, or *in cascade* as it is called, in the way shown in Fig. 45. The inner coating of the jar at one end of the series is connected with one electrode of the electrical machine, the outer coating of the jar at the other end of the series to the other electrode, and the outer coating of each of the rest to the inner coating of the jar that comes next in the series. If, in such a case, we may neglect the small capacity of the field existing between the outer surface of each jar and the walls of the room, by comparison with the much greater capacity of the field between the tinfoil coatings, the charge of each jar is the same, and writing V_o and V_n for the potentials of the first and last coatings

respectively, and V_1, V_2, V_3 . . . for the potentials of the intermediate coatings taken in order, we have for the charge of any one jar

$$Q = C_1 (V_o - V_1) = C_2(V_1 - V_2) = \ldots = C_n(V_{n-1} - V_n).$$

But (54), we have also

$$Q = \frac{C_1 C_2 \ldots C_n}{C_2 C_3 \ldots C_n + C_1 C_3 \ldots C_n + \ldots + C_1 C_2 \ldots C_{n-1}} (V_o - V_n)$$

where C_1, C_2, . . . are the capacities of the several jars, and the multiplier of $V_o - V_n$ has a numerator formed by multiplying together all the C's, and a denominator formed by adding together the series of terms obtained by striking out from the numerator each of its factors taken one at a time from C_1 to C_n. From this we can express the difference of potentials acting on any one jar when we know the whole difference of potentials and the separate capacities. Suppose, for example, that there are four jars, then the difference of potentials acting on the second is

$$V_1 - V_2 = \frac{C_1 C_3 C_4}{C_2 C_3 C_4 + C_1 C_3 C_4 + C_1 C_2 C_4 + C_1 C_2 C_3} (V_o - V_4).$$

If a number of jars combined in cascade are all of the same capacity, it follows that the difference of potentials acting on each jar is the same, and is equal to the whole difference divided by the number of jars. In this case also the sum of the separate charges is the same as any one jar would have received if it had been charged to the full difference of potentials $V_o - V_n$.

58. Charging and Discharging a Condenser.—Since the process of charging a condenser consists in establishing a difference of potentials between the boundaries of the corresponding electric field, a charge is produced whenever the opposed conducting surfaces of the condenser are respectively connected with other conductors at different potentials. In general the surfaces are connected with the positive and negative conductors of an electric machine. If a common friction machine, with the rubber uninsulated from the room, is used, a sufficient connection between the outer surface of a Leyden jar and the rubber is usually afforded by the table on which both jar and machine stand; a special conducting connection need be arranged in such a case only between the inner coating and the insulated conductor of the machine.

To discharge a condenser is to bring its two surfaces to the same potential. This may be done suddenly by connecting them by means of a *discharger* (Fig. 43), consisting of two metal rods hinged together like a pair of compasses and provided with glass handles, M, M′, and terminated by brass knobs. If one knob is put into contact with the outer coating of a charged Leyden jar, and

the other is gradually brought near the knob A connected with the inner coating, a bright loud spark passes before the knobs come quite into contact, and the jar is afterwards found to be discharged. In this case the whole process occupies only a very small fraction of a second, but it is, nevertheless, a somewhat complicated one, and usually consists of a succession of discharges of gradually decreasing magnitude, and alternately in opposite directions. The phenomenon is, in its general character, closely analogous to what happens when a bent spring is suddenly released, and, instead of simply returning to its position of equilibrium and remaining at rest there, is carried beyond it by its momentum and is bent in the opposite direction, then passes again through its position of equilibrium towards the side of the original bending, and so backwards and forwards through a gradually decreasing range until the whole of the energy due to the original strain has been converted into heat. In the electrical case we must imagine some kind of strain as existing in the charged dielectric, and that the sudden relaxation of this strain at the instant of discharge gives rise to a series of electric oscillations of progressively decreasing extent until the whole energy of the charge has here also been converted into heat.

If the two surfaces of a charged condenser are connected through a very long thin wire or through a damp thread, the resulting discharge consists simply of a comparatively gradual equalisation of potentials without reversal or oscillation. In this case the process may be compared with the result of releasing a bent spring which is immersed in a liquid so viscous that the elastic forces, urging the spring through it towards its position of equilibrium, are not able to make it arrive there with any velocity.

59. Discharge by Successive Contacts.—Discharge takes place in a different way if a condenser is charged, and its surfaces, which are both insulated from other conductors, instead of being connected with each other, are alternately connected, one at a time, with a third conductor, which may be the room containing the condenser. In order to follow out what happens in such a case, we will suppose the condenser to be a coated pane, and that its surfaces are alternately connected with the inside of the room. There is here a complex field, or rather there are three separate fields. Distinguishing the coatings as A and B (Fig. 46) we have a field of capacity, say C_A, bounded by the coating A and the inside of the room; next, a field of capacity C, extending through the glass between the coatings A and B; and, lastly, there is a field of capacity, C_B, bounded by the coating B and by the inside of the room.

FIG. 46.

We will refer to these in the order in which they have been named, as the first, second, and third fields respectively. Let us begin by supposing the coating B connected with the room; then the boundaries of the third field being at the same potential, its charge is zero, and the coating A becomes a common boundary of the first and second. Putting V for the potential of the room, and V_o for the initial potential of A, the charge on the first field is

$$C_A(V_o - V),$$

and that on the second field, the field of the condenser, is

$$Q_o = C(V_o - V).$$[1]

Now, after insulating B, connect A with the room, thus bringing its potential to the value V. The effect of this is that the charge on the first field vanishes, and that that on the second spreads into the third field, the coating B, whose potential we will denote by V_1', becoming a common boundary of the second and third fields. Expressing by an equation the fact that the combined charges on these two fields are the same as the previous charge on the second field, we have

$$(C + C_B)(V - V_1') = C(V_o - V)$$

which gives for the present difference of potentials between the coatings of the condenser—

$$V - V_1' = \frac{C}{C + C_B}(V_o - V).$$

That is to say, it is the original difference of potentials, $V_o - V$, multiplied by the proper fraction $\frac{C}{C + C_B}$. Denoting this, for shortness, by m, we may write

$$V - V_1' = m(V_o - V).$$

Next, let A be again insulated, and let B be connected with the room, as in the initial stage. The charge on the third field again disappears, and that on the second spreads into the first, the surface A being a common boundary. If we put V_1 for the resulting potential of A, the equation expressing identity between the combined charges on the first and second fields and the previous charge on the second is

$$(C_A + C)(V_1 - V) = C(V - V_1')$$

whence

$$V_1 - V = \frac{C}{C_A + C}(V - V_1') = n(V - V_1'), \text{ say,}$$

or, putting in the value just found for $V - V_1'$,

$$V_1 - V = mn(V_o - V).$$

[1] It may be noted that the electric force in both these fields is directed from the surface A, if V_0 is greater than V, and towards A, if V_0 is less than V—that is, in either case, the force is oppositely directed in space in the two fields.

The charge of the condenser is now $Q_1 = C\ (V_1 - V)$, and therefore

$$\begin{aligned} Q_1 &= mn\ C\,(V_o - V) \\ &= mn\ Q_o. \end{aligned}$$

That is, the charge of the condenser after each coating has been successively connected with the room and then insulated is equal to the original charge multiplied by the constant fraction

$$mn = \frac{C^2}{(C_A + C)\ (C + C_B)},$$

and by following out again exactly the same course of reasoning and calculation, we should find that the charge after repeating the same process would be

$$\begin{aligned} Q_2 &= mn\ Q_1 &&= (mn)^2\ Q_o \\ Q_3 &= mn\ Q_2 &&= (mn)^3\ Q_o \\ &\ \vdots && \ \vdots \\ Q_n &= mn\ Q_{n-1} &&= (mn)^n\ Q_o. \end{aligned}$$

It appears, then, that the charges remaining in the condenser after the surfaces have been uninsulated 1, 2, 3, . . . times, form the terms of a decreasing geometric series.

The successive decreases of charge due to repetitions of the process also form a similar series: thus, the first time, the loss of charge is

$$Q_o - Q_1 = Q_o\,(1 - mn);$$

the second time it is

$$Q_1 - Q_2 = Q_1\,(1 - mn) = Q_o mn\,(1 - mn);$$

the third time

$$Q_2 - Q_3 = Q_2\,(1 - mn) = Q_o m^2 n^2\,(1 - mn),$$

and so on.

The process of discharge by successive contacts may take place when the surfaces of a condenser are successively connected with any insulated conductor, and it then takes place essentially in the way that has been traced out. It is illustrated by various familiar experiments which need not be more than mentioned, as the electric chimes (Fig. 47), and Franklin's spider (Fig. 48), in which a small insulated conductor B or B′, suspended like a pendulum, strikes successively against knobs connected with the two coatings of a Leyden jar, being attracted in turn by each after coming in contact with the other.

The experiment known as *Lichtenberg's figures* is connected with the same kind of action. We take hold of a charged Leyden jar by the outer coating and draw the knob connected with the

inner coating along the surface of a cake of resin, tracing out with it figures of any kind. Then, putting the jar on an insulating stand, we take it by the knob and trace other figures on the resin with the outside of the jar. On afterwards dusting over the cake of resin a mixture of flowers of sulphur and red-lead, the particles of red-lead collect along the lines traced with one surface of the jar, and the sulphur along those traced with the other surface. By

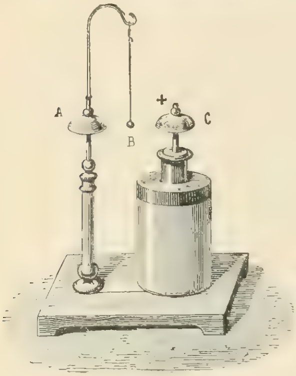


FIG. 47.

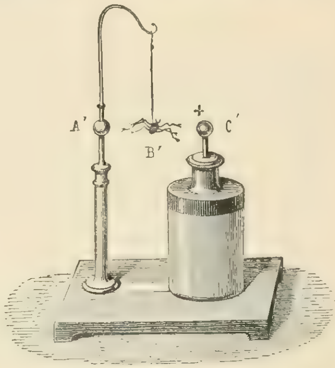


FIG. 48.

their mutual friction the particles of the two powders become electrified, the red-lead positively and the sulphur negatively, and they consequently attach themselves to parts of the surface of the resin which have been electrified in the opposite way to themselves. The lines formed present marked characteristics; the yellow lines of sulphur, corresponding to the positively charged parts of the resin, show very fine ramifications, while the red marks, corresponding to the negatively charged parts, are broader and often resemble drops or splashes.

60. Effect of the Insulating Material.—It was first observed by Faraday that the capacity of a condenser depends not only on the size and shape of its surfaces, but also on the nature of the insulating material which fills the space between them. He had two spherical condensers made as nearly alike as possible (Fig. 49), the inner sphere, A, being supported concentrically with the outer one by a thick cylinder of shellac, *l*, fitting into a cylindrical neck, and the outer sphere being formed of two hemispherical cups fitting each other air-tight when put together. When one of the condensers had been charged, the fixed ball of a torsion-balance was put into contact with the knob *a* connected with the inner sphere A, and the charge of the ball was measured. The charge of the condenser was next shared between the two condensers, by

connecting the two outer spheres by a wire, and then bringing the knob a of one into contact with the knob of the other. In this way the corresponding surfaces of the two condensers were brought to the same potentials. Then the fixed ball of the torsion-balance was again put into contact with the knob a and its charge measured as before. It was thus found that when both condensers were filled with air, the charge, measured by the torsion-balance after the sharing, was just half as great as that measured at first. On the other hand, if the space between the surfaces of one condenser was filled with some solid dielectric, such as glass, sulphur, or shellac, then the charge measured after sharing was more than half that measured before sharing, if the condenser first charged was that containing the solid dielectric; but it was less than half if the air condenser was the one first charged.

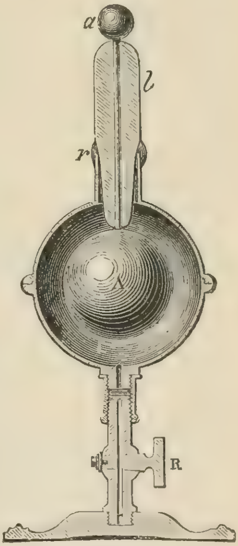


FIG. 49.

Throughout the experiments the outer surfaces of both condensers were connected with the room, so that the charge taken by the ball of the torsion-balance was in each case proportional to the excess of the potential of the inner sphere above that of the outer sphere, and the process was virtually a comparison of the difference of potentials between the surfaces of one condenser when first charged with the difference of potentials when it had given up part of its charge to the other. Putting V' for the constant potential of the room and outer surfaces of the condensers, V_1 for the initial potential of the inner surface of the charged condenser, and V_2 for its potential after sharing, we have

$$\frac{V_1 - V'}{V_2 - V'} = \frac{q_1}{q_2}$$

if q_1 and q_2 are the two charges measured by the torsion-balance. When the condensers are connected, the first loses a certain quantity of electricity, and the potential of its inner surface falls from V_1 to V_2; at the same time the second gains the same quantity, and the potential of its inner surface rises from V' to V_2. If now we call C_a' and C_b' the capacities of the two condensers respectively, and write an equation expressing that these are inversely as

the changes of potential caused by the loss or gain of the same quantity of electricity, we get

$$\frac{C_b}{C_a}=\frac{V_1-V_2}{V_2-V'}=\frac{(V_1-V')-(V_2-V')}{V_2-V'}=\frac{q_1}{q_2}-1.$$

In this way the ratio of the capacities of the two condensers was obtained, and Faraday's results showed that the substitution of a solid dielectric for air increased the capacity. He gave the name *specific inductive capacity* to that property of an insulating material on which the capacity of a condenser formed of this material depends, or, in other words, to that property of the material occupying an electric field, of given shape and size, which determines the charge required to produce a given difference of potentials between the boundaries of the field.

61. Specific Inductive Capacity, or **Dielectric Coefficient.**—The specific inductive capacity, or, as it is otherwise called, the *dielectric coefficient*, of an insulating or dielectric material, is the quantity denoted in our formulæ by the symbol K. To see how its numerical value may be determined, we may, in the first instance, go back to the fundamental formula of electric force (**15**), which in the case of two equal quantities of electricity concentrated at points, we may write

$$Q^2=fr^2K.$$

From this, if Q_1 is the quantity which repels an equal quantity at unit distance with unit force, we have

$$Q_1{}^2=K$$

and therefore K may be defined numerically as equal to the square of the quantity Q_1, just defined.

But when the dielectric medium is air, the quantity which repels an equal quantity at unit distance with unit force is, by definition (**15**), a unit of electricity. Hence, as already pointed out, our definition of the unit of electric quantity involves the numerical value $K=1$ for air.

Another way of arriving at the value of K is by considering the relation between charge, difference of potentials, and dimensions of an electric field. Taking the simple case of a field bounded by plane parallel surfaces (**53**), we have for the charge

$$Q=\frac{(V-V')\,KA}{4\pi e}$$

or, writing σ for Q/A, the surface-density, and transposing,

$$K=4\pi\sigma\,\frac{e}{V-V'};$$

or, again, if σ_1 be the surface-density for which the difference of potentials is unity when the distance between the boundaries of the field is unity

$$K = 4\pi\sigma_1.$$

In words, this is equivalent to saying that the dielectric coefficient of a given substance is equal to 4π times the charge per square centimetre on a field, bounded by large flat parallel surfaces one centimetre apart and filled by that substance, when the difference of potentials between the boundaries is unity.

The following are the approximate values of the specific inductive capacities of a few solid and liquid dielectrics, the inductive capacity of air being taken as unity :—

	Specific Inductive Capacity.
Paraffin wax	2·3
Rosin	2·6
Ebonite	2·8
Sulphur	4·0
Glass (plate or common flint)	6 to 7
Petroleum oil	2·1
Turpentine oil	2·2
Benzene	2·2

On the same scale, the specific inductive capacity of a vacuum is about 0·9994.

62. Residual Charge.—If the difference of potentials of a condenser with a solid dielectric is measured immediately after the condenser has been charged, and then again a few minutes or half-an-hour later, it is generally found, even when there is perfect insulation, that the second measurement gives a lower result than the first one. This diminution, when there is good insulation, does not go on continuously, but is comparatively rapid at first, and then becomes slower and slower, and at last imperceptible.

Again, if the surfaces of such a condenser, after it has been charged for some time, are connected for an instant by a thick wire, so as to discharge the condenser—that is, to bring its coatings to the same potential—it is found that, after disconnecting the coatings, a difference of potentials of the same sign as that previously existing makes its appearance, increasing comparatively quickly at first, and then more slowly, towards a constant value. This phenomenon is known as the *residual charge*, and is correlative with the decrease of difference of potentials spoken of above. The residual charge does not amount to more than a small proportion of the original charge, but a condenser that has been charged strongly for a long time will often allow of several successive discharges of progressively diminishing amount being obtained at intervals, one after another.

The formation of the residual charge implies that the ratio $Q/(V-V')$ for a condenser in which it occurs depends upon the time that has elapsed since the charge was imparted, or, more generally, upon the recent electrical history of the condenser, as well as upon its material and geometrical characters. This circumstance introduces complications into the experimental determination of the capacity of a condenser, which, as has been explained, is the name given to the ratio in question. The disturbance due to this cause may be avoided to a great extent by measuring the capacity of a condenser as soon as possible after it is charged,—that is to say, before the residual charge has had time to form to an appreciable extent.

63. Condenser with Removable Coatings.—According to the view we have adopted, the charging of a condenser, or of any electric field, consists in setting up a peculiar condition of stress within the dielectric. To do this the conducting surfaces which bound the dielectric must be connected with the conductor of an electric machine, or equivalent arrangement, between which an electric field is produced by friction or otherwise. By virtue of the connecting wires, the stresses existing in the original field are able to relieve themselves by causing an expansion and partial transfer of the field to the dielectric of the condenser. The function of the

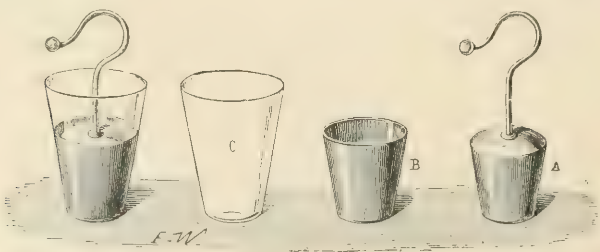


FIG. 50.

metallic coatings of a condenser appears to be simply to facilitate the transfer of electric stress to and from the dielectric material between them. So long as the stress exists, the dielectric is charged, even if the metal coatings are removed. This is usually illustrated by means of a Leyden jar with removable coatings of thin tin-plate (Fig. 50). If this has been charged in the ordinary way, and placed on a cake of resin, or other insulating support, the glass jar can be lifted out from the outer metal vessel, and then the inner conductor can be lifted out from the jar by a glass

rod put under the hooked stem. On bringing the two metal coatings together, they give only a slight discharge, if any; but if, after the metal coats have been shown to be electrically neutral, the jar is put together again, it will give a discharge almost as powerful as if it had never been taken to pieces.

The same experiment can also readily be made with an Æpinus's condenser (**56**).

64. Energy of an Electric Field.—As already pointed out (**9, 10**), an electric field constitutes a store of energy, the amount of which is measured by the work which the electric forces of the field can do during exhaustion of the field; or, what comes to the same thing, by the work that must be done against electric forces to establish the field.

We can estimate this amount of work as follows. We may suppose the total charge on the field to have been produced by the successive transfer of elementary portions of positive electricity, dq, from the negative boundary of the field to the positive boundary, until the total charge, Q, had been transferred; or we may suppose an equal amount of negative electricity transferred in successive quantities $-dq$ in the opposite direction; or lastly, we may suppose $\frac{1}{2}Q$ transferred from the negative to the positive boundary, and $-\frac{1}{2}Q$ from the positive to the negative boundary. These three operations would be physically identical and differ only in the mode of expression; we shall, therefore, for the sake of definiteness, adopt language appropriate to the first mode. If then v and v' are, at any stage of the process, the respective potentials of the positive and negative boundaries, the amount of work done in transferring an elementary quantity of electricity at this particular stage is

$$dW = (v - v')\, dq.$$

If C is the capacity of the field and q the quantity already transferred

$$v - v' = \frac{q}{C}$$

therefore

$$dW = \frac{q}{C}\, dq$$

and the whole amount of work, W, done in producing the final electrification is the sum of all the values of the last quantity as q varies from 0 up to its final value Q, or

$$W = \frac{1}{C}\int_0^Q q\, dq = \tfrac{1}{2} \cdot \frac{Q^2}{C}.$$

The same conclusion can be easily established geometrically. In Fig. 51, let the straight line OA be drawn so that the ratio of the ordinate of any point to the abscissa of the same point may be equal to the ratio of the difference of potentials of the boundaries of the field to be considered to the corresponding charge. Then if OQ represents the charge q at any stage of the electrification, QM will represent the corresponding difference of potentials, and the work dW due to an increment of charge, dq, represented in the figure by QQ′, will be expressed by the area of the trapezium QMM′Q′. The sum of all such areas standing upon the base OP is equal to the area of the triangle OAP $= \frac{1}{2}$ OP . PA; hence if OP represents the total charge Q, and PA the resulting difference of potentials $V - V'$, we have for the work spent in electrification

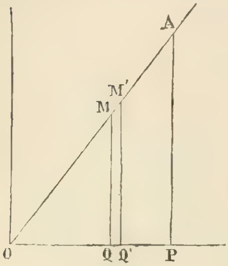


Fig. 51.

$$W = \tfrac{1}{2}\,\text{OP}\,.\,\text{PA} = \tfrac{1}{2}\,Q(V - V')$$

By combining this with the fundamental expression (51)

$$Q = C(V - V')$$

connecting charge, capacity and difference of potentials, we see that it is equivalent to the formula already obtained. In fact, the energy of an electric field may be expressed in terms of any two of the three quantities named, as follows:

$$W = \tfrac{1}{2}\,Q(V - V') = \tfrac{1}{2}\frac{Q^2}{C} = \tfrac{1}{2}\,C(V - V')^2.$$

65. Energy of a Condenser.—The energy of a condenser like that of any electric field (**64**), is represented by

$$W = \tfrac{1}{2}\,Q(V - V') = \tfrac{1}{2}\,C(V - V')^2 = \tfrac{1}{2}\frac{Q^2}{C},$$

or (**53**) by

$$W = \frac{AK}{8\pi e}(V - V')^2 = \frac{2\pi e}{AK}Q^2,$$

which shows that, for a given difference of potentials between its surfaces, the energy of a condenser is proportional directly to the area of the surfaces, and inversely to the thickness of the dielectric; while for a given charge, the energy is directly proportional to thickness and inversely proportional to surface.

It follows from these formulæ, that if a charged Leyden jar has its surfaces connected with the corresponding surfaces of a second uncharged jar (which for simplicity we will suppose exactly similar to the first), so that its charge is shared between the two, the combined energy of the two jars is now only half the original energy of the first jar. By combining the two jars, we have virtually a condenser of twice the capacity, and therefore, with the same charge, only half the energy. The electric energy that is lost during the transfer of part of the charge from one jar to another is converted into heat during the process.

If several jars with their similar coatings connected are charged simultaneously to a given difference of potentials, the energy of the combination is obviously the sum of the energies of the separate jars. Thus if there are n jars each having a capacity C, their total energy is

$$W = n \cdot \tfrac{1}{2} C (V - V')^2$$

or n times the energy of a single jar charged to the same difference of potentials.

If n similar jars, connected "in cascade," are charged so that the difference of potentials between the first and last coatings is $V - V'$, the difference between the coatings of each jar is $(V - V')/n$ and therefore its energy is $\frac{1}{2} C \left(\frac{V - V'}{n}\right)^2$, and the energy of the whole number is

$$W = n \cdot \tfrac{1}{2} C \left(\frac{V - V'}{n}\right)^2 = \frac{1}{n} \cdot \tfrac{1}{2} C (V - V')^2$$

or the nth part of the energy of a single jar charged to the same difference of potentials.

65.* Influence of Temperature.—The dielectric coefficient of an insulating material depends, among other conditions, on its temperature. It has been found, for example, that the dielectric coefficient of glass in the neighbourhood of 30° C. increases by about $\frac{1}{500}$ of its value for a rise in temperature of one degree. It follows that the capacity of a Leyden jar, or other condenser the dielectric of which is glass, must rise with rise of temperature, for (**53**) the capacity of any condenser is equal to the product of the dielectric coefficient K of the insulating material into a factor, say, a, depending only on the geometry of the conducting surfaces between which the insulator is contained. It also follows that the difference of potential, between the coatings of a charged Leyden jar and the energy stored up in it must both fall as temperature rises, since both, when charge

is constant, are inversely proportional to capacity. If we put β for the quantity $\frac{1}{K}\frac{dK}{dT}$, that is, for the rate of increase of dielectric coefficient with temperature expressed as a fraction of its value at a given temperature, the relations above stated can be conveniently expressed by the following formulæ—

Capacity	$C = aK$,	whence $\frac{1}{C}\frac{dC}{dT} = \beta$;
Difference of potentials	$V - V' = \frac{Q}{C}$,	" $\frac{1}{V - V'}\frac{d(V - V')}{dT} = -\beta$,
Electrical energy of charge	$W = \frac{1}{2}\frac{Q^2}{C}$,	" $\frac{1}{W}\frac{dW}{dT} = -\beta$.

A further consequence of the variation of dielectric coefficient with temperature, not so immediately obvious as those just pointed out, is that heat must be supplied to a Leyden jar, or glass condenser, to maintain its temperature constant when its charge is increased, or, what comes to the same thing, that its temperature must fall if its charge is increased without allowing it to gain heat at the same time. To show that this is the case, we will

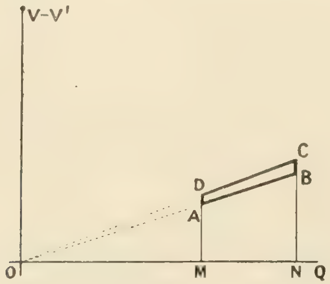


FIG. 51A.

suppose the following series of operations to be performed on the jar :—

i. The charge is increased from Q_o to Q_1 while the temperature is kept constant at T on the absolute scale. During the increase of charge, there will be a proportionate increase of the difference of potentials between the coatings, for the ratio $Q/(V - V')$ is the capacity, which is constant at the same temperature. Hence if the initial charge Q_o is represented by the abscissa OM of the

point A in the diagram (Fig. 51A) and the initial difference of potentials $(V-V')_0$ by the ordinate MA of the same point, the simultaneous changes of these quantities will be represented by a line such as AB, if the point B is taken so that its co-ordinates express the new values Q_1 and $(V-V')_1$ respectively.

ii. Now, without further increase of charge, let the temperature be *lowered* by the very small amount dT: this, in accordance with what has been said above, will cause a small *increase* in the difference of potentials between the coatings equal to

$$d(V-V')=(V-V')\beta dT$$

and denoted in the diagram by BC.

iii. Next, decrease the charge again from Q_1 to Q_0 and keep the temperature constant at the value $T-dT$. The corresponding changes of charge and potential-difference may be represented by the line CD.

iv. Lastly, the charge having now been restored to its initial value, let the temperature be raised by dT, so as to restore it also to the value it had at starting. The result of this will be to cause the difference of potentials to fall again to its initial value. This change is denoted in the diagram by the line DA which completes the closed figure ABCD.

During the first of the above operations the electrical energy of the jar is increased from

$$\tfrac{1}{2}Q_0(V-V')_0 \text{ to } \tfrac{1}{2}Q_1(V-V')_1$$

and electrical work equal to the difference of these quantities, represented in the diagram by the area MABN, must have been done upon the jar from without. During the second operation, the infinitesimal cooling by dT, the charge remains constant but the electric energy is infinitesimally increased and is now represented by the area of the triangle ONC, but no work is done since no displacement of electricity takes place. During the third operation, electric energy represented by the difference of the areas of the triangles ONC and OMD—that is, by the area MDCN—is withdrawn or is done by the jar on some external object. During the fourth operation, no work is done, but the electric energy diminishes to the value represented by the area OMA. The fourth operation completes a closed cycle, at the end of which the condition of the jar is in all respects identical with what it was at the beginning. Nevertheless, more work

has been obtained from the jar during the cycle than was done upon it; in other words, work has been *gained* without causing any permanent change in the condition of the jar, the amount of this work being represented by the area of the four-sided figure ADCB.

The whole series of operations that has been supposed to be performed upon the jar is thermodynamically "reversible"; that is, it does not involve the existence of a finite difference of temperature, or of electric potentials, between bodies in contact, and an infinitesimal change of conditions during any operation of the cycle would suffice to invert the sense in which that operation is going on. Hence we may apply the conclusions established by thermodynamic reasoning in reference to reversible cycles generally: those with which we are concerned are: (*a*) that, when work is gained as the result of such a cycle involving operations taking place at two fixed temperatures, heat must have been taken in at the higher of the two temperatures and a smaller amount must have been given out at the lower; (*b*) that the work done is the mechanical equivalent of the amount by which the heat taken in exceeds the heat given out; (*c*) that the work done during the cycle bears to the mechanical value of the whole amount of heat taken in at the higher temperature the same ratio as the difference of the two temperatures concerned bears to the higher temperature.

The application of these results to the case under consideration leads to the following conclusions. First from (*a*), that the jar must *receive* heat during operation i. and must *lose* heat, to a smaller amount, during operation iii.; or, what comes to the same thing, that, if a Leyden jar is not allowed either to lose or gain heat, its temperature will fall during increase of charge, and will rise during decrease of charge. If we put l for the average value, during increase of charge at T, of the ratio dH/dQ, where dH is the heat needed to maintain temperature constant during the very small increment of charge dQ, the whole quantity of heat taken in by the jar during operation i. is $l(Q_1 - Q_2)$. Similarly, if $l - dl$ stands for the average value of dH/dQ during decrease of charge at temperature $T - dT$, the whole quantity of heat removed during operation iii. is $(l - dl)(Q_1 - Q_2)$.

Secondly, we conclude from (*b*) that the excess of the heat taken in during operation i. over that given out during operation iii., namely $(Q_1 - Q_2)dl$ is, in dynamical measure, equal to the work gained during the whole cycle. This work is, as we

have seen, represented in Fig. 51A by the area of the quadrilateral ABCD; that is, it is the product of the increase of charge $Q_1 - Q_o$, represented by MN, into the mean amount by which the difference of potentials of the coatings at temperature $T - dT$ exceeds the difference at temperature T, this mean difference being expressed in the figure by $\frac{1}{2}$(AD + BC). Putting $V - V'$ for the arithmetic mean of $(V - V')_o$ and $(V - V')_1$ the excess in question is $(V - V')\beta dT$. Multiplying this by $Q_1 - Q_o$ and equating the electrical and thermal expressions for the work done, we have

$$(Q_1 - Q_o)(V - V')\beta dT = (Q_1 - Q_o)dl.$$

In the third place, the consideration (*c*) that the heat converted into work bears the same ratio to the whole amount of heat taken in as the range of temperature dT does to the absolute temperature T, gives the relation $dl/l = dT/T$. If we use this to eliminate dl from the last equation, we may deduce

$$l = (V - V')T\beta.$$

Accordingly, the heat that must be taken in by a condenser to keep its temperature constant during an increase dQ of its charge is

$$dH = ldQ = (V - V')T\beta dQ.$$

In this expression $(V - V')dQ$ is the amount of electric work dW that is done in increasing the charge of the jar; hence we may write

$$dH = T\beta dW.$$

The total increment of energy is the sum of the electric work done in charging and of the heat taken in to keep the temperature constant, or

$$dE = dW + dH = (1 + T\beta)dW.$$

Of this whole amount, only the electrical part dW is recovered when the jar is discharged.

The values of the coefficient β found for a few substances by Cassie are as follows at about 30° C.:—

Substance.	β.
Glass	0·002 to ·0012
Mica	·0003
Ebonite	·0004

Taking the value 0·002 for glass, we have $T\beta = 0{\cdot}6$ at 27° C.; consequently, by the above, $dW/dE = 1/1{\cdot}6$. Hence, when an ordinary Leyden jar is charged at constant temperature, the gain of electric energy is only about $\frac{5}{8}$ of the gain of total energy.

In the case of such liquid dielectrics as have been examined, the value of β is in most cases negative. For petroleum oil, however, it is positive. Fleming and Dewar have experimented on several congealed liquids, both saline solutions and organic compounds, at very low temperatures, down to that of liquid air, and find that, in a large number of cases, the dielectric constant at $-185°$ C. is not far from 2·5 or 3, and they conclude that, at sufficiently low temperatures, this would be the approximate value for nearly all the substances they examined.

66. Geometrical Representation of the Energy of an Electric Field.—If, as we suppose, the forces that act on an electrified body in an electric field are the result of stresses in the field itself, we may expect the energy of the field to be definitely related to the distribution of force within it as expressed by tubes of force and equipotential surfaces. This is, in fact, the case, and the energy of a field and the mode in which it is distributed admit of very simple representation. Suppose the positive boundary of a field, whose charge we will, for simplicity, assume to be some whole number of units, to be divided into areas such that each is charged with a unit of positive electricity, and let each of these areas be taken as the base of a tube of force extending to the negative boundary. In this way the whole field will be exhaustively divided into tubes of force whose number gives the numerical value of the total charge. If we now suppose a series of equipotential surfaces to be drawn in the field, the potential of each differing by unity from that of the next, every tube of force will be divided into a set of compartments or cells, the number of which gives the numerical value of the difference of potentials between the boundaries of the field. That is, the number of tubes of force gives the value of Q, and the number of cells into which each tube is divided by the equipotential surfaces gives the value of $V - V'$; consequently the whole number of cells gives the value of $Q\,(V - V')$; or *half* the number of cells into which the whole field is divided by tubes of force and equipotential surfaces drawn in the way described gives the electric energy, $\frac{1}{2}\,Q\,(V - V')$, of the field.

67. Conversion of the Energy of an Electric Field into Work.—If we imagine one or both of the boundaries of the field to be capable of yielding to the electric forces which tend to draw them together, the field would ultimately collapse, its energy being exhausted by the work done on the movable boundaries. During this process the difference of potentials between the boundaries would gradually fall from the initial value $V - V'$ to nothing when the surfaces were in contact, the energy remaining in the field at

any stage being the half-product of the then existing difference of potentials into the charge, and the work that has been done being equal to the amount by which this product falls short of its initial value. Thus the work done while the difference of potentials falls from an initial value $(V-V')_o$ to a smaller value $(V-V')_1$ is

$$W=\tfrac{1}{2}Q[(V-V')_o-(V-V')_1]=\tfrac{1}{2}Q^2\left(\frac{1}{C_o}-\frac{1}{C_1}\right).$$

If the field is bounded by parallel plane surfaces whose distance apart, e, is small in comparison with their linear dimensions, its capacity is $C=\dfrac{AK}{4\pi e}$ (**53**); hence, in this case the work done while the capacity changes from C_o to C_1 is

$$W=\frac{2\pi Q^2}{AK}(e_o-e_1).$$

But e_o-e_1 is the distance through which the bounding surfaces have approached each other, and if we put P for the average force pulling them together through this distance, we have

$$W=P\,(e_o-e_1),$$

and hence, for the case considered,

$$P=\frac{2\pi Q^2}{AK}.$$

It is to be noted that this value of P contains nothing that depends on the distance apart of the surfaces, so that, under the conditions supposed, P is the same at all distances.

Whatever the configuration of the field, its capacity must increase as its bounding surfaces are drawn nearer together, and consequently, with a constant charge, the difference of potentials of the surfaces must diminish. If, however, the field under consideration forms only a part of a connected electrical system—say, for instance, that its surfaces are each connected with one of the boundaries of another independent field of practically infinite capacity—it is possible for the difference of potentials to be kept constant while the surfaces approach and work is done by the force in the field. In this case it is evident that the charge must increase as the capacity increases, the increment required to maintain the initial difference of potentials being $Q'=(C_1-C_o)(V-V')$, and energy equal to the product of this quantity into the constant difference of potentials of the surfaces, or $Q'(V-V')=(C_1-C_o)(V-V')^2$, must be supplied to the field from without. The energy of the field, which was at first $\frac{1}{2}Q(V-V')$, is now $\frac{1}{2}(Q+Q')(V-V')$; that is to say, it has

been increased by $\frac{1}{2}Q'(V-V')$, or by *half* the amount of energy supplied from without, the other half of the energy supplied being spent in the mechanical work of drawing the surfaces of the field nearer together.

To see more in detail how this comes about, we may suppose the whole change of capacity to take place by an infinite number of successive infinitely small changes dC, the field being isolated during each, and then, before the next small change, being connected with the external system whereby the difference of potentials is restored to its original value. It follows from what has been proved above, that the work done during each elementary change of capacity is

$$dW = -\tfrac{1}{2}Q^2 d\left(\frac{1}{C}\right) = \tfrac{1}{2}\frac{Q^2}{C^2}dC.$$

In the last expression, both Q and C vary as the process goes on, but their quotient, $Q/C = V - V'$ is, by the circumstances of the case, constant. Hence the total amount of work done is equal to the product of $\frac{1}{2}Q^2/C^2$, or of $\frac{1}{2}(V-V')^2$ into the sum of all the infinitesimal changes of capacity dC. Hence we may write

$$W = \tfrac{1}{2}\frac{Q^2}{C^2}(C_1 - C_0) = \tfrac{1}{2}(V-V')^2(C_1 - C_0).$$

That is to say, the mechanical work done during the whole process is numerically equal to the amount by which, as we have already seen, the energy of the field is *increased*, each of these quantities representing one-half of the energy supplied from without.

Comparing the amounts of work done in the two cases considered, during a given change of capacity, we have, with charge constant—

$$W_Q = \tfrac{1}{2}Q^2\left(\frac{1}{C_0} - \frac{1}{C_1}\right);$$

with difference of potentials constant—

$$W_{V-V'} = \tfrac{1}{2}(V-V')^2(C_1 - C_0).$$

Hence
$$\frac{W_{V-V'}}{W_Q} = \frac{(V-V')^2}{Q^2}C_0C_1 = \frac{C_1}{C_0},$$

which is always greater than unity.

For the force pulling the boundaries of the field towards each other, we have, with difference of potentials constant—

$$P = -\frac{dW}{de} = -\tfrac{1}{2}(V-V')^2\frac{dC}{de}.$$

In the special case of parallel plane surfaces, where $C = \frac{AK}{4\pi e}$, we have $-\frac{dC}{de} = \frac{AK}{4\pi e^2}$, and therefore

$$P = (V - V')^2 \frac{AK}{8\pi e^2}.$$

Since in this expression everything is constant except e, it follows that the force between the surfaces is inversely as the square of their distance apart.

68. Stresses in an Electric Field.—The boundaries of an electric field are subject to stresses urging them towards one another; further, each of the electrified layers by which the field is bounded tends to spread as far as possible. We have expressed these facts by help of the conception of lines of force traversing the medium from the positive boundary to the negative boundary of the field, these lines being conceived of as in a state of longitudinal tension and as mutually repelling each other. This is equivalent to attributing electric attraction and repulsion, not to an action at a distance taking place between electrified conductors, but to a state of stress existing throughout the intervening dielectric medium. We have now to investigate a definite system of stresses to which the observed phenomena may be attributed.

We have seen (**45**) that the mechanical force acting on unit area of the surface of a charged conductor is $2\pi\sigma^2/K$, and if the force is due to a tension in the contiguous dielectric medium this must also be the value of the tension close to the surface. If we express this in terms of the value f_o of electric force at the surface, the tension p is given by

$$p = \frac{f_o^2 K}{8\pi},$$

since $f_o = 4\pi\sigma/K$ (**20, 44**); and, assuming that throughout the medium the tension is the same function of the electric force, we get for the tension at any point the expression

$$p = \frac{f^2 K}{8\pi} = \frac{Df}{2}$$

where D is the induction, and f the electric force at the point. The force on an area A, so small that p is uniform throughout, is $pA = \frac{DA}{2} \cdot f$, and since the product DA is constant throughout

a tube of induction, the force pA is greater for a cross-section of smaller area where f is relatively great than for a section of larger area where f is relatively small. Thus, if the figure (Fig. 52) represents a part of a tube, there is a pull from right to left applied at the smaller end which exceeds the pull from left to right applied at the larger end. Consequently, under these forces alone the portion of the medium occupying this portion of the field cannot be in equilibrium, and to produce equilibrium there must also be stresses applied at the lateral faces of the tube.

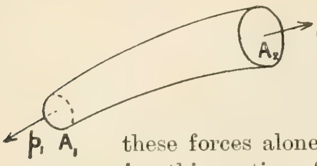


Fig. 52.

To investigate the relation between the longitudinal and transverse stresses acting on the medium in a tube of induction, consider as a special case a tube of rectangular section in a field bounded by two concentric spherical conductors.

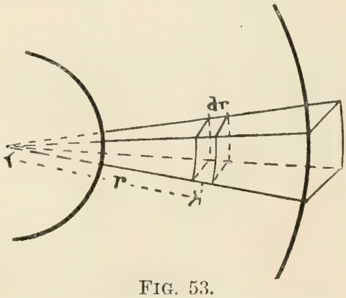


Fig. 53.

Let each pair of the four plane faces which bound the tube make a small angle θ with one another, all of them passing through a common centre (Fig. 53). Considering a small length dr of the tube at mean distance r from this centre, and bounded by spherical surfaces of radii $r+\frac{1}{2}dr$ and $r-\frac{1}{2}dr$, the force along the length of the tube on any such spherical surface within it is

$$pA=\frac{DA}{2}\cdot f=\frac{DA}{2}\cdot\frac{Q}{Kr^2}$$

where Q is the charge of the field (acting as though it were concentrated at the common centre), and the difference of the forces on its bounding curved surfaces is (since $DA=\text{constant}$)

$$d(pA)=\frac{DA}{2}\,df=-\frac{DA}{2}\cdot\frac{2Q}{Kr^3}\,dr=-\frac{2pA}{r}\,dr,$$

that is, a force of amount $\frac{2pA}{r}\,dr$ tending towards the centre.

Now $$A=r^2\theta^2;$$

hence this inward force is $2pr\theta^2dr$.

In order to maintain equilibrium, a stress p', the magnitude of which is to be determined, must be applied at the lateral faces, giving a resultant acting away from the centre equal to the inward force just found. The area of each lateral face is $r\theta\,.\,dr$; the

force applied to each is therefore $p' . r\theta dr$; each acts at right angles to the face, and consequently makes the same angle $90° - \frac{\theta}{2}$ with the central axis of the tube. The components perpendicular to this axis therefore mutually cancel one another, while the radial components add up to

$$4 . p' . r\theta dr . \sin\frac{\theta}{2}$$

$$= 2p'r\theta^2 dr, \text{ since } \theta \text{ is small.}$$

And in order that this may act outwards, p' must be an inward stress or *pressure*.

But comparing this with the previous result, we get

$$p' = p,$$

or, at each part of a tube there is a longitudinal tension and a transverse pressure of the same numerical amount. If the field is uniform, the tubes are cylindrical or prismatic, and in that case the longitudinal tensions applied to any portion of the medium are in equilibrium of themselves, and the transverse pressures also, and each has the constant value

$$\frac{Df}{2}.$$

The result obtained above for a tube bounded by lines radiating from a point may be extended to all cases, for they may all be regarded as resulting from the geometrical composition of forces arising from point-charges.

68.* Kerr's Electro-Optic Effect. Quincke's Experiments. —That the dielectric is modified when subjected to electric stress was demonstrated by Kerr, who found that if plane-polarised light is passed through glass across the lines of force, the light in general becomes elliptically polarised, indicating that the velocity with which the light is transmitted depends on the angle between its plane of polarisation and the direction of the electric field. This is what would happen if, instead of electric force being applied to the dielectric, it were subjected to mechanical pressure or tension in the same direction. Kerr has further shown that it is only the light which is polarised perpendicularly to the direction of the field that has its velocity affected. The effect may be shown by placing a rectangular vessel containing carbon bisulphide between two crossed Nicol's prisms, two parallel plates (connected to an electric machine) being immersed in the liquid. No light can pass through the combination if the plates are uncharged; if they are then charged the field grows bright unless the principal directions of the Nicols are either parallel to or at right angles to the lines of force. The effect is a maximum if the angle is 45°.

Quincke has shown that an air bubble in a liquid medium changes shape when placed in an electric field, expanding in the direction of the lines of force and contracting at right angles to them. At the same time the pressure in the bubble increases. He has also shown that if a glass bulb with a capillary stem be filled with conducting liquid the level of the liquid falls in the tube when the vessel is used as a condenser and charged. Since there is no stress in the conducting liquid the effect is due to the expansion of the glass.

68.** **Pyro-electricity.**—Certain crystals when heated become oppositely electrified at two extremities. These crystals, of which tourmaline and quartz may be taken as examples, possess one or more crystallographic axes the extremities of which are dissimilar. It is at the ends of such an axis that the opposite charges appear; these ends are known as antilogous poles and analogous poles according as negative or positive electricity appears at them on warming; and the line joining opposite poles is known as the electric axis.

Piezo-electricity.—The crystals which exhibit the above phenomenon (and only these) also produce opposite electricities when subjected to pressure or tension in directions parallel to an electric axis. To develop this effect a slice of the crystal may be cut with its faces perpendicular to the axis in question. Armatures of tinfoil are then gummed on opposite faces; and plates of hard rubber (for insulating purposes) are placed one on each outside surface. The whole can then be placed between the jaws of a press. If the armatures are connected, one to earth and the other to the needle of a quadrant electrometer, while the alternate pairs of quadrants are connected with the terminals of a cell, the production of the electrification can be observed. The sign is reversed when the pressure is changed to a tension. In all cases the sign of the charge produced by pressure is the same as that due to *cooling*.

To this phenomenon corresponds a reciprocal one. Take a parallelopiped of quartz, say, with two of its faces cut normal to the electrical axis and two of them normal to the optic axis of the crystal. If a difference of potential be produced between the former two faces, the quartz expands along the electric axis and contracts in the direction which is normal to both the electric and optic axes; or else the reverse, according to the sign of the difference of potential. The third direction does not change in length. The changes in the reciprocal and the direct phenomenon are connected together by a law of opposition (analogous to the law

of Lenz, **299**); that is to say, the difference in potential produces a change in length along the electric axis which would, if produced by pressure, give rise to a difference of potential of opposite sign. Moreover, in all cases, the magnitude of the change in length is given in centimetres for a difference of potential of 1 C.G.S. electrostatic unit by the same number as that which expresses in absolute value the charge set free by a force of one dyne in the direction in question.

CHAPTER VII

ELECTRICAL MACHINES

69. Electric Machines in General.—Any contrivance which serves for the continuous generation of electrical energy may be called an *electrical machine.* Such contrivances are frequently spoken of as sources of electricity, although no process is known which can be properly described as the production or generation of electricity. All that can be done by any electrical process is to *transfer* electricity in such a way that the quantity which enters a field across one of its bounding surfaces is equal to the quantity which simultaneously leaves the same field across the other bounding surface. Or, if we prefer to speak of two different kinds of electricity, then whenever a given quantity of positive electricity is transferred in one direction across a given surface, and an equal quantity of negative is simultaneously transferred in the opposite direction across the same surface, we must say that equal but contrary transfers are taking place at the other boundary of the field. It is thus impossible either to increase or decrease the absolute quantity of electricity.

The energy of an electric field (**64**), namely $\frac{1}{2}\,Q\,(V - V')$, being proportional jointly to the charge and to the difference of potentials, is increased when either of these factors separately, or their product, is increased. In the various kinds of electrical machines, the mechanical energy expended is partly expended in overcoming the ordinary friction of the moving parts, but in addition to this there is always an expenditure of energy in maintaining motion in opposition to electric force, and the electric energy generated by the machine is equal to the mechanical energy thus expended.

Electrical machines may be divided into two classes, namely, those in which the friction of heterogeneous substances plays an essential part in the electrical action, and those whose action depends on electrical influence. Machines of the former class are called *friction machines*, those of the latter class are called *influence machines.*

70. General Mode of Action.—In either case an electric field

is established, one boundary of which is formed by a moving part of the machine, which may be called the *carrier*, and the other boundary usually by the surface of the room. The motion of the machine brings the carrier into a position in which either the whole of it at once, or the various parts of it in succession, are more or less completely surrounded by an insulated conductor, the *collector* of the machine. Consequently, the field of force extending between the carrier and the room is cut into two parts, one extending from the carrier to the collector, and the other from the collector to the room. If the carrier is formed of conducting material and comes into connection with the collector as it passes, or if, being a non-conductor, its surface comes sufficiently near to a number of fine points projecting inwards from the inner surface of the collector, the first-mentioned part of the field is abolished, and the carrier passes away unelectrified and ready to be electrified afresh by friction or by influence. The result is thus to set up an electric field between the collector of the machine and the room. The charge, and therewith the difference of potentials of this field, increases with the continued action of the machine. By appropriate connections a greater or less portion of the charge and energy of the field may be transferred to a field bounded by any given conducting surfaces. In this way a Leyden jar or any other condenser may be charged.

71. Law of Increase of Charge.—In general, in a friction machine, each portion of the surface of the carrier is in the same electrical condition every time it comes under the influence of the collector, and consequently it increases the charge on the field established by the machine by the same amount every time; in other words, the charge on the field, and the difference of potentials which is proportional to the charge, increase in arithmetical progression. In other cases, however, particularly with influence machines, the charge frequently varies as the terms of a geometrical progression. Consider, for example, the following arrangement, which may be taken as a type of all influence machines. Two metal jars, A and B, are placed on insulating stands, the potential of A exceeding that of the room by some small amount V_0, and the potential of B exceeding that of the room by V'_0. A metal ball is hung by an insulating thread inside each jar, and while there is connected for a moment with the room. Each jar thus becomes one boundary of an electric field, the second boundary of which is formed partly by the ball hanging inside the corresponding jar, and partly by the surface of the room. If c is the capacity of the part of the field extending between the inner surface of a jar and its ball, and C the capacity of the part of the field extending between the

outer surface of a jar and the room, the total capacity of each field is $C+c$, and the electrification on the ball in the jar A is $-cV_o$, and that on the other ball is $-cV'_o$. Now transfer each ball to the other jar, letting it first go down to the bottom and then raising it a little by the insulating thread. The balls thus give up their electrification completely to the jars, which therefore acquire the potentials $V_1 = V_o - \frac{c}{C+c} V'_o$ and $V'_1 = V'_o - \frac{c}{C+c} V_o$ respectively, or, more shortly, $V_1 = V_o - mV'_o$ and $V'_1 = V'_o - mV_o$. This gives, for the difference of potentials

$$V_1 - V'_1 = (1+m)\,(V_o - V'_o).$$

Next let both balls be again connected for a moment with the room and then retransferred to the jars they were in before, allowed to touch the bottom and raised. The potentials of the jars thus become $V_2 = V_1 - mV'_1$ and $V'_2 = V'_1 - mV_1$ respectively, and the difference becomes

$$V_2 - V'_2 = (1+m)\,(V_1 - V'_1) = (1+m)^2\,(V_o - V'_o).$$

If the same series of operations be repeated over and over again, the difference of potentials becomes progressively greater and greater, being, after n times,—

$$V_n - V'_n = (1+m)^n\,(V_o - V'_o).$$

72. Frictional Machine.—This consists of a glass plate, P, movable about a horizontal axis (Fig. 54); of two pairs of cushions, C, placed at the ends of a vertical diameter; and of two U-shaped insulated conductors, S and S′, provided with points on the side towards the plate. These conductors are called *combs*. Electrification results from friction of the glass against the cushions, and as each part of the electrified glass comes between the arms of the combs, the electrification is transferred to the latter, and the plate passes on almost unelectrified; accordingly a quantity of electricity, equal to that which the plate has brought, is communicated to the insulated *prime conductor*, DD′, connected with the combs.

Glass is the best material for the plate, but as there is considerable variation in the quality of glass, some care is needed in selecting it; that which is least hygroscopic is the best. All parts of the machine should be kept thoroughly dry.

The cushions are ordinarily of leather stuffed with horse hair, and are kept pressed against the plate by springs. If the pressure is sufficient to bring the rubbers into good contact with the glass, further pressure does not increase the electrification. The surface of the rubber should be covered with a conductor such as that

variety of sulphide of tin known as *aurum musivum*, or, still better, with *Kienmayer's amalgam*, which is composed of zinc, tin, and mercury; this is reduced to powder, and a little lard having been applied to the cushion, some of the powder is spread evenly over it.

The contact between the glass and the rubber sets up a fixed

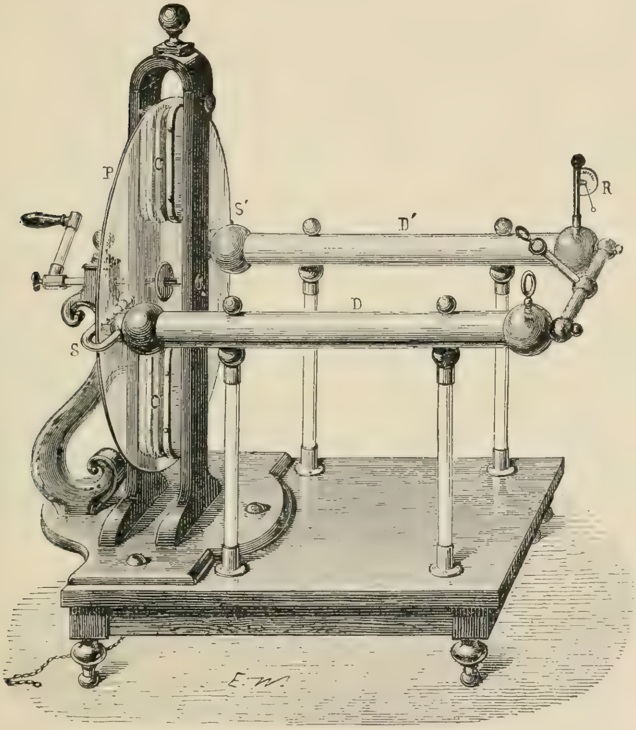


FIG. 54.

difference of potentials between them, the glass becoming positively electrified, and the rubber negative. This difference of potentials is at first very small, since the equal oppositely electrified surfaces are very close together. But as each portion of the glass moves away from the rubber, its potential acquires a very high value, and, in order to prevent loss, quadrant-shaped pieces of oiled silk, not shown in the figure, are often fixed to the rubbers so as to enclose on both sides the parts of the plate between the rubbers

and the combs. These are kept in contact with the plate by the electrical attraction.

The rubbers are usually put in connection with the earth by strips of tinfoil which pass down the support; the conductor of the machine thus gives positive electricity. But machines are also constructed in which the cushions are supported by glass legs, and connected with a second system of conductors. Either positive or negative electricity may thus be obtained at will. Whether the rubbers are insulated or not, the yield of the machine is the same for the same atmospheric conditions, as is also the difference of potentials between the rubbers and the conductors. It is only the absolute value of the potential which changes.

In order to take sparks from either the positive or the negative conductor, the experimenter must be electrically connected with the other conductor. If this is not insulated the table and floor generally serve as a sufficient connection.

Similarly, to charge a Leyden jar or battery, the rubbers of the machine must be connected with one coating and the prime conductor with the other. Here, again, if the rubber is uninsulated, the table generally forms a sufficient connection between it and the outer coating. But if either the rubber or the outer surface of the jar is insulated, some conducting connection must be provided.

In order to estimate the potential of the conductor, a small electroscope known as *Henley's electrometer* (R, Fig. 54) is used. This is a small pendulum formed of a thread with a pith-ball at the end, movable about an axis in front of a scale. The divergence increases with the potential. Similarly the diminution of the divergence indicates the loss of electricity.

In order to increase the capacity of conductors and obtain stronger sparks, the prime conductor used formerly to be connected with large hollow metallic cylinders suspended from the ceiling by silk, which were called secondary conductors. It is simpler and less cumbersome to connect the conductor with the inner coating of a Leyden jar, the outer coating of which is connected with the room. If the potential of the machine should exceed that which the jar can support, a cascade arrangement may be used (**57**).

The unit jar, to be described later (**88**), furnishes a very simple means of measuring the yield of the machine, and of investigating the conditions which modify it. We have already said that this yield is independent of the pressure of the rubbers when once there is sufficient contact between them and the glass. The yield

is likewise independent of the capacity of the conductors, provided the causes of loss are the same. It is proportional to the surface of the glass which passes between the rubbers; it is, therefore, proportional to the length of the rubbers; thus, if one pair of rubbers is used instead of two, the yield is reduced to one half. The yield is further proportional within wide limits to the speed of rotation of the plate.

In some machines only a single pair of rubbers and a single comb is used, with the object of pushing back the limit at which discharge is produced between the conductor and the rubber. A greater difference of potentials is obtained, but a lower yield.

73. Influence Machines—Replenisher.—We shall commence the description of these machines by that of Lord Kelvin's re-

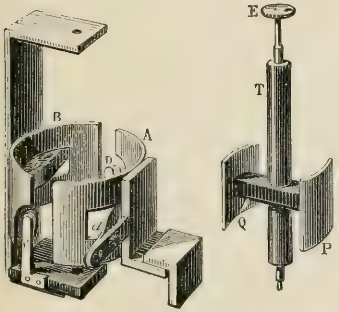


FIG. 55.

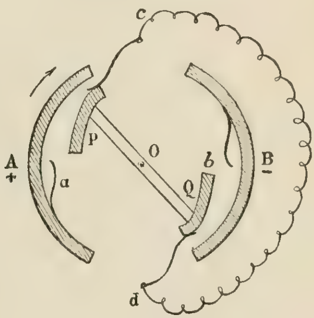


FIG. 56.

plenisher, which Figs. 55 and 56 represent in about its ordinary size.

A and B (Figs. 55 and 56) are two portions of a cylinder with the same axis, O, both of which are insulated; these take the place of the metal jars in the arrangement described in (**71**). In the interior are two curved metal plates, P and Q, which are fixed by a cross-piece of ebonite to an ebonite rod, T, so that by means of a milled head, E, they can be made to rotate within the cylinders A and B. Four springs, *a*, *b*, *c*, *d*, placed at the same distance from the axis, are touched in succession by the projecting parts of the plates P and Q; *a* and *b* communicate with A and B respectively, while *c* and *d* are connected with each other. The action of the replenisher is essentially the same as that discussed in (**71**).

74. Water Dropping Machine.—Kelvin has also constructed a machine in which the carriers are drops of water.

A and A′ (Fig. 57) are insulated metal cylinders into which dip the nozzles from which the drops fall. If A has a slight positive charge, the drops within it being, up to the moment of breaking away, connected with the earth, must be at zero potential and thus carry away a negative charge. They then fall into an insulated funnel, B′, to which they give up their charge completely, as the point of the funnel by which they escape is surrounded by the receiver itself. The negative charge thus given to B′ is shared with A′ through a wire connecting them; hence the drops through A′ will carry a positive charge to B, which in turn shares it with A, thus increasing the charge on A. This increase goes on until the falling drops are diverted from their course by the attraction of the influencing cylinders and the repulsion of the collecting cylinders. In this instrument the energy of the resulting electric field is drawn from that of the falling drops.

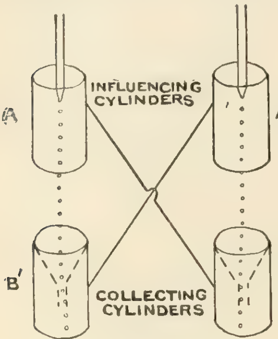


FIG. 57.

It is instructive to calculate the law of increase of charge. Let V be the instantaneous potential of the system AB; and V' that of the system A′B′; let c be the capacity of a drop, n the number of drops which fall per second, C the capacity of each of the equal insulated systems AB and A′B′, and l the coefficient of leakage of each $\left(i.e.\ l = \frac{1}{V}\frac{dQ}{dt} = \frac{1}{V}\frac{CdV}{dt}\right)$. Since the charge on each drop inside A′ is $-cV'$ the increment of charge of the system AB in time dt is

$$-ncV'dt - lVdt,$$

and the corresponding increase of potential is this quantity divided by C. Similarly for the system A′B′ the increment of potential is

$$dV' = (-ncV - lV')\frac{dt}{C}$$

Subtracting we obtain

$$d(V - V') = (nc - l)(V - V')\frac{dt}{C}$$

whence by integration

$$\log_e (V - V') = \frac{nc - l}{C}t + A$$

where A is a constant. To find the value of this constant assume that when $t = 0$, $V' = 0$, and $V = V_o$ then

$$A = \log_e V_o$$

and
$$V - V' = V_0 \exp.\left(\frac{nc - l}{C}t\right)$$
showing that for $V - V'$ to increase there must be an initial difference of potentials V_0, and that nc must be greater than l.

75. Holtz's Machine.—The action of Holtz's machine is analogous, except that the increase of the charge is continuous instead of intermittent.

It consists of two glass plates (Fig. 58) placed parallel at a small distance from each other; one is fixed by means of wooden

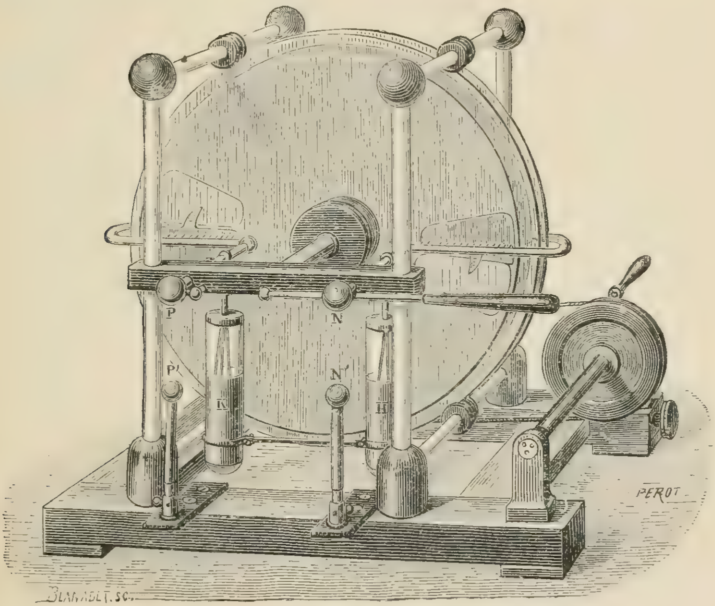


FIG. 58.

knobs supported by glass rods, the other is movable about a horizontal axis. Besides the central aperture, through which passes the axle of the movable plate, the fixed plate has two apertures or *windows* at the ends of the same diameter. Along one edge of each of these windows is an armature of paper provided with a *tongue*, also of paper, terminating in a point in the centre of the opening. Opposite the armatures, on the other side of the free movable plate, are two combs connected with two small insulated conductors, P and N. By an insulated handle it is possible to vary at will the distance of the two knobs, P and N,

which will be called the poles of the machine. Two sliding knobs, P′ and N′, may be brought into contact with the corresponding poles, so that either one or the other may be put to earth.

To get the machine to work, it must first be *primed*. The two poles, P and N, are brought into contact, and the movable plate being put in rotation by the handle, so as to move towards the points of the tongues, an ebonite plate or stick of sealing-wax charged by friction with negative electricity is brought near one of the armatures. When the machine is in action a peculiar hissing sound is heard; from the comb near which the ebonite was held, a luminous sheet spreads in a direction opposite to the motion of the plate, this is the positive comb (**19**); every point of the negative comb is tipped by a small star of light. The electrified ebonite may then be taken away, and the two poles can be separated; a brush discharge then passes between the two poles as long as the rotation continues. It is readily shown that the electricity of each of the poles is of the opposite kind to that which escapes from the corresponding comb. The action stops if the knobs P and N are drawn too far apart.

The brush discharge depends upon the small capacity of the conductors. In order to increase this capacity, the two conductors, P and N, may be connected with two Leyden jars, K and H, the outer coatings of which are connected with each other. The two jars form thus a cascade between the two poles, and each of them has half the difference of potentials of the two poles (**57**). If the outer coatings are connected to earth, the two conductors are at equal and opposite potentials. When jars are used, the sparks are not continuous, as with the brush, but they follow each other at regular intervals, and are much more luminous, denser, and produce a louder crack. With a machine the plate of which is 60 cm. in diameter they may attain a length of 20 cm.

Most of these machines, as now constructed, are double; they have two fixed plates close to each other. The two movable plates, which are mounted on the same axle, rotate on the outside. The armatures are placed opposite each other, those which are similarly electrified being together; the same U-shaped comb surrounds the two movable plates. The action of the machine is in no way altered, except that the yield is double.

It is sometimes difficult to start the action of the machine, especially in damp weather; it then will not begin unless the ebonite plate is strongly electrified. The glass plate is usually made to rotate with a velocity of eight to ten turns a second.

In order to explain the action of the machine and to make the

figures clearer, we will suppose the glass plates replaced by concentric cylinders (Fig. 59).

The circumference represents the movable cylinder; A′ and B′ the two paper armatures; the fixed cylinder is not shown, as it has scarcely any other function than that of supporting the armatures; A*a* and B*b* are the two conductors represented as in actual contact.

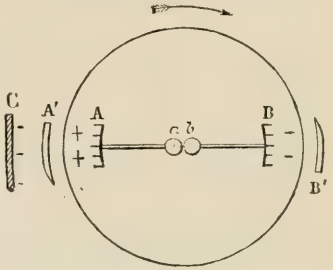


FIG. 59.

Suppose the plate to be at rest. The ebonite plate C charged with negative electricity is brought near the armature A′. It acts by influence on the conductor AB; positive electricity is attracted towards A, and negative repelled towards B; but owing to the points the positive electricity escapes on to the glass at A and negative at B until equilibrium is established.

When the plate is turned, the positively electrified glass is carried away from A, and the negatively electrified glass from B, so that the conductor AB is again left subject to the action of the ebonite, which, acting as before, causes a stream of positive electricity to escape from the comb A, and a stream of negative from B. Each part of the plate, as it passes A, thus receives a positive charge, and, as it passes B, a negative charge, until, after half a turn, the whole plate is electrified as indicated by Fig. 60.

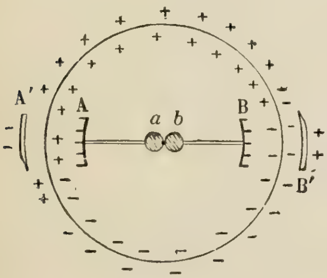


FIG. 60.

Henceforward the armatures come into play: the paper tongues that project from them through the windows in the fixed plate graze the surface of the approaching parts of the revolving plate, and thus the armature A′ acquires a negative charge, and B′ a positive charge. The electric force due to these charges acts in the same way as that due to the electrified ebonite, so that the electric influence on the conductor AB is increased, and a greater discharge of electricity—positive at A, negative at B—takes place on to the surface of the revolving plate. The consequence is that the armatures, in their turn, become more strongly charged, or acquire a greater difference of potentials, and that the action already described goes on still more actively, so that, after a few

turns of the plate, the electrified ebonite may be removed and the machine continues to act. As described, the upper half of the plate carries positive electricity from left to right, and the lower half carries negative electricity from right to left, which is electrically the same thing. This transfer, due to the motion of the plate, is compensated by a flow of positive electricity from B to A through the conductor, and of negative electricity from A to B. When the difference of potentials between the armatures has attained a sufficiently high value, the electric flow through the conductor continues, even if the knobs P and N are separated, and it then becomes evident as a stream of sparks. If the knobs are separated too far, the action of the conductor ceases, and there being now no electrical interchange between the combs and the revolving glass-plate as it passes them, negative electricity is carried round to the positive armature B′, and positive electricity to the negative armature A′. The electrification of the armatures is thus lessened, and may be destroyed altogether, or even inverted. When the knobs are quickly put into contact again, if the electrification has only been diminished, or if it has been inverted in sign, the action of the machine recommences, either in the same way as before or in the opposite, as the case may be.

If the jars are connected with the conductors, the flow, instead of passing continuously across the gap *ab*, charges the jars until the potential reaches the value corresponding to the striking distance; a spark passes and the same succession of phenomena is repeated.

When the jars are used, if the distance of the two poles is too great for the spark to pass, it sometimes happens that the action stops, then starts again, the signs of the poles being reversed, and the same series of alternations are reproduced. Each jar is periodically charged and discharged by the corresponding comb. This is specially liable to occur with batteries of large capacity.

Holtz's machine affords a means of making a curious experiment on *reversibility*. If the two combs of a machine in the ordinary state are connected with the poles of a second one, from which the driving band has been removed so as to allow the plate to turn freely, and if the first machine is then set in action, and a slight impulse is given to the plate of the second machine, it continues to rotate; the electricity thus transmits the motion of the first machine to the second.

76. Diametral Conductor.—Various attempts have been made to prevent the machine from ceasing to act or reversing its poles. A useful contrivance for this purpose is the *diametral conductor*

(Fig. 61). Each armature occupies almost the whole of a quadrant, and an insulated conductor, A′B′, with combs at the ends, can be placed between them along any diameter. So long as the two poles are in contact, or when they are separated, so long as the machine continues in normal action, the diametral conductor has no effect, the conditions of its equilibrium between the two armatures being those which have been established by the principal conductor. But if the poles are separated so far that the principal conductor ceases to act, the diametral conductor replaces it, and the machine continues to work.

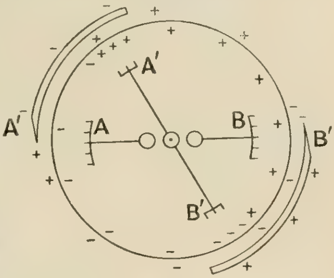


FIG. 61.

77. Voss's Machine.—Holtz's machine has been modified in many ways. Voss's machine (Fig. 62), which is now widely used, is a machine with a diametral conductor, and has some of the features of the Holtz machine and of the replenisher.

The movable plate is provided with metal studs, which act like

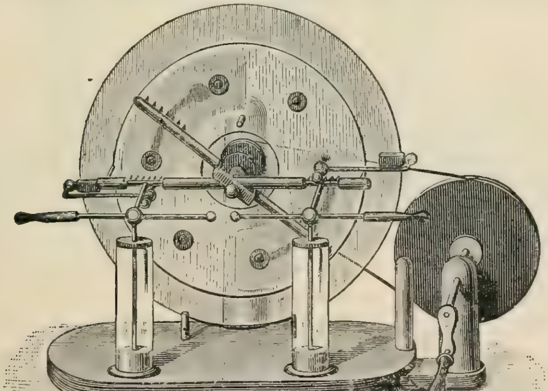


FIG. 62.

the carriers of the replenisher. The influencing conductors are represented by two strips of tinfoil fixed to the centre of the paper armatures. They are connected with two small brushes which touch the studs just before their passing under the combs; these brushes play the same part as the two springs *a* and *b* (Figs. 55,

56). Two other brushes placed in the middle of the comb of the diametral conductor act like the two springs, c and d.

78. Wimshurst's Machine.—This consists of two identical plates of glass which are mounted on horizontal spindles so as to rotate in contrary directions, and are provided on the outside with narrow strips of tinfoil arranged radially at equal distances apart (Fig. 63). Two horseshoe-shaped conductors provided with points enclose the two plates at opposite ends of the horizontal diameter, and are in connection with the two poles, and also with two Leyden jars. Diametral conductors, making an angle of from 60° to 90°

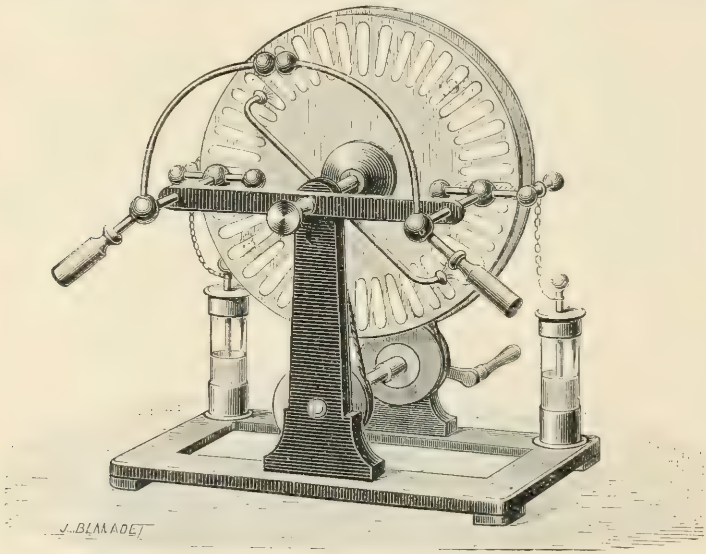


FIG. 63.

with each other, are placed, one opposite each plate; they have small brushes at their ends which graze against the bands of tinfoil.

The direction of rotation of the plates is such that a radius, which at any instant is horizontal, becomes parallel to the corresponding diametral conductor by turning through an acute angle. To explain the action, suppose that, by the approach of an electrified body or otherwise, the potential near one end of one of the diametral conductors,—say at the left-hand upper part of the figure,—is made higher than elsewhere. A tinfoil sector passing through this region is connected by the conductor with the sector

at the opposite end of the same diameter, and the two are, therefore, brought for the moment to the same potential, by the passage of positive electricity through the conductor from the first to the second. The motion of the plates brings these two sectors simultaneously opposite two sectors of the other plate, which are for the moment connected by the second diametral conductor. This second pair of sectors thus assume the same potential, although one is near a negatively charged sector of the first plate, and the other near a positively charged sector. This equalisation of potentials involves a passage of positive electricity through the second conductor from the lower left-hand end to the upper right-hand end. These actions taking place with all the sectors in turn as they pass the diametral conductors, the result is a continuous succession of positively and negatively charged sectors on one plate, arriving simultaneously opposite the two ends of the diametral conductor applied to the other plate. As we have described the action, the sectors arriving opposite the left-hand ends of both conductors would be positively electrified, and those to the right negatively. There would thus be a continuous flow of positive electricity through both conductors from left to right, charging the sectors positively which pass simultaneously through the left-hand collecting comb. Similarly, those which pass simultaneously through the right-hand comb would be negatively charged. The combs take up the electrification of the sectors that pass them, and there is, consequently, a flow of positive electricity from the left-hand comb and discharging knob to the right-hand comb, which compensates the transfer from right to left due to the motion of the plates and the action already described. The action of the machine is such as to increase the initial inequality of potential, and it, therefore, acts more powerfully as the motion goes on.

As a general rule, a machine like that of Holtz, where the electricity has to strike across an interval of air, can only begin to work with an appreciable charge. On the other hand, a machine with direct metallic contact or with a brush may start itself, however small the initial charge. It is very seldom that there is not sufficient difference of potentials between the different parts of a Wimshurst's machine to make it begin work.

79. Yield and Energy of a Machine.—By means of Lane's unit-jar (**88**), it may easily be shown that the yield of influence machines is proportional to the velocity of rotation, independent of the capacity of the conductor, and of the absolute value of the potential of the poles; it does not alter when one of these is connected with the ground; it diminishes as the difference of potentials between

the poles increases. The yield of influence machines is ordinarily much greater than that of frictional machines. With a Lane's jar with a striking distance of 1 mm., the yield for one turn of a plate machine, 98 cm. in diameter, being taken as 1, that of a Holtz machine of 55 cm. was 0·86. But the former made one turn in a second, and the latter ten turns, so that the yields per second were as 1 : 8·6.

The following numbers will give an idea of the absolute value of the yield: seven turns of a double plate Holtz machine charged a battery, the electrostatic capacity of which was 22,500 cm., so as to give a spark of 0·1 cm. in length.

As the capacity of the battery was $\frac{22{,}500}{3^2 \times 10^5} = 0{\cdot}025$ microfarads (1 microfarad being equal to $3^2 \times 10^5$ electrostatic units of capacity), and since a striking distance of 0·1 cm. corresponds to a potential difference of 4830 volts (**95**), we get for the charge of the battery

$$Q = VC = 4830 \times 0{\cdot}025 \times 10^{-6} = 120{\cdot}75 \times 10^{-6} \text{ coulombs,}$$

and for the coresponding work (**39**)

$$W = \tfrac{1}{2} QV = \tfrac{1}{2} \times 4830 \times 120{\cdot}75 \times 10^{-6} = 0{\cdot}742 \text{ joules.}$$

These numbers correspond to seven turns of a machine, which makes ten a second; so that to get the numbers for one second we must multiply by 10 and divide by 7. We thus find that the machine would give per second 0·00017 coulombs of electricity, and 1·06 joules of work.

The *power* of a machine is defined by the work which it does in unit time. The *watt*, which corresponds to one joule per second, has been taken as unit of power. We should accordingly say that the power of the machine was 1·06 watts.

CHAPTER VIII

APPARATUS FOR ELECTROSTATIC MEASUREMENT

80. Coulomb's Balance.—This fundamental apparatus, by which a quantity of electricity may be measured in absolute value, has been previously described. Much skill and many precautions are needed to obtain exact numerical results by means of a torsion-balance, and, as we shall see later, equivalent measurements can be made by the help of more modern instruments, which, although the theory of their action is less simple, are much easier to use. Nevertheless, on account of the importance of the matter, it is worth while to point out how—in principle at least—the total electrification of any electrified body can be deduced from measurements that can be made with a torsion-balance.

Suppose the electrified body to be placed inside an insulated hollow metal sphere (which for this purpose may be made in two halves), then, as we have seen (**24***b*) the outer surface of this sphere will assume an electrification exactly equal to that of the body enclosed within it. If now the fixed ball of the torsion-balance (*a*, Fig. 12) be put in contact with the outside of the sphere, it will share its electrification in a proportion depending on the comparative diameters of the ball and the hollow sphere. This proportion has been calculated so as to satisfy the conditions specified in (**18**), and the result is given in mathematical treatises on electricity: let it be expressed by n, so that if Q is the original electrification of the sphere and q that of the ball when put in contact with it, we have

$$Q = (n + 1)\, q$$

where n can be obtained from the published tables.

For simplicity we will suppose the movable ball b of the torsion-balance to be exactly equal to a, the fixed ball; then when the two come into contact the electrification of the latter will be equally divided between them and each will have electrification

$$\tfrac{1}{2}\, q = \frac{Q}{2(n + 1)} .$$

Applying Coulomb's law, we get

$$fr^2 = \frac{1}{K} \cdot \frac{Q^2}{4(n+1)^2}$$

and, by the formula in the footnote to **(14)**

$$\frac{1}{K} \cdot \frac{Q^2}{4(n+1)^2} = 4\ Cl\ (A + \alpha) \sin\frac{\alpha}{2} \tan\frac{\alpha}{2},$$

or

$$Q = 4(n+1)\sqrt{KCl\ (A+\alpha) \sin\frac{\alpha}{2} \tan\frac{\alpha}{2}}\ .$$

The factors C and l in this formula depend on the construction of the torsion-balance, and can be determined once for all, n depends on the relative diameters of the ball and hollow sphere, and A and α are angles observed during the experiment. Hence it may be said that Q is expressed in terms of known quantities and the factor K.

It is to be remembered that C in the above formula stands for a couple per unit angle **(14)**, or the product of a force into an arm divided by an angle. If the force is expressed in dynes, the arm and the length l in centimetres, the angle involved in the value of C in the same unit (degrees or radians) as that in which the angles A and α are expressed, and if K is taken $= 1$, the value of Q, obtained by putting their proper numerical values for the quantities on the right-hand side of the above formula, is said to be expressed in *absolute centimetre-gramme-second* (or C.G.S.) *electrostatic measure.*

81. Gold-Leaf Electroscope.—This instrument has already been described **(11)** as well as the manner in which it is used. A few words may here be added to complete the theory.

In order that the indications of the instrument may be definite, the cage which contains the leaves should be at a known potential—that of the earth, for instance. It is advisable to make the bell-jar a conductor by lining the interior to within a convenient distance from the stem with strips of tinfoil, or with brass gauze of somewhat large mesh. This as well as the metal rods is in connection with the ground. An excellent earth contact is obtained by connecting the apparatus with gas or water pipes.

The charge of the leaves is proportional to their capacity and to the difference of potentials $V - V_o$ **(51)**, where V is the potential of the leaves and V_0 that of the cage. The divergence, which depends on the charge, serves to measure either the charge or the potential. If the capacity were independent of the divergence, the potential and the charge would vary proportionally to each other, and the same graduation would serve for both cases.

82. Condensing Electroscope.—We have sometimes to deal with electrification which is very great as to quantity, but which is so distributed as not to produce a difference of potentials sufficient to cause a visible divergence of the leaves of an electroscope employed as hitherto described. In such cases it is advantageous to use a *Volta's condenser* (Fig. 64) in connection with the electroscope. This consists of two metal plates varnished on the surfaces facing each other, the two layers of varnish forming an insulating plate.

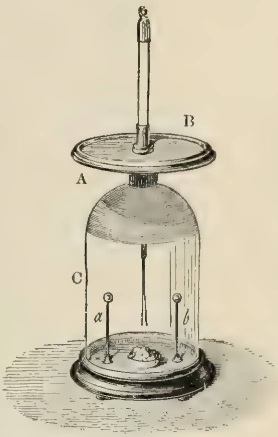


FIG. 64.

The two boundaries of the electrified field to be tested are connected, one with one of the plates of the condenser, and the other with the other. If, as is the commonest case, one of the boundaries consists of an insulated conductor and the other of the surface of the room, the former is connected, say, with the lower plate, while the upper plate is connected with the room either through the body of the experimenter, who touches it with his finger, or, still better, by a wire. The connections with both plates are removed after a moment or two, and then the upper plate is raised by the insulating handle. The capacity of the Volta's condenser, which is very great when the plates are together, is thus greatly diminished, and the potential of the lower plate and leaves, which was previously nearly the same as that of the room, now differs from it enough to make the gold leaves diverge.

The phenomenon of electrical absorption (**62**) explains why it is necessary that the insulating layer should consist of two parts: it is necessary that the upper plate should remove, with the adjacent half of the insulating layer, the charge which the latter may have absorbed.

83. Quadrant Electrometer.—This instrument (Fig. 65), which was invented by Lord Kelvin, serves for the same purpose as the gold-leaf electroscope, but is far more sensitive and is better adapted for accurate quantitative measurements.

It consists of two pairs of quadrants AA′, BB′ (Fig. 66), which together form a flat cylindrical metal box cut into four equal sectors by two diametral sections at right angles to each other. Each quadrant is supported by an insulating glass stem, and the alternate pairs of quadrants, AA′ and BB′, are in conducting communication with each other. Each pair is connected also with a small insulated

rod, which is called the *electrode*, and which serves to connect up with bodies on the outside.

In the middle of the quadrants is suspended what is called the

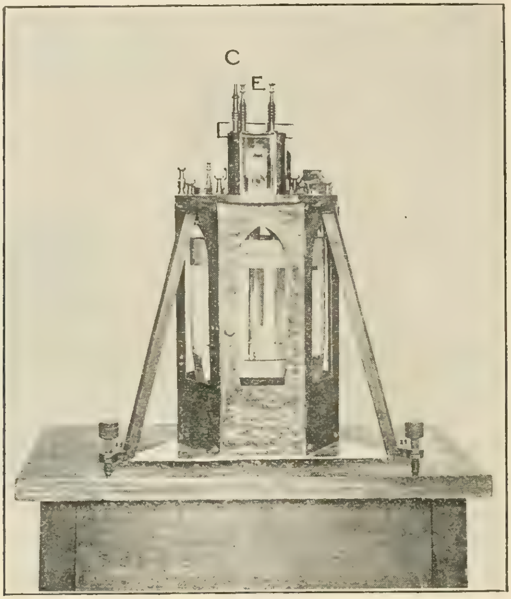


FIG. 65.

needle, which has the shape represented in the figure, and consists essentially of two quadrants of thin sheet aluminium joined at the centre, parts being cut out, as shown in the figure, for still greater lightness.

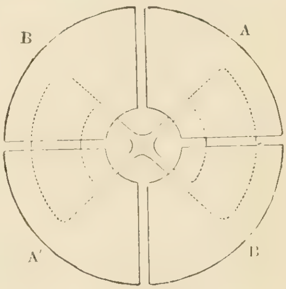


FIG. 66.

The needle usually hangs by two cocoon fibres, which constitute what is called a *bifilar suspension.*

The position of mechanical equilibrium of the needle is that in which the two fibres are in the same plane. For a small deflection the force which tends to bring the needle back to its position of equilibrium is proportional to the angle of deflection. The suspension must be adjusted so that the position of equilibrium of the needle is symmetrical in reference to the space which separates the quadrants.

We may regard the electrometer as consisting of two condensers the inner coatings of which are portions of the needle at potential V_o, while the outer are the two pairs of quadrants AA′ and BB′ at potentials V_1 and V_2 respectively, while their charges are Q_1 and Q_2; we shall assume that $V_o > V_1$ and $V_1 > V_2$. The capacities of these condensers depend upon the position of the needle, and may each be expressed as the sum of two terms, one of which, representing the capacity of that part near a radial edge either of the needle or the quadrant, is constant for all usual positions of the needle, while the other represents the capacity of the portion which is distant from any such edge, and increases if the needle moves further into the quadrant by an amount proportional to the angular movement, say, by an amount $cd\theta$. In order to calculate the couple acting upon the needle in any position we shall suppose a small displacement $d\theta$ to take place about this position from A toward B′. If we keep the quadrants insulated during this displacement (so that no energy may be gained from external sources) we may equate the work done by the electrical couple to the decrease of energy of the system. Let C_1 and C_2 be the capacities of the quadrants in the first position; in the second position they will be $C_1 - cd\theta$ and $C_2 + cd\theta$. Recalling the general expression for energy, viz.:— $\frac{1}{2}\frac{Q^2}{C}$ (**64**) we may write

$$\text{Initial energy} = \tfrac{1}{2}\frac{Q_1^2}{C_1} + \tfrac{1}{2}\frac{Q_2^2}{C_2}$$

$$\text{Final energy} = \tfrac{1}{2}\frac{Q_1^2}{C_1 - cd\theta} + \tfrac{1}{2}\frac{Q_2^2}{C_2 + cd\theta}$$

Hence, since $d\theta$ is supposed to be very small, the decrease of energy equals

$$\tfrac{1}{2}\left\{-\frac{Q_1^2 cd\theta}{C_1^2} + \frac{Q_2^2 cd\theta}{C_2^2}\right\}$$

But $$\frac{Q_1}{C_1} = (V_o - V_1) \text{ and } \frac{Q_2}{C_2} = (V_o - V_2);$$

therefore the decrease of energy equals

$$\tfrac{1}{2}cd\theta\,\{(V_o - V_2)^2 - (V_o - V_1)^2\},$$

or $$cd\theta\,(V_1 - V_2)\left\{V_o - \frac{V_1 + V_2}{2}\right\},$$

and this must equal the work done by the electrical couple during the displacement, that is, $Gd\theta$ if G is this couple.

The position of equilibrium in any case will be that for which the electrical couple and the mechanical couple due to the bifilar suspension are equal and opposite. The latter is very nearly proportional to $\sin\theta$ for moderate angles; or, as stated above, for small angles it may even be taken as proportional to θ the angle of deflection itself. Hence we have

$$\theta = A\ (V_1 - V_2)\left\{V_o - \frac{V_1 + V_2}{2}\right\}$$

where A is a constant depending upon the construction of the instrument.

As the electrometer is commonly used, the potential V_o of the needle is positive and very much higher than that of either pair of quadrants. Consequently, the part of the above formula enclosed within long brackets does not differ much from V_o, and we may write, approximately, in ordinary cases

$$\theta = A\ (V_1 - V_2)\ V_o.$$

Again, this simple formula will be rigorous if the arithmetic mean of the potentials of the quadrants is zero, that is, if they are equal but of opposite signs.

In order to keep the potential of the needle as nearly constant as possible, it is connected with the inner surface of a Leyden jar, generally formed by a glass vessel forming part of the outer case of the electrometer. In the more complete forms of instrument, there is a small electrical machine (called a *replenisher*) whereby the potential of the jar and needle can be brought to any required value; and also a subsidiary electrometer (the *gauge*) which shows when the right value is attained. These two organs are referred to in more detail in **73** and **87**.

Any two conductors, the difference of whose potentials is to be measured, are connected one to each of the electrodes belonging to the respective pairs of quadrants and the resulting deflection θ is observed (see **84**). The required difference of potentials would then be given by

$$V_1 - V_2 = \theta/(A\,V_o)$$

if V_o and the factor A were known for the electrometer used. But as these hardly admit of being accurately determined directly, the usual plan is to observe the deflection, say θ_{ab}, corresponding to some known difference of potentials $V_a - V_b$, and then to eliminate A and V_o from two equations similar to the last. We thus get the required difference of potentials in the form

$$V_1 - V_2 = (V_a - V_b)\ \theta/\theta_{ab}.$$

If it is required to determine the amount by which the potential of an insulated conductor differs from that of the earth, the electrode of one pair of quadrants is connected with the room and the other with the conductor to be tested.

A different mode of employing the quadrant electrometer is often adopted by Continental experimenters. The two pairs of quadrants are maintained at equal opposite potentials (so that their mean potential is zero) by connecting them respectively with the opposite terminals of a large number of small galvanic cells connected in series, the middle of the series being put to earth. Thus arranged, the electrometer is adapted for indicating the difference of potentials between an insulated conductor and the earth. The conductor is connected with the needle, and its potential, reckoned from that of the earth (or the room) taken as zero, is given by

$$V = \frac{\theta}{A(V_1 - V_2)}.$$

If a difference of potentials between two insulated conductors is to be measured, each is connected separately with the needle and the corresponding values of θ are observed. Their difference gives the required difference of potentials, thus

$$V - V' = \frac{\theta - \theta'}{A(V_1 - V_2)}.$$

As before, the factor A can be eliminated by observing the effect due to some known difference of potentials.

Another way of using the electrometer is to connect the needle with one pair of quadrants, and with one of the two bodies whose difference of potentials is required, while the second of these is connected with the second pair of quadrants. In this case, $V_1 = V_o$, and the deflection becomes

$$\theta = \tfrac{1}{2} A (V_o - V_2)^2$$

and is thus always in the same direction, whether the difference $V_o - V_2$ is positive or negative. This circumstance makes this method of employing the quadrant-electrometer of great value when, as sometimes happens, a rapidly alternating difference of potentials has to be measured.

84. Reading by Scale and Mirror.—The deflections are measured by the method of reflection. A small concave mirror (Fig. 65) is supported by the rod of the needle, and is movable with it. A vertical slit of light and a scale divided into milli-

metres (Fig. 67) are placed in the vertical plane passing through the centre of curvature and perpendicular to the axis of the mirror, at equal distances above and below the centre of curvature. The

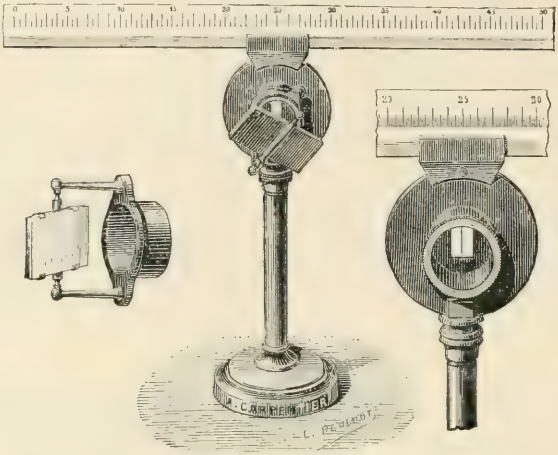


FIG. 67.

image of the slit of light is produced in its real size on the scale. Instead of a narrow slit, a wider one may be used, in the middle of which an opaque wire is stretched vertically (Fig. 67). The image of the opening moves along the scale, and, as the region where the shadow of the wire falls is bright, the divisions are easily read off. Very good results are also obtained with a scale on ground glass or celluloid, which is read from behind.

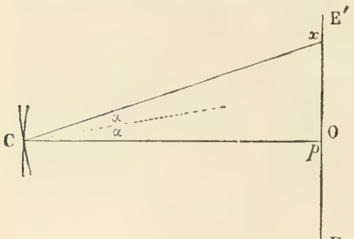


FIG. 68.

Let p (Fig. 68) be the division of the scale which corresponds to the position of equilibrium of the needle; if the needle is deflected through an angle α, the image is formed on the division x, such that

$$x - p = d \tan 2\alpha$$

d being the distance of the scale from the mirror. As the angles are always very small, we may take the arc for the tangent and replace $\tan 2\alpha$ by 2α, which gives

$$\alpha = \frac{x-p}{2d}.$$

85. Difference of Potentials between a point in the Air and the Earth.—If an insulated conductor is placed in an electric field, the surface of the conductor is everywhere at the same potential, and becomes continuous with some one equipotential surface of the field. Those parts of the surface of the conductor which lie to one side of this equipotential surface extend into regions where the undisturbed potential of the field would be higher than that of the conductor, and the other parts of the surface extend into regions where the potential would be lower. The electrification assumed by the conductor is such as to keep the potential of the first-mentioned parts down, and to bring that of the remaining parts up to the actual uniform potential possessed by the conductor as a whole. That is to say, the parts of the surface of the conductor which extend into the region of higher potential acquire a negative charge, and those parts which extend into the region of lower potential acquire a positive charge. If now a portion of the surface of the conductor were to become detached, say from the part which is negatively electrified, carrying its electrification with it, the potential of the whole remaining part of the conductor would rise; and if the part that has been detached were in some way to grow again, a conductor of the original form would be reproduced, but it would have a somewhat higher potential than before. Now, imagine this process of successive separation and reproduction of a portion of the surface of the conductor to go on, over and over again, always at the same part: it is obvious that the potential of the conductor as a whole will go on rising (if the portions that become detached are negatively electrified, or, in the opposite case, that it will go on falling) until the density of the surface-charge of the detached portions becomes zero, when there will be no further change. The whole conductor will then have been brought to the potential of the equipotential surface passing through the part from which the successive portions break away.

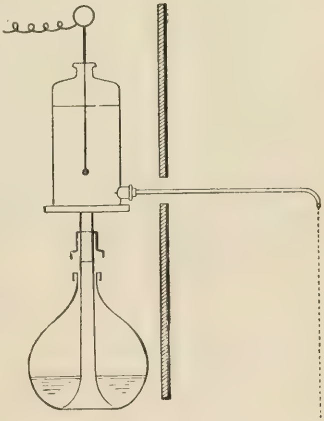


FIG. 69.

To apply this principle to the experimental determination of the potential at a point in the air, a vessel of water is placed on an insulating stand (Fig. 69), and provided with a long, very narrow

tube, from the end of which the water escapes in a fine stream. The whole vessel is thus gradually brought to the potential of the point at which the stream of water breaks into separate drops. To observe this, one pair of quadrants is connected with the water-vessel, and the other pair with the earth. The flame of a spirit-lamp connected by a fine wire with the electrometer acts in the same way as a water-jet, or a piece of touch-paper prepared by soaking paper with a solution of nitrate of lead may be used.

86. Absolute Electrometer.—This is based on the attraction which is exerted between a movable plate and an infinite plane parallel to each other and at different potentials (Fig. 70).

Let V_1 be the potential of the movable plate, V that of the plane, and e their distance apart. Neglecting perturbations arising from the edges, the lines of force are straight, perpendicular to the planes. In other words, the tubes of force extending between the opposed surfaces have the same cross-section from end to end, and the intensity of force is therefore uniform throughout and equal to $4\pi\sigma/K$, where σ is the equal surface-density at each end of the tubes, positive at one end and negative at the other (**43, 44**). But the electrification at each end of the tube contributes equally to the force within it, hence the force due to each end is $2\pi\sigma/K$, and this represents the pull exerted by one plate on each unit of electricity on the other (for the mutual forces exerted by the various portions of the charge on either plate cannot have a resultant tending to move that plate). The pull is, therefore, $2\pi\sigma^2/K$ per unit of surface, or $F=2\pi\sigma^2 A/K$ for the whole surface A.

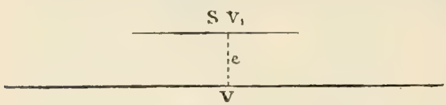


Fig. 70.

Since, further (**52**),

$$\frac{\sigma}{K}=\frac{V_1-V}{4\pi e},$$

we have (comp. **68**.)

$$F=\frac{2\pi\sigma^2 A}{K}=\frac{AK}{8\pi e^2}(V_1-V)^2.$$

In order to avoid the disturbing effect of the edges, the movable plate S is cut out of a larger plate, AB, which is called the *guard ring* (Fig. 71). The plate S can move freely in this, and is in metallic connection with it, and therefore at the same potential. The plate AB is the bottom of a box ABCD, which forms a closed conductor and prevents any action which might be exerted from the outside on the upper face of the plate.

The plate, S, is supported by a spring which keeps it when not electrified a little above the plane of the ring. The experiment consists in raising the lower plate A (Fig. 72), which represents the infinite plane and is at the potential V, until the plate S, as observed by help of the lenses l and l', appears exactly in the plane of the guard ring. On the other hand, it is easy to determine the mass P, whose weight causes the plate S to sink through the same distance. The attraction is, then, equal to Pg dynes.

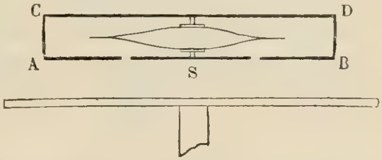


FIG. 71.

We thus have $F = Pg$, and therefore the difference of potentials becomes

$$V_1 - V = e\sqrt{\frac{8\pi Pg}{AK}}.$$

It is somewhat difficult to measure accurately the distance e of the two plates. This difficulty is got rid of by the following device: The lower plate is put to earth so that its potential $V = 0$, and the reading of the micrometer m, which satisfies the condition above stated, is noted: suppose the corresponding distance between the surfaces is e'. The plate is then put in connection with the body at the potential V, and the distance is again adjusted: if it is now e, we have

$$V_1 = e'\sqrt{\frac{8\pi Pg}{AK}}$$

$$V_1 - V = e\sqrt{\frac{8\pi Pg}{AK}},$$

from which by subtraction

$$V = (e' - e)\sqrt{\frac{8\pi Pg}{AK}}.$$

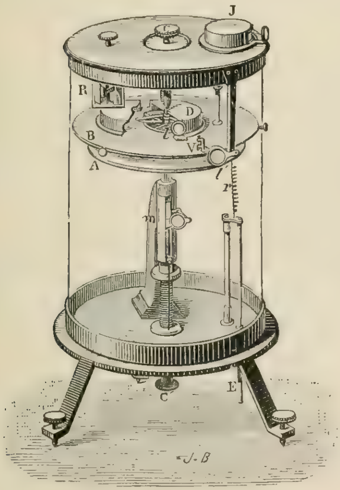


FIG. 72.

It is thus sufficient to measure the difference $e' - e$ of the two positions of the plate A.

87. In order to keep this plate at a constant potential, V_1, the whole system consisting of the plate, the guard ring, and the box is connected with the inner coating of a Leyden jar, which is

formed by the glass case of the instrument. One of two accessory pieces of apparatus serves to maintain, and the other to verify, the constancy of the potential, V_1. The first, R, which is called the *replenisher* (**73**), increases the charge of the jar when the knob is turned in one direction, and lessens it when it is turned in the contrary direction. The second, which is called the *gauge*, J, makes it possible to recognise when the potential has acquired a given value. This is an apparatus which is based on the same principle as the electrometer itself, and consists of a movable plate of aluminium, p (Fig. 73), with its guard ring, and a plate which attracts it, the whole being contained in the box, J (Fig. 72). Both the plate p and the guard ring are connected with the outer coating of the Leyden jar; the lower plate is at a fixed distance from the plate p, and is in connection with the inner coating of the Leyden jar, and thus the plate p always comes back to the plane of the guard ring for the same value of the potential difference.

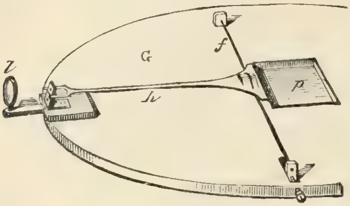


FIG. 73.

The plate p is attached to a stretched platinum wire, f, which it twists slightly in coming into the plane of the guard ring; a projecting tongue, h, carries at its end a fine wire which can be observed by means of the lens l, and the plate p is in the plane of the ring when this wire appears exactly mid-way between two marks on a white card.

A modified form of quadrant electrometer has been introduced by Dolezalek, and this has practically replaced the older forms. It is of the ordinary quadrant type, the quadrants being mounted on short insulating pillars. The "needle" is of thin silvered paper, and in consequence of its lightness any motion given to it rapidly dies out. It is suspended by a very fine quartz fibre; in the earlier forms this was made conducting (to enable the needle to be charged) by dipping it into calcium chloride solution; in the later forms a metal rod can be brought into contact with the central metal frame of the needle, and contact is made through this. The electric capacity of the instrument (*i.e.* the ratio of the charge on the positive quadrants to the difference of potential between the terminals) is about 50 electrostatic units. The sensitiveness passes through a maximum when the needle is charged to about 100 volts, and a deflection of 10,000 millimetre divisions per volt of the spot of light on the scale can be then readily obtained.

88. Lane's Unit Jar.—This apparatus is used to measure comparatively large quantities of electricity, such as the charge of a Leyden battery. It consists of a Leyden jar (Fig. 74), the outer coating of which is in conducting communication with a knob, B, which by means of a micrometric screw, F, may be set at any measured distance from the knob A, connected with the internal coating. If the knob A is connected with the source of electricity,

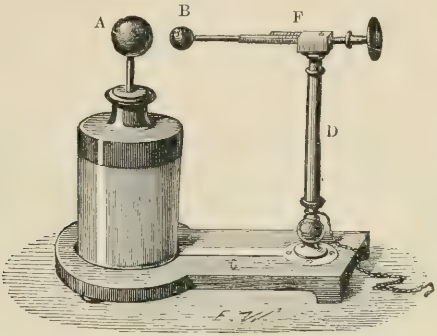


Fig. 74.

a spark passes between the two knobs whenever the difference of potential between them reaches a value depending on their distance apart. For the same distance between the knobs, every spark that passes represents the same quantity of electricity.

In order to measure the charge of a Leyden battery, the unit jar may be insulated and interposed between one terminal of the machine and the inner coating of the battery; or the battery may be insulated and the unit jar interposed between the outer coating of the battery and the second terminal of the machine. In either case the number of sparks serves as a comparative measure of the quantity of electricity given to the battery.

To follow the action in fuller detail: suppose the whole arrangement shown in Fig. 74 to be insulated, let the knob A be connected with the positive conductor of the charging electrical machine, while the knob B is connected, through the metal rod D, with the inner coating of the battery to be charged, the outer coating of the battery being connected to the negative conductor. Then, on working the machine, while the inner surface of the unit-jar receives a positive charge, an equal quantity leaves the outer surface and passes to the inner surface of the battery. This goes on until the difference of potentials between the knobs A and B has attained the value, say v, required to cause a spark to pass

between them. At the same time a difference of potentials $= vc/C$ is established between the coatings of the battery, if c is the capacity of the unit-jar and C that of the battery. The effect of the spark is to equalise the potentials of the coatings of the unit-jar; therefore, to cause a second spark between A and B, the same quantity of electricity as before must pass from the machine to the inner coating of the unit-jar and also from the outer coating of the jar to the inner coating of the battery. The battery will thereby be charged to potential $2vc/C$. Similarly, the passage of n sparks will indicate that the battery has received a total charge $Q = n.vc$ and has been charged to potential $V = n.vc/C$. The charge will reach its limit when the potential of the inner coating of the battery, and therefore also that of the outer coating of the unit-jar, has reached such a value that the machine employed is unable to cause the potential of the inner coating of the jar to exceed it by the amount v.

88.* Measurement of Specific Inductive Capacity.—Nearly every phenomenon in which the properties of a dielectric medium are concerned depends, as to its quantitative aspects, on the value of the specific inductive capacity of the medium. Hence every such phenomenon which admits of being accurately measured may be made, in principle, to furnish the basis of a method for determining the numerical value of this property for the medium concerned. We shall indicate here the general principles of two of the simplest and most direct methods that are available for this purpose.

1. *Condenser Method.*—As has been pointed out (**51**), the capacity of a condenser depends jointly on a factor determined by the geometrical relations of its conducting surfaces and on the specific inductive capacity of the medium between them. If this medium is in one case air, and if in another case the air is entirely displaced by another dielectric, the ratio of the capacity of the condenser in the second case to its capacity in the first gives at once the specific inductive capacity of the dielectric referred to, that of air taken as unity. But this method, in the form indicated, is applicable only in the case of liquids or gases, for it is not practicable to displace the air between the plates of a condenser accurately and completely by a solid dielectric. By a modification, however, the same general principle can be applied in the case of solid bodies.

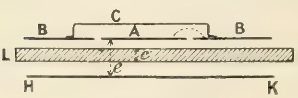


FIG. 74A.

Let A, in Fig. 74A, be a metal disk surrounded by a guard-plate B, the lower surfaces of the disk and guard-plate being exactly in the same plane, and suppose the two so arranged that they can be connected or disconnected at will, and let C be a metal cover, electrically continuous with the guard-plate. Further, let HK represent a second metal plate, parallel to the disk and guard-plate, at distance e. This plate and the disk taken together form a condenser of which the capacity is very accurately represented by the formula

$$C = \frac{AK}{4\pi e}$$

if A is the area of the disk and K the specific inductive capacity of air.

Now, let a plate LM, formed of a solid dielectric material, whose specific inductive capacity K' is to be compared with that of air, be introduced between the two metal surfaces and parallel to them, care being taken that it does not come in contact with either of them either when in its place or while being put in. The effect of this is to alter the capacity of the condenser, and the ratio of the specific inductive capacity of the solid dielectric to that of air can be deduced from the ratio of the capacities of the condenser in the two states.

If e' is the thickness of the solid dielectric, the thickness of the remaining layers of air is $e - e'$, and if f is the intensity of electric force in the air and f' that in the solid, the difference of potentials between the plates of the condenser is

$$V - V' = f(e - e') + f'e'.$$

But, putting σ for the surface-density of the charge of the positive plate and Q for the charge, we have—

$$f = \frac{4\pi\sigma}{K}, \quad \frac{f'}{f} = \frac{K}{K'}, \text{ and } \quad Q = A\sigma.$$

Putting in these values, the capacity of the condenser becomes

$$C' = \frac{Q}{V - V'} = \frac{AK}{4\pi\left[e - e'\left(1 - \frac{K}{K'}\right)\right]}$$

Comparing this with the capacity of the same condenser with air only between the plates, we have

$$\frac{C'}{C} = \frac{e}{e - e'\left(1 - \frac{K}{K'}\right)}$$

whence, for the ratio of the two inductive capacities—

$$\frac{K'}{K}=\frac{1}{1-\left(1-\frac{C}{C'}\right)\frac{e}{e'}}.$$

2. *Electrometer Method.*—If the disk in Fig. 74A is in permanent electrical connection with the guard-ring and is suspended from the arm of a balance, or from a spring, whereby the force in dynes can be ascertained which is required to counterbalance the attraction between it and the lower plate, when a given difference of potentials is established between them and the lower surface of the disk is exactly in the same plane as the lower face of the guard-ring, the whole arrangement becomes an absolute electrometer (**86**).

If m is the mass whose weight, mg, balances the pull between the disk and the lower plate, with air only between them, when they are electrified to a difference of potentials $V-V'$, we have (**68**)—

$$mg=(V-V')^2\frac{AK}{8\pi e^2}=f^2\frac{AK}{8\pi}$$

since $V-V'=fe$.

If now part of the air is replaced by a slab of a solid dielectric of thickness e', the electric intensity in the air, for the same difference of potentials between the plates, is altered: let it be f_1 and put m_1 for the corresponding value of m. The difference of potentials is now

$$V-V'=f_1(e-e')+f_1'e'=f_1\left(e-e'+\frac{K}{K'}e'\right),$$

and the pull on the disk becomes

$$m_1g=f_1^2\frac{AK}{8\pi}.$$

Hence $m_1/m=f_1^2/f^2$ and, equating the two values of $V-V'$, we get

$$\frac{f_1}{f}=\frac{e}{e-e'\left(1-\frac{K}{K'}\right)}=\sqrt{\frac{m_1}{m}}.$$

From this we get, for the ratio of the dielectric coefficients,

$$\frac{K'}{K}=\frac{1}{1-\frac{e}{e'}\left(1-\sqrt{\frac{m}{m_1}}\right)}.$$

To obtain accurate results, either by this method or by the con-

denser method, it is essential that the solid dielectric examined should be quite free from superficial charge. One of the most effectual ways of ensuring this condition is to pass the flame of a Bunsen lamp over all parts of the surface in succession. For the same reason, the dielectric slab should be supported between the metal plates without coming into actual contact with either of them.

CHAPTER IX

EFFECTS OF THE DISCHARGE

89. Conductive Discharge and Disruptive Discharge.—The energy which is accumulated in an electric field is expended during the discharge in various forms, which will now be briefly discussed.

When the boundaries of the field are put in conducting communication with each other, a spark is always produced. The experiment may be made in such a manner that this spark absorbs either the greater part of the available energy, or, on the contrary, only a very small portion. In the latter case the work of the discharge is chiefly expended in the conductors, and the discharge is said to be *conductive;* in the former case it is said to be *disruptive.*

90. Resistance of Conductors.—Whatever may be the nature of the conductors by the aid of which the discharge is effected, they always offer a certain *resistance* to the motion of electricity, and a greater or less part of the work available is expended in overcoming this resistance, and in so doing it produces an equivalent quantity of heat.

The quantity of heat disengaged in any portion of the conducting wire is readily measured, at any rate comparatively, by means of Riess's thermometer (Fig. 75). The portion, S, of the conductor to be investigated is placed between binding screws, A and B, in a glass globe to which is connected a slightly inclined capillary tube terminating in a much wider vertical tube, E. The whole rests on a hinged board, which can be fixed at any desired angle by the screw K. The capillary tube contains a coloured liquid, and when the discharge is passed, the increase of pressure produced by the heating of the gas is manifested by a displacement of the top of the capillary column. The heating takes place so rapidly that there is not time for appreciable loss of heat to occur: and as the volume of gas does not sensibly alter, the displacement is proportional to the change of pressure, which in turn is proportional

to the rise of temperature, and therefore to the quantity of heat disengaged in the wire s.

In order to compare two conductors, they are both placed simultaneously in the circuit, each in turn, in the bulb of the thermometer, and the other outside, and the same quantity of electricity is discharged through the circuit. Experiment shows that the quantity of heat liberated, and consequently also the resistance, is, for a given metal, directly proportional to the length of the wire, and inversely proportional to its area of cross-section. It varies, moreover, with the material of the wire. In this way the resistance of the metals may be determined in comparison with any one taken as a standard. If ρ is the *specific resistance*

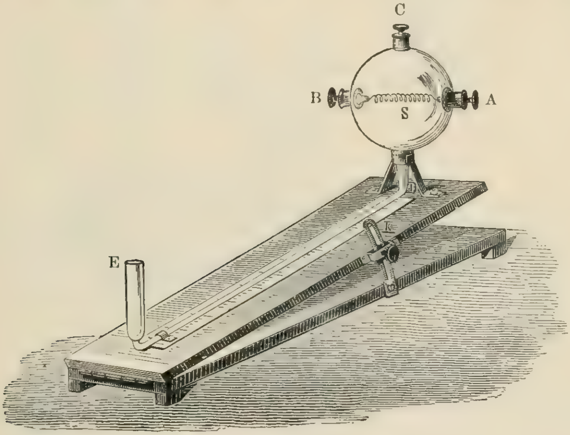


FIG. 75.

of a metal—that is to say, the resistance expressed in any units which it offers for unit length and unit section—the resistance, R, of a conductor of this metal, of length l, and section s, in terms of the same unit, will be

$$R = \rho \frac{l}{s}.$$

We shall hereafter become acquainted with methods more convenient than that of Riess's thermometer for determining the resistance of a conductor, and of obtaining it not merely in relative, but in absolute value. The practical unit of resistance is the *ohm*. This will afterwards be defined theoretically; for the present it will be sufficient to know that it is the resistance presented at 0° C. by a column of mercury a square millimetre in cross-section and 106·3 centimetres in length.

91. Fusion and Volatilisation of Metals.—In the conductive discharge all the energy is converted into heat in the conductor itself, provided no external work is done. We have already seen **(39)** that if we express quantity of electricity in coulombs, and difference of potentials in volts, electric energy is expressed in joules. On the other hand, a *gramme-degree*,—that is, the quantity of heat necessary to raise the temperature of one gramme of water at 0° through *one* degree,—is equal to 4·18 joules. This number is called the *mechanical equivalent of heat*, and is de-

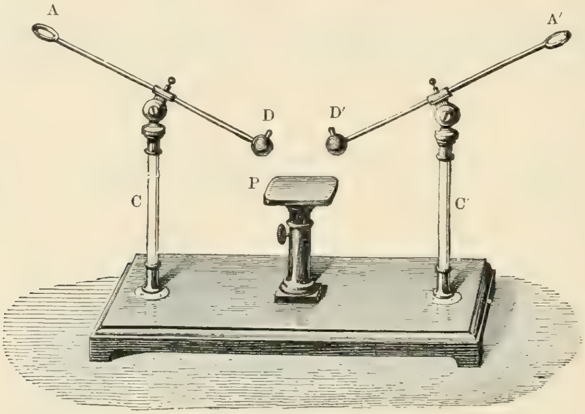


FIG. 76.

noted by the letter J. If H is the number of gramme-degrees disengaged, we have

$$JH = W = \tfrac{1}{2}\,Q\,(V - V') = \tfrac{1}{2}\,C\,(V - V')^2 = \tfrac{1}{2}\,\frac{Q^2}{C}.$$

The quantity of heat corresponding to a given discharge is a fixed quantity; it is independent of the conditions in which the discharge takes place, and in particular of the nature of the conductor. If this latter is made up of two parts, R and R', experiment shows that the quantity of heat divides between them in proportion to their respective resistances. Accordingly, by taking very thick conductors of very small resistance, and interposing between them a wire of great resistance, we may concentrate in the latter almost the whole of the heat produced by the discharge. Thus with an apparatus called the *universal discharger* (Fig. 76), we may raise the temperature of a fine wire of any material stretched between the two knobs D and D′ sufficiently to melt it, or even to volatilise it.

If H is the quantity of heat available, m the mass of the metal, d its density, and c its mean specific heat, the temperature, t, to which it will be raised from 0° by the quantity of heat, H, is

$$t = \frac{H}{mc} = \frac{H}{sldc}$$

if we disregard any losses by radiation.

Hence, for a given metal, the temperature depends only on the mass of metal; it is higher as the mass is less—that is to say, as the wire is finer and shorter. For two wires of the same mass but of different metals, it is inversely as the specific heat; lastly, for two wires of different metals, of the same section and the same length, it is inversely as the product of the density into the specific heat—that is to say, the thermal capacity of unit volume.

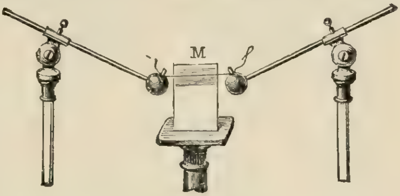


FIG. 77.

If the wire is to be fused by a given quantity of heat, H, the mass of the wire may be determined by the equation

$$m\,(cT + \lambda) = H$$

in which T is the melting point and λ the heat of fusion of the metal.

All metals may thus be either melted or volatilised; iron is suddenly reduced to droplets, which burn in the air, throwing out sparks, and producing oxide of iron. Silk threads interwoven with fine gold, silver, or copper wire, are very well suited for these experiments. A quite moderate discharge is sufficient to volatilise them with explosive violence. The metal disappears completely in fumes, which rapidly settle. They are easily collected on a card placed against the wire (Fig. 77), leaving a deposit on it which is characteristic of the metal, both in shape and in colour. The silk usually remains unchanged. The explosion which accompanies the volatilisation of the metal is an indication that the air is violently disturbed. If the experiment is made by immersing the ends of the rods between which the wire is stretched in a vessel of water (Fig. 78), the disturbance which takes place when the discharge is passed, being transmitted

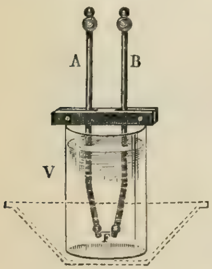


FIG. 78.

through an incompressible liquid, smashes the glass with a great noise. The effect resembles that of a torpedo.

In what is known as *Franklin's experiment* (Fig. 79), a piece of gold-leaf, FF′, is applied against a piece of paper, B, in which a portrait of Franklin is cut out; on the other side is placed a white silk ribbon, C, and the whole is strongly clamped between two slips of wood, P. When a discharge is passed, the gold is volatilised, its vapours pass through the parts cut out, and reproduce the design in a violet-brown colour on the ribbon.

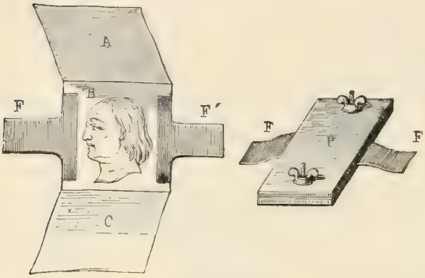


FIG. 79.

92. Passage of the discharge through Bad Conductors.—If a bad conductor is interposed in the circuit, the greater part of the energy is expended in the mechanical operations of tearing

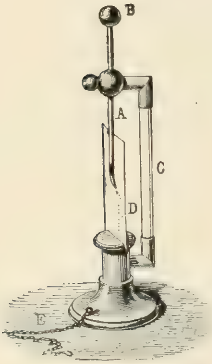


FIG. 80.

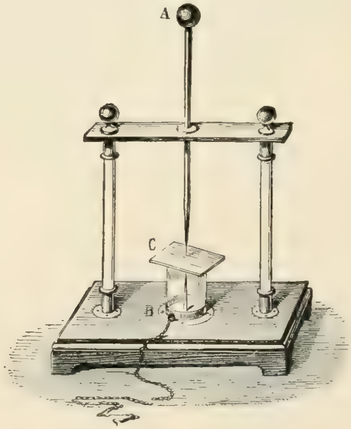


FIG. 81.

and of rupture. A sheet of paper, a card, a thin sheet of glass are easily perforated even by rather weak discharges.

In the experiment with the card represented in Fig. 80, when the discharge is passed, the perforation produced exhibits a kind of welt round the edges, and if the two points are not at the same height, the hole is nearer the negative point.

Electricity passes less easily through glass. The experiment may be made by means of the apparatus represented in Fig. 81. The greatest difficulty in this experiment is to prevent the spark from travelling over the surface of the glass plate. This should be well warmed, and a drop of oil placed round the upper point. With powerful batteries glass plates of some centimetres in thickness may be perforated; in this case the two points should be completely imbedded in a bad conductor, such as resin or wax, or, still better, a mixture of the two.

93. Passage of the discharge over the surface of Bad Conductors.—The property which the discharge possesses of travelling over the surface of bad conductors gives rise to remarkable luminous phenomena. When a battery is so strongly charged that it is on the point of discharging spontaneously between the two coatings, a crackling sound is heard, and at the same time numerous lines of light spread over the uncoated surface of the jars.

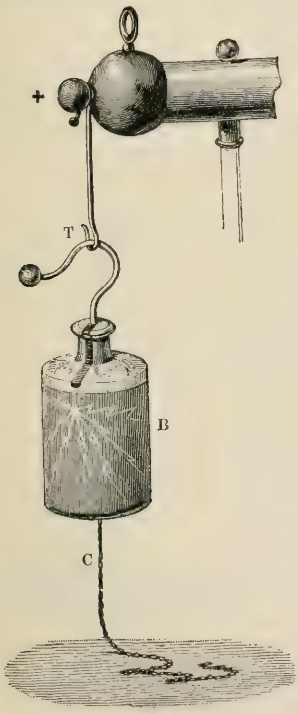


Fig. 82.

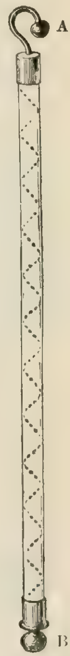


Fig. 83.

Very beautiful effects are obtained by covering a glass plate with tinfoil on one side only. One of the limbs of the discharger is placed in contact with the tinfoil, and the other at a small distance in front of the uncoated surface. Each discharge gives rise to very beautiful ramifications.

The discharges which travel thus over the surface of the glass leave a permanent trace, which can be made visible by breathing on the surface so as to deposit a little moisture.

The propagation of the spark along the surface of badly conducting bodies is facilitated by portions of conducting material dispersed over them. Very beautiful effects are thus obtained with what is known as the *luminous jar* (Fig. 82); this is a

Leyden jar, the outer coating of which, instead of being continuous, consists of metal filings fastened on with varnish. The inner coating is prolonged by a tongue of tinfoil to within a short distance from the outer coating. Every spark which passes between the two armatures branches out over the latter.

With these phenomena may be associated the luminous tubes and panes (Fig. 83), obtained by fastening on the surface of glass a series of small lozenges of tinfoil, between which sparks pass simultaneously, producing the effect of a continuous line of light.

94. Disruptive Discharge.—The disruptive discharge (**8, 89**) presents a great variety of forms, which may be referred to three types—the *spark*, the *brush*, and the *glow*.

All these forms have one common character, which is that, notwithstanding their short duration, they are not simple phenomena. When examined by means of a rotating mirror, a discharge is often seen to be an intermittent phenomenon consisting of a great number of successive discharges in opposite directions, which have a well-marked oscillatory character, as if the electricity went from one armature to another until it had lost all its energy.

FIG. 84.

The *spark* when it is short appears as a bright line of light, which is the more dense the greater the quantity of electricity. In proportion as the distance increases or as the capacity of the condenser diminishes, the line is less defined; it becomes narrower and less luminous, and acquires a zigzag form (Fig. 84), passing into that of a brush.

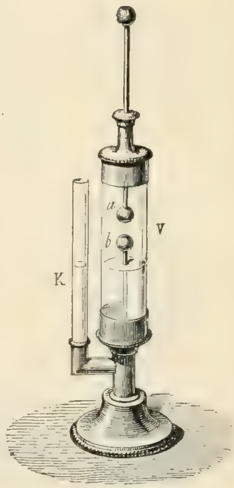


FIG. 85.

The production of the spark is always accompanied by a hard crackling sound, due to a violent mechanical disturbance of the medium, as may be illustrated by means of *Kinnersley's thermometer* (Fig. 85). When the discharge passes between the two knobs, the water is driven out of the larger tube into the smaller one.

If the spark is passed through water in the apparatus represented in Fig. 86, the disturbance is sufficient to break the tube.

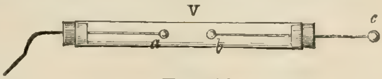


Fig. 86.

The *brush* (Fig. 87) has a pale violet tint, and is accompanied by a peculiar continuous hissing sound, very different from the sharp crack of the spark discharge. The whole has an ovoidal form, and the ramifications which proceed from the positive pole are connected with it by a kind of brightly luminous stem. The negative pole appears as though it were clothed with a luminous layer; if it ends in a point, this is tipped with a small brilliant star.

When the spark passes between terminals sealed into glass vessels, and the pressure of the contained gas is altered, very various effects are produced. Thus at a pressure, which varies from 4 cm. for air to over atmospheric pressure for helium, the discharge forms an unbroken flexible column, passing without noise between the terminals. This can be investigated by means of the *electric egg* (Fig. 88). At a smaller pressure the column becomes more nebulous and breaks up into detached portions (*striæ*) (Fig. 89), which commence at the positive pole and extend through almost the whole length of the tube, whatever its length (*e.g.* 40 feet). These striæ are convex outwards on the side facing the negative pole, but nearly flat on the other side. The negative pole is coated with a blue velvety glow (the *negative glow*), between which and the striæ a dark space exists (the *Faraday dark space*). As the pressure of the gas is made still smaller the striæ disappear, the luminosity of the positive column decreases, while a second dark space (the *Crookes space*) is seen to separate the velvet glow from the negative terminal; the size of this space increases as the exhaustion proceeds; its shape is that of the locus of equal-length normals drawn from the electrode. The negative glow starts abruptly from the outer boundary of the Crookes space but ends diffusely in the

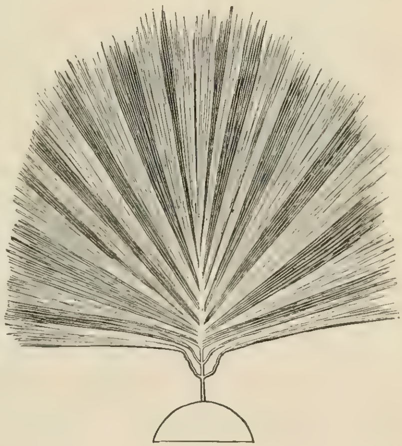


Fig. 87.

Faraday space. At the same time, especially if the terminal be concave, a stream of blue light (or green light if the gas is laden with mercury vapour) emanates in straight lines from the electrode, traverses the dark space and negative glow, and excites phosphorescence on any objects upon which it may fall. The phosphorescence is blue if it falls upon lead glass, green if on soda glass, brilliant green if on uranium glass. This stream is called the Cathode stream: its properties will be further described in the concluding chapter (**481, 482**).

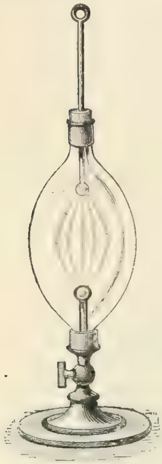


Fig. 88.

Finally, if the exhaustion be pushed as far as possible, all luminosity in the tube gradually vanishes, and any discharge which takes place probably does so by leakage over the surface of the glass vessel itself. To obtain the necessary vacuum very prolonged pumping is required, and the vessel has to be heated at the same time to drive off absorbed gases.

The difference of potentials necessary to produce a discharge in a partial vacuum varies from many causes. Under ordinary circumstances a few hundred volts at least are required; but if the terminals are heated, or if another discharge is already passing through the tube between independent terminals, the required voltage is much less; while in the highest stage of exhaustion 100,000 volts may be ineffectual.

Spectroscopic examination of the luminous phenomena produced by the discharge throws some light on their nature. The light of the spark in air at ordinary pressure shows lines which are characteristic of the metals of the electrodes between which the spark passes, and at the same time of the gas in which it is formed.

Fig. 89.

The light of the former predominates especially in the case of strong sparks. If the electrodes are of two different metals, the lines due to both metals are found A portion of the energy is therefore expended in detaching and volatilising particles of metal from the two electrodes. It is observed, for instance, that after a somewhat powerful discharge between a gold knob and a silver one, silver is found on the gold, and gold on the silver knob.

In the light of the glow discharge no trace of metallic lines is met with; the spectrum is the same whatever the nature of the metal, and is due solely to the luminous gas.

95. Striking Distance.—The length of the spark in air under the ordinary pressure, or what is known as the *striking distance*, depends on the difference of potentials of the two electrodes. It varies with the shape and size of the electrodes; thus, for the same striking distance, the difference of potentials required between two spheres increases when their radii increase. It must be observed that in the case of two parallel plates, which are large in proportion to their distance apart, if the difference of potentials increased in proportion to the striking distance, the discharge would always take place with the same electrical force, and consequently for the same density of the electric layer (**44**). As a matter of fact, both for planes and spheres, and for striking distances greater than 1 mm., the difference of potentials increases less rapidly than the distance, the rate of increase becoming less rapid the greater the distance. This is seen from the following table, which refers to the discharge between two knobs each one centimetre in diameter:—

Length of Spark. Cm.	Difference of Potentials. Electrostatic Units.	Volts.
0·1	16·1	4,830
0·5	56·3	16,890
1·0	84·7	25,440
1·5	97·8	29,340
2·0	104·5	31,350
3·0	124·0	37,200
5·0	153·0	45,900
10·0	187·8	56,100
15·0	206·0	61,800

For the case of brass balls, diameter of each 2·54 cm., the following values have been obtained:—

Length of Spark. Cm.	Difference of Potentials. Electrostatic Units.	Volts.
0·1	16·0	4,800
0·5	60·9	18,270
1·0	108·9	32,670
1·5	146·2	43,860
2·0	175·7	52,710

The following values (taken from more complete tables quoted by Dr. A. Russell; see **95***) have been found for the case of equal

brass balls of 5 cm. diameter. The three first results are due to Heydweiller, the remainder to Algermissen :—

Length of Spark. Cm.	Difference of Potentials. Electrostatic Units.	Volts.
0·5	61·2	18,360
1·0	109·5	32,850
1·5	154·1	46,230
1·5	154	46,200
2·0	194	58,200
2·5	230	69,000
3·0	260	78,000
3·5	288	86,400
4·0	314	94,200
4·2	324	97,200

The striking distance in air is sensibly the same for the spark properly so called as for the brush. This is readily demonstrated by a Holtz machine, with or without the Leyden jars, which increase the capacity of the conductors (**75**). Without the jars, the brush is formed continuously and the difference of potentials between the two electrodes is sensibly constant. When the jars are used the sparks pass intermittently; the difference of potentials increases until the spark is produced, when it falls almost to zero.

When the pressure is diminished, the difference of potentials corresponding to a given striking distance diminishes rapidly to a certain limit, beyond which it again increases with extreme rapidity. There appears to be one particular pressure at which the resistance to the discharge is a minimum. This varies in different gases, and with one and the same gas it varies with the dimensions of the tube. With air it is about 3 mm. in a space of an ovoidal form, but it is considerably smaller in tubes. Lastly, as the pressure is gradually diminished a point is reached at which the spark does not pass, whatever may be the difference of potentials. It seems to result from this, that matter is necessary for the transmission of electricity.

95.* Electric Strength of Dielectrics.—Before a spark passes in a dielectric medium, an electric stress, measured by the rate of change of potential dV/dx, is set up in it, and the spark passes when the stress reaches such a value that the medium can no longer support it, but breaks down, more or less as a shell bursts if it is subjected to too great a pressure. The limiting value of the stress (which we have elsewhere (**18**) called the electric force or intensity) that a medium can support is called its *electric strength* (**57**). For air at ordinary pressure and temperature,

it is about 127 electrostatic units. It is greater at higher pressures. As the pressure is reduced, the electric strength of air falls to about 1·1 electrostatic unit at about one-hundredth of an atmosphere and rapidly rises again at still lower pressures.

Suppose two metal spheres, each of radius a, with their centres at a distance d, and having charges $+Q$ and $-Q$ respectively: if d is much greater than $2a$, the charges will act externally as though concentrated at the centres of the respective spheres. In this case the electric stress at a point between them, on the line of centres, at distance x from the surface is $\dfrac{Q}{(a+x)^2}+\dfrac{Q}{[d-(a+x)]^2}$. This has its least value when $a+x=\frac{1}{2}d$—that is, at a point half-way between the spheres—and increases from this point, either way, and reaches a maximum when $x=0$ — that is, at the surface of the spheres where it becomes $\dfrac{Q}{a^2}+\dfrac{Q}{(d-a)^2}$. If therefore the charges Q and $-Q$ are simultaneously increased, a point will be reached at which the medium will break down, yielding first at the point on the surface of each sphere which is nearest the other sphere.

Even in the simple case we have supposed the limiting stress required to give a spark is not measurable by direct experiment, for it is hardly practicable to measure directly the charges $+Q$ and $-Q$. Under the given conditions, the potential of the positive sphere is approximately $Q\left(\dfrac{1}{a}-\dfrac{1}{d}\right)$, and that of the negative sphere is approximately $-Q\left(\dfrac{1}{a}-\dfrac{1}{d}\right)$, and therefore the difference of potentials $2Q\left(\dfrac{1}{a}-\dfrac{1}{d}\right)$. This difference of potentials can be measured and hence the value of Q, and therefore of the stress needed for a spark, can be calculated. Results deduced from this formula would, however, only be comparatively rough approximations. An accurate calculation requires that the distribution of the charges on the two spheres resulting from their mutual influence should be taken into account. The question has been carefully discussed by Dr. A. Russell[1] by a method founded on the principle of electrical images (**38**). He gives tables whereby the disruptive stress can be found from the difference of potentials required to give a spark whose length is in a known ratio to the radius of the spheres. When the spark-length l is small as com-

[1] *Philosophical Magazine*, [6] vol. xi. (Feb. 1906).

pared with the radius a, and the spheres have potentials $V/2$ and $-V/2$, the disruptive stress is given very nearly by the formula

$$V\left(\frac{1}{l}+\frac{1}{3a}\right).$$

When l/a is 4 or more, one sphere being at potential V and the other at 0, the sparking stress is given very accurately by the formula

$$V\left(\frac{1}{a}+\frac{a}{(a+l)l}\right).$$

95. Maximum Surface-Density of Charge—Maximum Charge.**—The relation $f=4\pi\sigma/K$ (**44**) determines the maximum surface-density of the charge that can exist on a conductor in contact with a dielectric of inductive capacity K when the electric strength of the dielectric is known. Thus, for air, we have $K=1$ and, taking the maximum value of $f=127$, as stated in the last section, we get

$$\sigma=\frac{fK}{4\pi}=\frac{127}{12\cdot57}=10\cdot1.$$

Hence the maximum charge that can be given to a metal sphere surrounded by air and at a great distance from other conductors is $=10\cdot1\times$ surface-area $=10\cdot1\times12\cdot57\,a^2$, or generally,

$$Q_{max.}=\sigma_{max.}\,A=\frac{f_{max.}\,K}{4\pi}\,.\,4\pi a^2=f_{max.}\,Ka^2.$$

This applies to a condenser of any form if it is such that the surface-density of the charges is uniform. With any such condenser and air as dielectric, the maximum charge is

$$\frac{127}{12\cdot57}A=10\cdot1\,A.$$

The electric strength of non-conducting solids and liquids is in general much greater than that of air. Some data bearing on this point will be found in a table at the end of this book.

96. Chemical Effects of the Spark.—The spark in air ignites combustible bodies like alcohol and ether; it ignites mixtures of oxygen with combustible bodies like hydrogen and its compounds, carbonic oxide, &c. The eudiometer is based on this principle. In this, as in *Volta's pistol* (Fig. 90), a spark is caused to pass through a detonating mixture of gases. The rod, DE, is insulated from the side of the vessel, and the moment a spark passes between the electrified body and the knob D, a spark also strikes between the knob E and the side of the vessel which is connected with the earth.

A succession of sparks decomposes ammoniacal gas into its elements, nitrogen and hydrogen, until a certain proportion of these gases has been produced. If hydrogen and nitrogen are present in larger proportion, the passage of sparks causes the gases to combine and reform ammonia. By means of an apparatus such as that shown in Fig. 91, it was proved by Cavendish that the passage of electric sparks through a mixture of nitrogen and

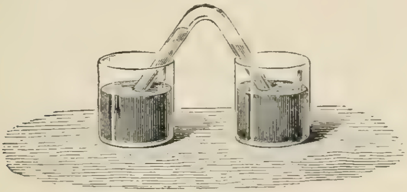


FIG. 91.

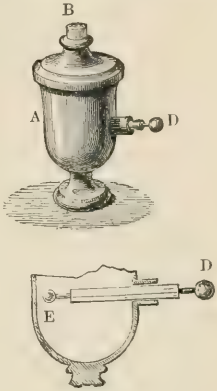


FIG. 90.

oxygen produced nitric acid. The ends of the V-tube filled with the mixture of the two gases are placed in two vessels of mercury, one of which is connected with the prime conductor of an electrical machine, while the other is put to the rubber. The gases should be slightly moist, and when the machine is worked the mercury rises in the tube in proportion as the gases combine to form the acid.

The brush discharge, or *electric effluvium*, produces in some cases characteristic effects different from those of a sudden spark discharge. An important example is the transformation of oxygen into ozone, which takes place when a current of oxygen is passed between two glass plates, or two glass tubes one inside the other, each coated on the farther side with tinfoil connected with the terminals of an induction-coil or of a Holtz machine. The proportion of ozone formed is increased when the temperature is kept low. If concentric tubes are employed, the tinfoil coatings may with advantage be replaced by strong sulphuric acid. Fig. 92 shows a convenient form of apparatus.

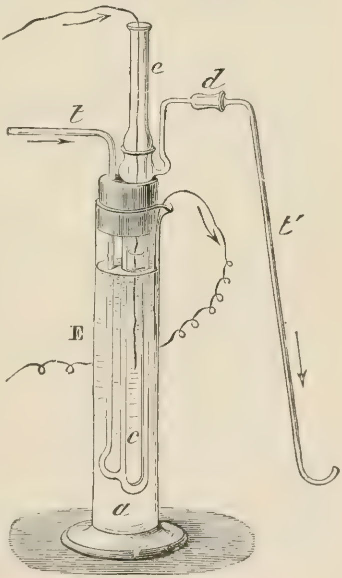


FIG. 92.

97. Physiological Effects of the Discharge.—When the discharge is passed through the human body, a shock is felt which, according to the energy of the charge, may be either a simple prickling or a disabling shock. The experiment is made by holding a charged Leyden jar in one hand by its outside coating while the other hand is brought to touch the knob connected with the inner coating. The shock may also be taken by a large number of persons at once, by joining hands in a chain, while the first grasps the outside, and the one at the end brings his hand to the knob.

Experiment shows that the physiological effect depends on the energy of the discharge, that is to say, jointly on the fall of potential and on the quantity of electricity. Thus we may take with impunity sparks from an ordinary machine 20 to 30 cm. in length, while one of only a few millimetres from a battery of great capacity could not be borne. In like manner, for the same quantity of electricity, the discharge by cascade gives a more violent shock than that of a single jar. Another circumstance which comes into play is the duration of the discharge and the manner in which it is effected; thus a battery which would give a violent shock under ordinary circumstances only gives a very feeble one when the discharge takes place through a moist cord forming part of the circuit. The effect of this is to increase the duration of the discharge, and at the same time, without altering the total energy of the discharge, to cause a considerable part of it to be converted into heat in the cord itself by virtue of the resistance which it offers.

CHAPTER X

GALVANIC OR VOLTAIC BATTERY

98. Galvanic Element.—If a plate of zinc and a plate of copper dip side by side, without touching, in a vessel of water acidulated with sulphuric or hydrochloric acid, two pieces of copper wire attached one to each plate are found to differ in potential. If the wires are connected with the alternate pairs of quadrants of a quadrant-electrometer, a deflection is produced indicating a difference of potentials of about one volt, the potential of the wire from the copper plate being the higher.

Similar results would be obtained if almost any two metals were substituted for copper and zinc in the above experiment, and instead of acidulated water, ordinary tap-water, or water containing a small quantity of some salt in solution, may be used. The only difference in the result would be in the magnitude of the observed difference of potentials. The material of the connecting wires is of no consequence; copper is mentioned as being the material most likely to be at hand, and, on the whole, the most convenient.

Results equivalent to those just stated were observed in 1789 by Ludovico Galvani, professor of anatomy at Bologna. An arrangement such as we have described, or any modification of it producing like effects, is consequently often spoken of as a *galvanic element* or *galvanic cell;* and the study of the numerous phenomena for the discovery of which Galvani's observation formed the starting-point constitutes the science of *galvanism.*

99. Voltaic Pile—Galvanic Battery.—The difference of potentials between the wires of a galvanic cell depends only on the materials employed, and, to some extent, on their temperature; it is not affected by the size, shape, or relative positions of the plates, nor by their absolute electrical potential. If placed on an insulating stand, a galvanic cell may be strongly electrified either positively or negatively, but, however high or however low the potential of the cell as a whole may be, the potential of the wire from the copper plate remains about one volt higher than that of the wire connected with the zinc.

A consequence of this is that, if several cells are arranged one after another, and the zinc plate of the first is connected by a wire with the copper plate of the second, the zinc of that with the copper of the third, and so on throughout, the zinc of the last cell but one being joined to the copper of the last, the difference of potentials between the wire from the first copper plate and that from the last zinc plate is equal to the difference found in the case of a single cell multiplied by the number of cells. To see how this follows, let us assume for simplicity that the difference of potentials in the case of a single cell is exactly one volt, and that the wire connected with the last zinc plate is in contact with the inside of the room where the experiment is made. We may assume then (**36**) that its potential is zero. Accordingly, the potential of the copper plate of the same cell and of the wire attached to it will be one volt. But this wire is fastened to the zinc of the last cell but one, and therefore the copper of this cell will have a potential = 2 volts. In the same way that of the next preceding cell will have a potential = 3 volts, and so on to the beginning of the series, the potential rising by one volt for every cell. Proceeding in the opposite direction along the series of cells, we should find a fall of potential equal to one volt at a time as we passed from one cell to the next; so that, if the first copper plate were kept at potential zero, the second would have a potential = -1, the next -2, and so on.

The fact that the difference of potentials, resulting from a single pair of metals dipping into an aqueous liquid, can be thus multiplied by arranging in order a number of such pairs connected together, was discovered in 1800 by Alessandro Volta, professor of physics at Pavia. In order to produce the effect in a marked degree, he arranged a number of alternate discs of copper and zinc, each pair being separated from the next by a round of cloth soaked with acidulated water, as shown in section in Fig. 93, and in perspective in Fig. 94. This arrangement is known as *Volta's pile*. A corresponding combination of pairs of metal plates in separate vessels of liquid is called a *voltaic battery* or *galvanic battery*, and in general the terms voltaic and galvanic, and words formed from them, are used almost interchangeably in relation to the class of phenomena we have now to study.

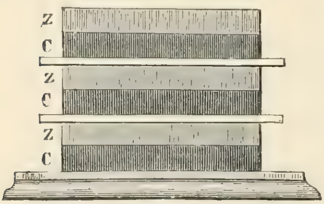


FIG. 93.

100. Volta's Theory of Electrification by Contact.—In order to explain facts of the kind referred to in the last two paragraphs, Volta supposed that the surface of contact of two metals, or more generally of any two different substances, is the seat of a peculiar force, which he termed *electromotive force*, the effect of which is to transfer positive electricity across the surface in one direction and negative in the other, until the tendency of the two electricities to reunite balances the force which tends to separate them. He supposed the electromotive force in any given case to depend solely on the nature of the materials in contact, and not upon the size or shape of the bodies, or the extent of the surface of contact, or on their electric state.

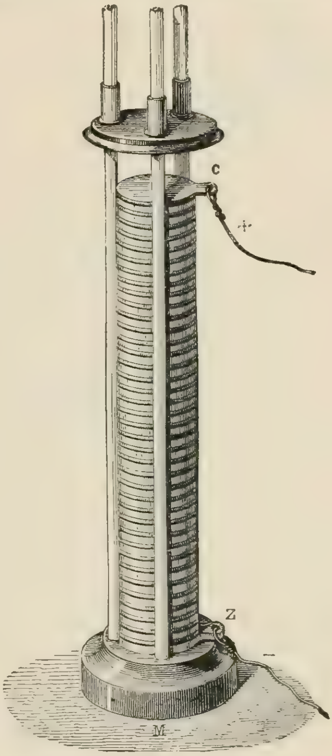


FIG. 94.

Among the numerous experiments which he devised to establish this principle the following may be cited :—

1. Two plates, one of zinc and the other of copper, supported by insulating handles (Fig. 95), and at first in the neutral state, are placed in contact and then separated. On applying these, each separately, to a delicate electroscope, it is found that they are each electrified, the zinc positively and the copper negatively.

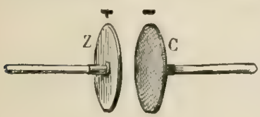


FIG. 95.

The amount of the charge is proportional to the extent of the surfaces in contact. The two plates are charged as if they were the plates of a condenser between which a difference of potentials has been established. The two electricities form on the surfaces in contact two equivalent layers, which remain separated in conformity with the electromotive force of contact of Volta, notwithstanding the absence of an insulating plate.

2. A zinc strip joined to one of copper is held in the hand, and

the copper end is applied against the lower of the two copper plates of a condensing electroscope, the top plate of which is touched by the other hand (Fig. 96). If the connections are broken and the top plate is raised, the gold leaves diverge, with negative electricity; this proves that the potential of the copper was negative, for as the zinc was held in the hand, it was at the potential of the earth —that is, at zero. If the experiment is repeated in exactly the same manner, except that the copper is held in the hand and the zinc applied to the electroscope, the leaves do not diverge. The fall of potential from zinc to copper is evidently the same in both cases, and the potential of the plate is zero like that of the copper held in the hand.

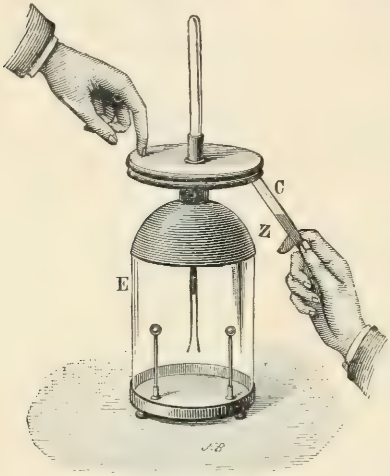


FIG. 96.

The experiments may be modified by interposing between the plate of the condenser and the strip of copper a disc of cloth or paper wetted with acidulated water. Nothing is changed in the former case; the divergence of the gold leaves is still negative. But in the second case, in which the copper is held in the hand, the leaves diverge with positive electricity, instead of remaining unaffected.

101. Objections to Volta's Theory.—While no question is raised as to the results of the fundamental experiments adduced by Volta and his supporters in confirmation of his views, there is by no means universal agreement among physicists as to the true interpretation of these results. The most serious objection to Volta's theory is that it does not afford a satisfactory account of the source of the energy the production of which it implies. Suppose S to be the capacity of the condenser used in either of the forms of experiment described in (**100**), and V and V' to be the potentials acquired by the two plates respectively, then the production of the difference of potentials $V-V'$ would indicate a generation of electrical energy $=\frac{1}{2}\ S\ (V-V')^2$. If the plates of the condenser have a radius of 5 cm. and are $\frac{1}{80}$ cm. apart, the capacity is (**53**) $S=\dfrac{\text{area}}{4\pi\times\text{distance}}=500$ in C.G.S. units. If the

difference of potentials is one volt, or $\frac{1}{300}$th of an electrostatic unit, the energy is $\frac{1}{2}\times 500\times\frac{1}{300}\times\frac{1}{300}=2{\cdot}78\times 10^{-3}$ erg. The only source from which this energy could be derived, consistently with Volta's hypothesis, is the heat contained in the zinc and copper. It is true that the withdrawal of the quantity of heat required to furnish the quantity of energy in question,—rather less than three-thousandths of an erg,—would cause an utterly imperceptible fall of temperature in a compound bar of zinc and copper large enough to be used in the experiment, so that the suggested explanation cannot be rejected on this ground. Indeed, it is known from other experiments (see Chapter XIV.), that when a much larger quantity of electricity than there is any question of in the present case, is caused to pass from copper to zinc across the common surface, an ascertainable lowering of temperature really ensues. But when a numerical comparison is made, it is found that the quantity of heat which thus disappears is inadequate to account for the difference of potentials developed in Volta's experiment. It is further found, when other pairs of metals are tried, that not only is the fall of temperature due to the passage of electricity across the separating surface generally insufficient to account for Volta's results, but that there is no recognisable relation between the magnitude of the change of temperature in the one set of experiments and the difference of potentials observed in the other set. In fact, there appears to be a real difference of potentials between different metals in contact, arising in some way solely from their own properties, which enable them to transform thermal energy into electric energy; but this difference is, in general, much too small to account for the effects now under consideration, and it does not always agree with these even in direction.

102. Alternative Theory.—If we no longer look for the source of the required energy in the surface of contact of the two metals, we may inquire whether it is not possible for it to be furnished by chemical action taking place between one or both of the metals and the air or liquid with which they are in contact. It is well known (see Chapter XV.) that chemical combination or decomposition is associated in a definite way with the transfer of electricity between the bodies taking part in the action, and that the occurrence of a fixed amount of chemical action involves the transfer of a fixed quantity of electricity. In order to charge a condenser, such as that supposed in (**101**), of capacity 500, to a difference of potentials of 1 volt, or $\frac{1}{300}$th of an electrostatic unit, a charge of $\frac{500}{300}=1\frac{2}{3}$ electrostatic unit would be required. Now

the oxidation of one gramme of zinc would correspond with a charge of 9×10^{12}, or nine millions of millions of such units. From this it follows that if the electrification observed in such experiments as those of Volta were really furnished by chemical action (oxidation of zinc), the required amount of such action would be so small as to be utterly imperceptible by any direct mode of investigation.

The connection above alluded to between chemical action and transfer of electricity naturally suggests the idea that a definite electric charge is uniformly associated with each atom of a chemical substance. If, in addition to this, we admit that the tendency to combination which is recognised by chemists under the name of chemical affinity is capable of counterbalancing electric forces so as to produce a condition of mutual equilibrium, we may give some such account as follows of the action that results from the contact of two metals.

When a piece of zinc is exposed to oxygen or air, we may suppose that atoms of oxygen, each carrying with it a definite charge of negative electricity, unite with the metal and at the same time give up their charge to it, so as to lower its potential below that of the air, until the resulting electric force counterbalances the chemical force tending to cause further combination. On grounds that we cannot conveniently discuss at this stage, we infer that equilibrium is reached when the potential of the zinc is about 1·8 volt below that of the air. We may further suppose that when a piece of copper is exposed to oxygen, oxidation goes on in a similar way until again a condition of equilibrium is reached, this time when the potential of the metal is about 0·8 volt below that of the gas. We assume here that the force tending to cause combination of copper with oxygen is less than that tending to make zinc combine with oxygen, which is in harmony with the generally recognised chemical characters of the two metals.

If now a piece of zinc and a piece of copper, both in contact with air, and therefore, according to what we have just said, differing in potential by something like one volt, are brought into contact with each other, we must suppose equalisation of potentials to take place between them in accordance with the general properties of conductors. (We leave out of account at present the small difference of potentials referred to at the end of **101.**) The potential of the zinc is thus raised, so that it no longer causes equilibrium between the chemical and electrical forces, and we may therefore suppose a further oxidation to occur until the difference of 1·8 volt is again produced. Or we may suppose the same result

to be produced by a *deoxidation* of the copper; or, again, we may imagine that both these actions occur simultaneously. Either way, the result would be *uniformity of potential throughout the two connected pieces of metal*, with a *difference of potentials between the portions of the gas* in contact with the zinc and copper respectively. The potential of the gas just outside the oxidised portion of the zinc would exceed that of the metals by 1·8 volt, and that of the gas just outside the copper would exceed it by 0·8 volt. That is to say, there would be an electric field established in the gas, the difference of potentials between the boundaries being one volt, and the positive side of the field would adjoin the zinc. We must suppose that it is this difference of potentials which is observed in many experiments that are usually cited as evidence of a difference of potentials between the metals.

A point of considerable importance remains to be discussed in connection with the account just given of the probable origin of the phenomena referred to in the preceding paragraphs. We have assumed that the oxidation of a metal is accompanied by a transfer of negative electricity to the metal. This seems to imply that oxygen gas as such is negatively electrified, a conclusion inconsistent with the observed electrical neutrality of oxygen or air under ordinary conditions. In reference to this we may call to mind the fact that, on purely chemical grounds, chemists have concluded that a *molecule* of oxygen gas consists of *two atoms* which are in some way held together, and that the connecting force has been likened to, if not actually identified with, electrical attraction, a molecule of oxygen being represented by the symbols $\overset{+}{O}\,\overset{-}{O}$. If we follow out such ideas, we must look on each molecule as constituting in itself an electric field of almost infinitesimal extent, but the gas as a whole would be electrically neutral. If now combination of oxygen and zinc takes place, we must suppose that the *negative* member of a molecular pair unites with the metal, leaving the positive member in the state of a so-called dissociated atom. It seems likely that the metallic surface, with which a number of negative atoms have entered into chemical combination, would have the corresponding positive oxygen atoms adhering to it as an external layer held on by electric force. If so, we should have two parallel oppositely electrified surfaces separated by a molecular distance, and comparable with the surface of a condenser. The difference of potentials between the positive and negative layers would be the electric force multiplied by the distance, or, in the symbols we have commonly used—

$$V - V' = 4\pi\sigma e/K.$$

If we put $K=1$, $V-V'=1{\cdot}8$ volt $=\frac{1{\cdot}8}{300}=0{\cdot}006$ electrostatic unit, and $e=1$ cm./10^8 (one hundred-millionth of a centimetre), a value which there is reason to believe is somewhere near the truth, we get for σ the charge per square centimetre

$$\sigma=\frac{{\cdot}006 \times 10^8}{12}=5 \times 10^4.$$

Now we have already said that the oxidation of one gramme of zinc corresponds with the transfer of nearly 9×10^{12} electrostatic units of electricity. Hence, to produce a difference of potentials of 1·8 volt between a zinc plate and the adjacent air, we should require, according to the view we have been discussing

$$\frac{5 \times 10^4}{9 \times 10^{12}}=5{\cdot}5/10^9$$

grammes of zinc to be oxidised per square centimetre of surface. Or the oxidation of $5\frac{1}{2}$ milligrammes would suffice for 100 square metres of surface.

The charge per unit surface of a copper plate in air or oxygen would be proportionately less than that of zinc, as its potential differs less from that of the gas.

103. Résumé.—The two theories we have been discussing may be distinguished as the contact theory and the chemical theory respectively.

According to the contact theory, the seat of the action that gives rise to the difference of potentials is at the separating surface of the two metals, and this action would still go on if it were possible to make the experiment in an absolute vacuum. According to this theory, the source of energy is the heat contained in the metals, whose temperature must accordingly be lowered by an immeasurably small amount when they are put into contact.

According to the chemical theory, the seat of the action is the surface of contact between the metals and the surrounding air (oxygen), and the metals themselves are at the same potential. The energy is derived from the energy of chemical combination between one of the metals and the surrounding gas.

Various attempts to decide between the two theories have been made by trying experiments, such as are usually quoted in proof of the contact theory, in a space from which the air has been exhausted, or where it has been replaced by an inert gas; but these have not led to a decisive result. A more effectual way of displacing a film of gas or moisture adhering to the surface of the metals is to heat them in heavy petroleum oil to about 145° C. A pair of copper and zinc plates that have been thus treated do not show the usual

contact-effects, and even when the oil has been carefully wiped off these effects reappear only gradually in the course of some hours or days, the metal surfaces apparently retaining for a considerable time an invisible film of oil sufficient to protect them from the action of the atmosphere. This result seems to be clearly in favour of the chemical theory, which is further supported by the apparently conclusive facts that the energy of the galvanic battery, the action of which Volta's hypothesis was intended to explain, is admittedly derived from chemical action (**159**); and that, in the case of thermo-electric action (**144**, **148**) where heat is undoubtedly the source of electric energy, Volta's theory of contact action has no application.

104. Electromotive Force of a Cell or Battery.—According to the contact theory, when a plate of copper and a plate of zinc dip into acidulated water without touching each other, there is no appreciable difference of potentials between them, because there is no metallic contact; while, according to the chemical theory, there is a difference of potentials of about one volt. If a copper wire is attached to the zinc plate, then, according to either view, there is a difference of about a volt between this wire and the copper plate or a copper wire attached thereto. According to the contact theory, this result is attributed to the introduction of a copper-zinc contact; according to the chemical theory, the copper wire merely takes the potential of the zinc to which it is joined.

In (**98**) and (**99**), in order to avoid raising the questions we have just been referring to, we were careful to speak of the difference of potentials between wires of the same metal (copper) attached to the two plates of a single cell, or to the two terminal plates of a battery or pile. But in any case, these questions are of theoretical and not practical importance: in practice the plates of a cell or a battery are always connected either with wires of the same metal (usually copper), or with binding-screws of the same metal (usually brass), and there is then no controversy as to the difference of potentials existing between these wires or binding-screws, as the case may be.

This difference is necessarily equal to the algebraic sum of the differences occurring at all the surfaces of contact of different materials intervening between the two extremities of the cell or battery; for since all the materials are conductors, and there is electric equilibrium, there is no difference of potentials except at the contacts. The wires or binding-screws connected with the first and last plates of a battery, or with the two plates of a single cell, are called the *poles* or *terminals* of the battery or cell; that one which is connected with the copper plate and has the higher

potential is the *positive* pole or terminal, and that connected with the zinc is the *negative* pole.

The algebraic sum of the differences of potential occurring at all the surfaces of contact of different materials intervening between the terminals of a cell is called the *electromotive force* of the cell. As we have just said, this is the same as the difference of potentials between the terminals when there is electrical equilibrium. If, however, the terminals are joined by a wire, a condition of statical equilibrium no longer obtains, and the difference of potentials between the terminals is then *less* than the electromotive force of the cell. The relation between these quantities is more fully discussed in Chapters XI. and XII.

When several cells are connected one after another to form a battery, the algebraic sum of the electromotive forces of the several cells is the electromotive force of the battery. By connecting the terminals of each separate cell, and then those of the battery, with the quadrant electrometer, it is easily verified that the difference of potentials between the terminals of the battery is the sum of the differences between the terminals of the cells composing it.

105. A Galvanic Battery Comparable with an Electric Machine.—The fundamental property of a galvanic battery, by virtue of which it maintains a constant difference of potentials between its terminals, or between any conductors connected with them, makes it exactly analogous to an electric machine. By its action an electric field is established, the boundaries of which are the surfaces of the conductors attached to its terminals. If the theory set forth in (**102**) is correct, at least in its main outlines, the charge of the field is derived from the atomic charges of the molecules which take part in the chemical action that goes on in the cell. Expressing the same thing in other words, we may say that the lines of force that extend between the boundaries of the field are transferred to them from the atoms. Fig. 97 may suggest the electric field set up when the terminals of a battery, B, are connected by wires with the surfaces of a condenser, C: the fine lines in the figure represent roughly the distribution of lines of force. If now the plates of the condenser be connected for an instant with each other, the lines of force between them disappear, but new lines immediately take their place. Lines previously extending between the connecting wires close in upon the condenser and are followed by fresh lines generated by the battery, the ends of the lines travelling along the wires. We have already seen (**22**) that a surface from which lines of force start is what is otherwise called a positively electrified surface, and that a surface

on which lines terminate is negatively electrified. It follows that when lines of force flow through the dielectric medium from the battery to the condenser, their ends at the same time moving along the connecting wires, that the process may be otherwise described as a flow of positive electricity from the positive terminal of the battery along one wire to the corresponding plate of the condenser, and of negative electricity from the negative terminal along the other wire to the second plate of the condenser. But, as we cannot distinguish the transfer of positive electricity in one direction from the transfer of an equal quantity of negative electricity in the opposite direction, the process referred to may be equally well described as a flow of positive electricity from one pole of the battery to the condenser, and of an equal quantity of positive electricity from the condenser to the negative pole of the battery.

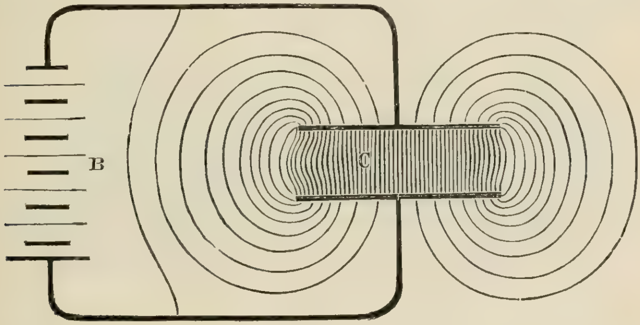


FIG. 97.

If the plates of the condenser are connected permanently, instead of only for an instant, or, more simply, if a continuous wire (sometimes called an *interpolar* wire) is used to connect the terminals or poles of the battery with each other, a continuous procession of lines of force must be thought of as spreading out from the battery, their ends travelling in opposite directions along the wire until they meet, and the lines, as it were, shrink into the wire. This process constitutes what is known as a continuous *electric current*, which may be described as the passage of equal quantities of positive and negative electricity in opposite directions round the circuit formed by the battery and wire, or as the circulation of a double quantity of positive electricity in one direction. The ordinary phraseology employed in reference to electric currents is founded on the conception of a single kind of electricity traversing the circuit. If we employ the conception of a double circulation of the two kinds of electricity in opposite directions,

any statement of the quantity of electricity passing through the battery or along a wire must be taken as referring to the arithmetical sum of the quantity of positive going one way and of negative going the other way. This conception of two electricities in some ways connects itself most easily with the notion of an electric field employed in the previous part of this work; but, whatever the mode of expression employed, it is important to remark that, while what is called an electric current is maintained in a circuit, the battery is continuously giving off energy to the surrounding dielectric medium, and that the conducting wire is continuously absorbing energy from the medium. While this is going on, both the conducting wire and the dielectric medium exhibit special properties that we have to study in what follows.

106. Decrease of Electromotive Force by the Passage of a Current.—During the passage of a current through a galvanic cell, chemical action goes on among the materials. With cells

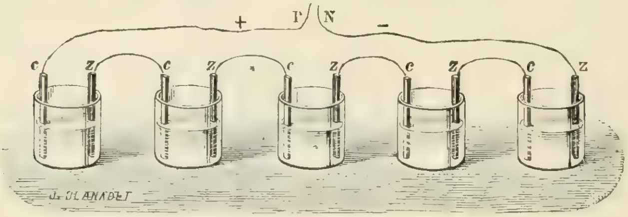


FIG. 98.

of the construction we have hitherto spoken of, formed of two metal plates dipping into dilute acid, the result is a change in the chemical nature of the substances in contact, producing a diminution of electromotive force, and a consequent weakening of the action of the cell. Usually, on interrupting the conducting circuit and allowing the cell to rest for a while without producing a current, there is a more or less complete recovery of the original electromotive force, unless indeed the chemical action has gone so far as nearly to use up some essential constituent. This temporary decrease of electromotive force is often referred to as *polarisation* of the cell or battery. We shall discuss the effect more fully in connection with the chemical effects of the current (160).

107. Modifications of Volta's Pile.—This decrease of power is shown in a marked degree by a Volta's pile if it is used to give a continuous current. It was at first attributed to the squeezing out of the liquid from the pieces of cloth at the lower part of the column by the weight of the metal discs above them, and to

remedy it Volta adopted the arrangement shown in Fig. 98, and known as the *crown of cups*, consisting of a number of separate vessels containing dilute acid, with a strip of copper and a strip of zinc inserted in each and connected as before described. To make a battery of many cells more compact and portable, and to facilitate filling in and removing the acid, a further early device was Cruickshank's *trough battery* (Fig. 99), consisting of a number of double plates of zinc and copper cemented into grooves cut in the opposite sides of a wooden trough so as to divide the trough into a number of water-tight cells, each having one side formed of copper and the other of zinc.

The electromotive force of a Volta's cell, as measured on open circuit on the quadrant electrometer, is about one volt, but in

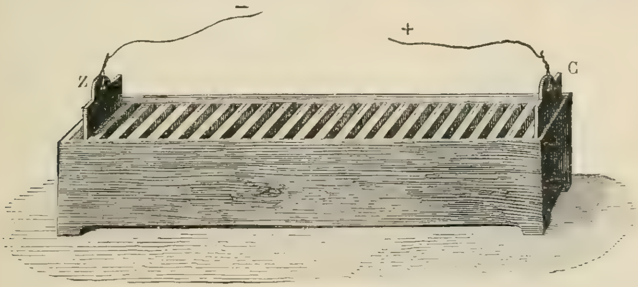


FIG. 99.

whatever form it is constructed the electromotive force falls rapidly when the circuit is closed.

108. Use of Amalgamated Zinc.—If ordinary commercial zinc is used in the construction of a voltaic cell, it dissolves in the acid even when the circuit is not closed. But pure zinc, or common zinc that has been amalgamated, does not dissolve by itself in dilute sulphuric or hydrochloric acid. When used in a voltaic cell, it is attacked only while the current is passing, and the quantity that then dissolves in a given time is, as we shall see (**156**), proportional to the current produced. To amalgamate a zinc plate, its surface should be cleaned by dipping it in dilute acid, and then, while it is still wet with the acid, pouring on to it a few drops of mercury, which readily spread over it.

109. Constant Batteries.—The so-called polarisation of a Volta cell appears to be mainly due to the formation of a layer of hydrogen on the surface of the copper plate, to which it adheres with considerable obstinacy. To prevent this many expedients have been employed, mostly consisting in the employment of such

materials that the chemical action taking place among them does not lead to the separation of hydrogen. In order that there may be no polarisation, in other words, that the electromotive force may be the same whatever the strength of the current, the materials of the cell should be such that the passage of the current and the accompanying chemical action cause no change in the nature of the substances in contact. In proportion as polarisation is avoided, a battery or cell is called *constant*. No perfectly constant form of battery has yet been devised, but there are two or three forms which make a fair approach to constancy when properly constructed and employed.

Daniell's battery (1836) was the earliest form of constant battery, and is still one of the most efficient. The metals employed are amalgamated zinc and copper, the zinc dipping in a dilute solution of zinc sulphate or in dilute sulphuric acid, and the copper in a saturated solution of copper sulphate. The two liquids are kept from mixing by a porous partition of unglazed earthenware, or sometimes by putting the heavier saturated copper solution in the bottom of a vessel and the lighter liquid above. A cell arranged in the former way is represented in Fig. 100, where C and Z are the copper and zinc plates respectively, and D is the porous partition. During the passage of the current, metallic zinc is converted into zinc sulphate, which dissolves in the liquid in contact with the zinc plate, and metallic copper separates from the solution of copper sulphate and is deposited on the copper plate. The metallic surfaces thus retain permanently their original character, but the concentration of the solutions changes, especially in close contact with the metal plates, and this causes a small decrease of electromotive force while the current is passing. The normal electromotive force of a Daniell's cell is about 1·07 volt.

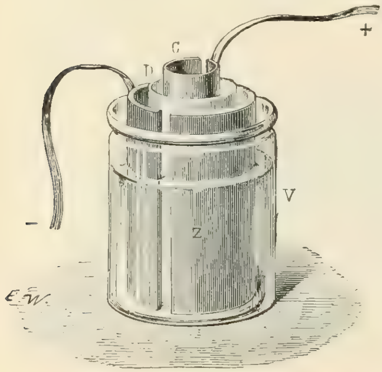


FIG. 100.

In *Grove's* battery (1840) the evolution of hydrogen is suppressed by the employment of strong nitric acid, which gives up part of its oxygen and converts the hydrogen into water. One of the metals is again amalgamated zinc dipping into dilute sulphuric acid (1 vol. of acid to 8 or 10 of water), which is separated from

the nitric acid by a partition of porous earthenware. The use of nitric acid necessitates the replacement of copper by a metal, platinum, which the acid does not attack.

Bunsen's battery (Fig. 101) is a modification of Grove's, in which rods of dense gas-graphite are substituted for the expensive platinum plates. All else remains as in Grove's battery.

The electromotive force of a Grove's or Bunsen's cell is about 1·8 volt.

Latimer Clark's cell consists of mercury in contact with a paste of mercurous sulphate, upon which is a saturated solution of zinc sulphate (containing crystals of this salt) into which a zinc rod dips. The electromotive force is 1·434 volts at 15° C. and remains very constant, provided no sensible current is allowed to flow through the cell; moreover, it recovers its original value very completely even if a small current has passed through. Owing to its great constancy it is largely used as a standard of electromotive force. Unfortunately, however, the value decreases considerably with rise of temperature (about a millivolt for each degree centigrade).

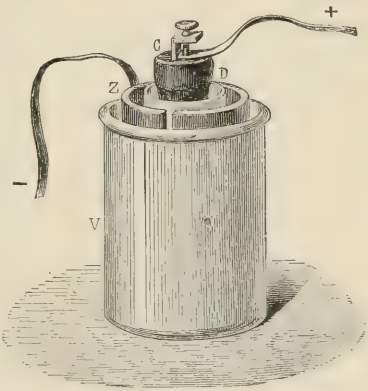


Fig. 101.

Among the many other forms of cells that are in more or less common use, we may mention the *bichromate* and the *Leclanché* cells. In both of these the plates consist of amalgamated zinc and graphite respectively: in the former, the plates dip in a solution of potassium bichromate acidulated with sulphuric acid; in the latter, the liquid is a saturated solution of ammonium chloride (sal ammoniac), and the carbon plate is closely packed round with black oxide of manganese in small fragments. The electromotive force of a bichromate cell is about 2 volts, but it has the disadvantage that the liquid acts on the zinc even when no current is passing; it must not, therefore, be left to stand with the plates in the liquid. A Leclanché cell has an electromotive force of not quite 1·5 volt on open circuit. It falls considerably when a current passes, but recovers when left to itself, and the plates may be left in contact with the liquid for long periods without injury.

In all the cells mentioned the *zinc* terminal is *negative*.

Fig. 102 represents the mode of connecting a number of cells

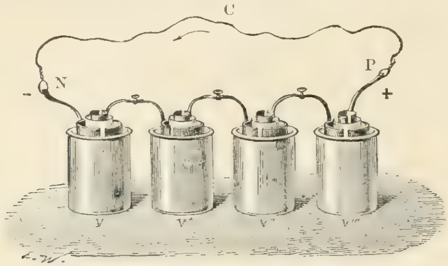


FIG. 102.

to form a battery, and shows a conducting wire joined to opposite terminals of the first and last cells.

CHAPTER XI

ELECTRIC CURRENTS—RESISTANCE—OHM'S LAW

110. The Electric Current.—We have already said (**105**) that when the terminals of a battery are connected with the plates of a condenser, positive electricity passes to the plate in connection with the positive terminal, and negative electricity to that connected with the negative terminal. This accumulation of electricity on the two plates continues until each of them is brought to the same potential as the pole of the battery with which it is connected. When this state of things has been reached, there is a condition of equilibrium in which one condenser-plate, the corresponding wire, and the battery terminal to which it is attached are all at one potential, and the second condenser-plate, wire, and terminal are all at another potential, differing from that of the first plate by an amount equal to the electromotive force of the battery,—that is, the same change of potential occurs between the two sides of the dielectric of the condenser as between the two terminals of the battery, and the tendency of the condenser to discharge exactly balances the electromotive force of the battery.

If, however, the terminals are connected by a continuous wire, a condition of statical equilibrium is never established. The wire tends to bring the two terminals to the same potential, while the battery tends to maintain between them a difference of potentials equal to its own electromotive force. These two tendencies cannot be satisfied simultaneously: the result is that an intermediate state of things is reached, in which there is a continuous passage of positive electricity in one direction through the wire, and of negative in the opposite direction, tending to equalise the potential at the two ends; while, at the same time, the battery continues to supply positive electricity to, and to remove negative from, that end of the wire which has the higher potential, and to supply negative and remove positive at the other end, thus keeping up a difference of potentials between the ends. This difference of potentials falls short of equality with the electromotive force of the battery to an extent to be investigated later (**111**).

As long as these conditions are maintained, the wire possesses certain special properties which are briefly indicated by saying that it is traversed by an *electric current*. These properties are not in all respects the same in both directions—that is, some of them differ according as, when we pass along the wire in a given direction, we are proceeding from the positive terminal of the battery to the negative, or the reverse. The same thing is commonly expressed by saying that the properties of the wire depend on the *direction of the current*, which is conventionally understood to mean the direction in which *positive* electricity is conceived of as passing round the circuit, namely, through the battery from the negative terminal to the positive, and through the connecting wire from the positive terminal to the negative.

Experiment shows that the characteristic properties associated with what is called the passage of a *current* are possessed by every point in the *circuit* formed by the battery and the conductor by which the terminals are connected, and, further, that the intensity with which these properties are exhibited, as well as their direction, is the same all round the circuit. It is found also by experiment that the potential at any one point of the circuit remains constant, which shows that there is no accumulation of electricity at any part. Hence it is concluded that the current, of which the circuit is the seat, is of the nature of a continuous circulation such that every section of the circuit is traversed simultaneously by the same quantity of electricity. The quantity of electricity which passes any section of the circuit in a given time determines what is called the *strength of the current*. The unit of current-strength, or, as it is also called more shortly, the unit of current, commonly employed for practical purposes, is that which corresponds to a coulomb per second. Such a current is called a current of one *ampere*. The strength of a current expressed in amperes therefore gives the number of coulombs which pass every cross-section of the circuit in one second.

If the strength of a current is constant, it is represented by the formula

$$C = q/t$$

where q is the quantity of electricity which traverses any section of the circuit in t seconds, and since the quantity which traverses every section of the circuit is the same, we may speak of this as the quantity which traverses the circuit. If the current is variable, its strength *at a given instant* is still expressed by the same formula if we understand by q the small quantity that passes

during an indefinitely short interval of time, t, including the instant in question.

111. Variation of Potential along the Circuit.—A characteristic property of the current is that on each of the conductors which make up the circuit, the potential, instead of being uniform throughout, as in the condition of statical equilibrium, varies from one point to another, always decreasing in the direction of the current.

Let us suppose that the terminals of the battery are connected by a series of homogeneous cylindrical wires, AB, BC, CD . . . of different metals (Fig. 103). Take a quadrant electrometer, the needle of which is kept at a fixed potential, and put one pair of quadrants in connection with the point A, which may be, for instance, the positive terminal of the battery, and the other with a variable point, M, of the interpolar wire.

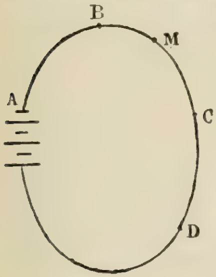


Fig. 103.

From A to M there will be a fall of potential which will vary with the position of the point M. So long as this point is on the wire AB, the fall is proportional to the distance of the point M from A, and its value in volts is equal to the product of the strength of the current C, expressed in amperes, into a factor, that we will denote by ρ_1, depending on the length AM measured along the wire, on the material of the wire, its thickness, and to some extent on its temperature and other physical conditions. Thus, letting A and M stand respectively for the potentials at the corresponding points, we may write

$$A - M = C\rho_1.$$

If we put r_1 for the value which ρ_1 assumes when M comes to B, the end of the first wire, we have for the difference of potentials between the points A and B

$$A - B = Cr_1.$$

Again, if the point B is connected with one pair of quadrants, and the point M connected with the other pair, is taken on the second wire, BC, we have, as in the previous case, for the difference of potentials between B and M

$$B - M = C\rho_2$$

ρ_2 being a factor depending on the material and dimensions of the second wire, or, when M coincides with C

$$B - C = Cr_2$$

and so on for each wire of which the circuit is made up.

If N be the potential at the negative terminal of the battery, the difference of potentials between the terminals, or $A - N$, is the sum of the differences $A - B$, $B - C$, ... ; that is

$$A - N = C\,(r_1 + r_2 + r_3 + \ . \ . \ . \ ).$$

If we examine, similarly, by means of the electrometer, the difference of potentials between the terminals of the several cells of the battery, we find that in each case this is less than the corresponding difference when the circuit is not closed, and when therefore no current is passing, by an amount proportional to the strength of the current and to a factor, say ρ', depending on the nature of the materials forming the cell, and on the size and shape of its different parts. Thus if e be the electromotive force of a cell, or the difference of potentials between the terminals when no current is passing, the difference f when it is traversed by a current of strength C, is

$$f = e - C\rho'.$$

If the battery consists of n identical cells connected in series, its total electromotive force, E, is equal to n times the electromotive force, e, of one cell; and when the circuit is completed, so that the battery is producing a current of strength, C, the difference of potentials between its terminals is n times the difference between the terminals of a single cell, or

$$A - N = nf = ne - nC\rho' = E - Cr'$$

if we put r' instead of $n\rho'$.

Equating this value of $A - N$ with that previously found, we have

$$E = Cr' + C\,(r_1 + r_2 + r_3 + \ . \ . \ . \ )$$

or

$$E = C\,(r' + r) = CR$$

when r is put for the sum of the separate quantities, r_1, r_2, r_3, . . . , and R for the total sum $r' + r$.

112. Definition of Resistance.—Writing the expressions for the difference of potentials between two points of a circuit as follows:—

$$r_1 = \frac{A - B}{C}, \quad r_2 = \frac{B - C}{C}, \quad r_2 = \frac{C - D}{C}, \text{ \&c.},$$

we see that the factors r_1, r_2, . . . applicable to the metallic parts of the circuit may be defined as the ratio for each conductor of the difference of potentials between its ends to the strength of the

current which traverses it. By experiment it is found that this ratio is constant. If, for example, various differences of potential, $A-B$, $A'-B'$, $A''-B''$, . . . be maintained between the ends of a conducting wire, the wire is in each case traversed by a current of definite strength, such that if C, C', C'', . . . represent the currents in the several cases, we have

$$\frac{A-B}{C}=\frac{A'-B'}{C'}=\frac{A''-B''}{C''}=\ldots=\text{a constant}=r_1.$$

This ratio, which for every conductor has a definite value characteristic of that conductor, is called its *resistance* (**90**).

In order that any conductor which is not itself the seat of electromotive force may be traversed by a current, it is essential that, as the result of an external electromotive force applied to it, the potential at the part where the current enters should exceed that at the part where the current leaves—that is to say, there must be a *drop of potential* as we pass with the current through the conductor. The resistance of the conductor may then be shortly defined as being numerically equal to the *drop of potential corresponding to a current of unit strength* traversing the conductor.

A galvanic cell is a conductor which is the seat of electromotive force. When there is no current, there is a difference of potentials, say e, between the terminals equal to the electromotive force of the cell. When the terminals are joined by a wire, a current traverses the cell from the terminal which has the lower potential to that which has the higher, and the difference of potentials assumes a value, say f, less than the electromotive force of the cell, so that there is a decrease in the difference of potentials, in consequence of the passage of the current, equal to $e-f$. This decrease, divided by the strength of the current, is again a constant quantity, characteristic of each particular cell, and is called its *resistance*. Using accents to distinguish the values in different cases, this may be expressed in symbols as follows:—

$$\frac{e-f}{C}=\frac{e-f'}{C'}=\frac{e-f''}{C''}=\ldots=\text{a constant}.$$

Accordingly, the resistance of a galvanic cell may be defined as being numerically equal to the *decrease of difference of potentials between its terminals due to the passage of unit current.*

113. Resistance of Combinations of Conductors.—If several conductors are connected end to end, or *in series*, so as to be traversed by the same current, the resistance of the combination is, by definition, the drop of potential between the points where the current enters and leaves the series divided by the strength of

the current, provided that none of the conductors is the seat of an electromotive force; and, as appears at once on writing down the values, it is equal to the sum of the resistances of the separate conductors. Thus, using the same notation as previously, we have for the resistances of the first, second, third, . . . conductors taken by themselves

$$r_1 = \frac{A-B}{C}, \quad r_2 = \frac{B-C}{C}, \quad r_3 = \frac{C-D}{C}, \text{ \&c.}$$

and for the resistance of the combination

$$r = \frac{A-N}{C}$$

or

$$r = r_1 + r_2 + r_3 + \ldots$$

since

$$(A-B)+(B-C)+(C-D)+\ldots = A-N.$$

Similarly, for a battery formed of a number of cells connected in series; if $e_1, e_2, \ldots$ are the electromotive forces of the separate cells when no current is passing, and $f_1, f_2 \ldots$ the differences of potential between their terminals when the circuit is closed, and if E and F have corresponding meanings for the whole battery, the resistances of the separate cells are

$$r_1 = \frac{e_1 - f_1}{C}, \quad r_2 = \frac{e_2 - f_2}{C}, \quad r_3 = \text{\&c.}$$

and the resistance of the battery is

$$r = \frac{E-F}{C}.$$

Now it is found by experiment (**99**, **111**) that $E = e_1 + e_2 + \ldots$, and $F = f_1 + f_2 + \ldots$, hence

$$r = \frac{e_1 - f_1 + e_2 - f_2 + \ldots}{C} = r_1 + r_2 + \ldots;$$

or, again, the resistance of the combination is the sum of the separate resistances.

If a number of conducting wires whose separate resistances are $r_1, r_2, r_3, \ldots$ are employed to connect the same points, as A and B (Fig. 104), the total current, of strength say C, divides among the wires, so that the sum of the currents in the separate branches is equal to the undivided current, or

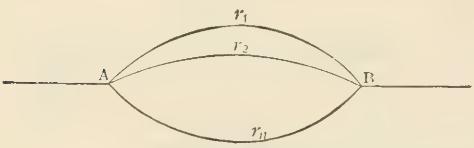


FIG. 104.

$$C = c_1 + c_2 + c_3 + \ldots$$

If we put $A - B$ for the difference of potentials between the points A and B, the resistances of the separate wires are given by

$$r_1 = \frac{A-B}{c_1}, \quad r_2 = \frac{A-B}{c_2}, \quad r_3 = \frac{A-B}{c_3}, \text{ \&c.}$$

and the resistance of the combination is

$$r = \frac{A-B}{C} = \frac{A-B}{c_1 + c_2 + c_3 + \ldots}$$

whence,

$$\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \ldots$$

Conductors connected in this manner are variously spoken of as being connected *abreast*, or in *multiple arc*, or in *parallel circuit.* A case that occurs in practice very frequently is where two conductors are so combined. In this case the resistance of the combination is $r = \frac{r_1 r_2}{r_1 + r_2}$, the product of the separate resistances divided by their sum.

If two galvanic cells, whose separate resistances are r_1 and r_2 respectively and whose electromotive forces are e_1 and e_2, are connected in multiple arc, their positive terminals being joined at A (Fig. 105) and their negative terminals at B, the difference of potentials between the points A and B may be found from the following consideration. The two cells form a closed circuit, round which there will in general be a current which will traverse one of the cells, namely, that one which has the greater electromotive force, from the negative to the positive terminal (this may be called the *natural* direction), and the other from the positive to the negative terminal (or in the *inverse* direction). Let c denote the strength of this current; then, according to (**111**), the difference of potentials between A and B will be less than the electromotive force, say e_1, of the stronger cell, and greater than that, e_2, of the weaker cell, by the product of the current strength into the resistance of the corresponding cell. If E be this difference of potentials, we have accordingly

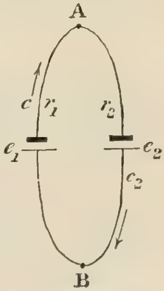


FIG. 105.

$$E = e_1 - cr_1 \text{ and } E = e_2 + cr_2.$$

Or, eliminating c between these two values,

$$E = \frac{e_1 r_2 + e_2 r_1}{r_1 + r_2}.$$

Now suppose the points A and B connected by a wire of resistance r (Fig. 106). This wire will be traversed by a current flowing from A to B of strength $C = c_1 + c_2$, if c_1 and c_2 denote the currents now flowing from B to A through the two cells respectively. As the result of this current, the difference of potentials between A and B will now be less than before: let it be denoted by F. Then, by the definition given at the end of **(112)**, we have for the resistance of the single cell which would be equivalent to the given combination of two cells

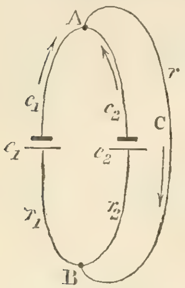


FIG. 106.

$$R = \frac{E - F}{C}.$$

To determine F, we may apply the same definition to the two cells taken separately; we thus get

$$r_1 = \frac{e_1 - F}{c_1} \text{ and } r_2 = \frac{e_2 - F}{c_2},$$

while the consideration of the connecting wire gives

$$r = \frac{F}{C}.$$

Remembering that $C = c_1 + c_2$, these equations give

$$F = \frac{e_1 r r_2 + e_2 r r_1}{r r_1 + r r_2 + r_1 r_2}$$

$$\text{and } C = \frac{F}{r} = \frac{e_1 r_2 + e_2 r_1}{r r_1 + r r_2 + r_1 r_2}.$$

Putting these values into the expression for R, we have

$$R = \frac{E}{C} - \frac{F}{C} = \frac{r r_1 + r r_2 + r_1 r_2}{r_1 + r_2} - r = \frac{r_1 r_2}{r_1 + r_2}.$$

Hence it appears that when two conductors are connected in multiple arc, the resistance of the combination, or, in other words, the resistance of the single conductor by which the two given conductors can be replaced without altering the strength of the current in any other part of the circuit, is in all cases equal to the *product divided by the sum* of the separate resistances, whether these are simple metallic wires, or whether they are galvanic cells, or other arrangements which form the seats of electromotive forces. In the first case, the single equivalent conductor must not contain an electromotive force; in the second case, it must be the seat of an electromotive force $E = (e_1 r_2 + e_2 r_1) / (r_1 + r_2)$.

114. Ohm's Law.—The nature of electric resistance, and the

fact that the resistance of any given conductor has a definite numerical value characteristic of that conductor, were first clearly recognised by G. S. Ohm. The properties of the electric circuit were discussed by him from this point of view in a work published in 1826, and the relations that he established between the electromotive force or forces acting in a circuit, its resistance, and the strength of the current by which it is traversed, form the starting-point of all calculations involving these magnitudes. This relation is usually referred to as *Ohm's Law*, and may be expressed in symbols by the following equations—

$$R=\frac{E}{C}, \quad C=\frac{E}{R}, \quad E=CR$$

which may be applied either to a single conductor or to a complete circuit. R is the resistance of the conductor, or the total resistance of the circuit, as the case may be; E the difference of potential between the ends of the conductor, or the resultant electromotive force of the circuit; and C the strength of the current.

The unit, in terms of which resistances are commonly expressed, or the *practical unit of resistance*, is called an *ohm*, and is such that an electromotive force of one volt applied to a homogeneous metallic conductor having this resistance will maintain in it a current of one ampere. The ohm will be more fully discussed in the chapter on Electrical Units (Chapter XXXI.), but we may state here that it is very nearly equal to the resistance at 0° C. of 14·452 grammes of mercury in the form of a column of uniform cross-section and 106·3 cm. long.

Ohm not only showed that the resistance of a given conductor in a given condition is a constant quantity depending on its material and dimensions, but he also showed that the resistance of a conductor of uniform cross-section s, and length l, is expressed by

$$R=\rho\frac{l}{s}$$

where ρ is a constant characteristic of the material, called its *specific resistance*. The full statement of Ohm's law must be taken as including this equation.

115. Specific Resistance.—If we put R_1 for the resistance of a conductor of length 1 cm., and cross-section 1 cm.^{2}, l and s in the above expression both become unity, and we have $R_1 = \rho$, which gives us the definition of the specific resistance of a material as being numerically equal to the *resistance of a conductor of that material* 1 *cm. long and having a cross-section of* 1 *cm.*^{2}. The shape

of the cross-section makes no difference, so that, for the purposes of this definition, we may suppose that we take 1 cm. length along a square rod measuring 1 cm. across each way; such a piece would be a centimetre-cube, and its resistance would be the specific resistance of the material. But it is important to notice that, when specific resistance is defined as the resistance of a centimetre-cube, it is to be understood that the flow of electricity takes place between opposite faces and is uniformly distributed throughout the cross-section. If we measure resistance in ohms and lengths in centimetres, the unit of specific resistance will be an ohm × 1 square centimetre and divided by 1 cm.; *i.e.* it will be one ohm-centimetre.

The specific resistances of the metals increase with temperature, and therefore also the actual resistances of conducting wires. If the resistance of a wire at 0° C. is denoted by R_0, its resistance at $t°$ may be represented to a first approximation by

$$R = R_0 (1 + at),$$

where a is a coefficient depending on the material. For several pure metals, the value of a has been found to be from 0·0036 to 0·0038, or nearly equal to the coefficient of expansion of gases. For alloys, the value of a is much smaller, and solids are known (*e.g.* manganin) for which its value is actually negative at least within a certain range of temperature.

Experiments by Professors Fleming and Dewar show that at very low temperatures, such as can be obtained by the evaporation of liquid oxygen, the specific resistances of the pure metals are very much less than they are at ordinary temperatures. The following numbers are taken from their results:—

Approximate Temperature.	Specific Resistance.			
	Platinum.	Silver.	Copper.	Iron.
0° C.	10974	1489	1564	9115
– 200°	3339	390	289	1220
– 220°	2439	...	144	660

The unit employed in this table is the one-thousand-millionth of an ohm-centimetre; to obtain the values expressed in ohm-centimetres we must therefore *divide* the given values by 10^9.

When a long range of temperature is regarded or great accuracy is required a parabolic formula must be employed

$$R = R_0 (1 + \alpha t + \beta t^2);$$

and even this must be used with caution. Experiments by E. P. Harrison on nickel show that such a formula fits very well between $-191°$ C. (*i.e.* the temperature of liquid air) and $+370°$ C. but that at this point a change occurs and thereafter the resistance increases much less rapidly and almost as a linear function of the temperature up to 1050° C. Similar results were obtained with iron, the change not occurring till nearly 800° C.

The specific resistances of non-metallic liquids are enormously greater than those of metals, and they decrease with rise of temperature.

A table giving numerical values for a number of substances, both metallic and non-metallic, is given at the end of this volume.

CHAPTER XII

APPLICATIONS OF OHM'S LAW

116. Current in a Simple Circuit.—Ohm's law enables us to calculate immediately the strength of the current that a given battery will produce through any conductor of known resistance which may be connected with its terminals. For instance, suppose the battery to consist of n cells connected in series, each having electromotive force e and resistance r', and that its terminals are connected by a wire of resistance r. Then the total electromotive force of the circuit is $E = ne$, and the total resistance is $R = nr' + r$; consequently the current is

$$C = \frac{E}{R} = \frac{ne}{nr' + r}.$$

If the cells composing the battery, instead of being all identical, have various electromotive forces and resistances, e_1, e_2, . . . and r_1', r_2', . . ., we shall have $E = e_1 + e_2 + \ldots$, and $R = r + r_1' + r_2' + \ldots$ Again, if any of the cells are connected with their poles in the opposite direction to the rest, the values of e for them must be taken as negative—that is to say, E must be taken as the *algebraic* sum of the separate electromotive forces. The resistance of a cell comes into account in the same manner, whichever way it is connected in the circuit; hence the value of R is not affected by the reversal of any of the cells.

117. Geometrical Representation.—The influence on the strength of the current of the electromotive force of the battery, and of the relative resistance of the battery and the rest of the circuit, can be conveniently expressed geometrically by the construction of Fig. 107. Here horizontal distances represent resistances, and vertical distances electromotive forces or differences of potential. MN represents the resistance of the battery, NO the external resistance, MK the electromotive force, and NL the difference of potentials between the battery terminals.

The strength of the current is given by the slope of the line KO, that is, by the tangent of the angle at O, or if we draw KP

parallel to MO, to represent one ohm (represented in the figure by 1·5 cm.). and PQ at right angles to KP, the current is represented by the length of the single line PQ. For, by construction

$$C = \frac{E}{R} = \frac{\text{MK}}{\text{MO}} = \tan \text{MOK} = \tan \text{PKQ} = \frac{\text{PQ}}{\text{PK}} = \text{PQ}.$$

The figure also shows clearly the relation between F, the difference of potentials at the terminals of the battery, and the electromotive force $E = ne$. For NL, which represents F, is equal to MK − LL′, and LL′ = MN × $\frac{\text{MK}}{\text{MO}}$ which is equivalent to

$$F = E - bC = E\frac{r}{b+r},$$

where the single symbol b is put for nr', the resistance of the

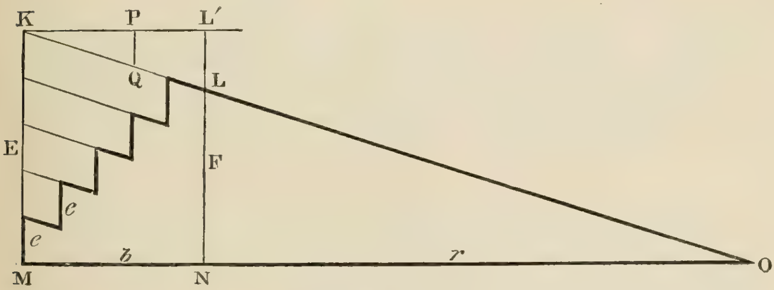


FIG. 107.

battery, in order to recall the fact that, as long as the battery remains unaltered, its resistance is a constant quantity.

It is easy from the same figure to see, at least generally, how the strength of the current depends on the external resistance. If this is great, O will be very far from N, the line KO will be nearly parallel to MO, and PQ, which is proportional to the current strength, will be small; obviously, however, PQ does not vanish for any finite value of NO or r. On the other hand, if r diminishes, O approaches N, and finally coincides with it. In this case the current attains its maximum value, $C = E/b = e/r'$, which is the same as that of the maximum current obtainable from a single cell. When the terminals of a cell or battery are connected through a conductor of no appreciable resistance, the battery is often spoken of as being *short-circuited*. Fig. 108 represents the effect of short-circuiting a battery of twelve similar cells in series, the resistance of each cell being represented by an arc of a circle instead of by a distance along a straight line.

In Fig. 107 a vertical ordinate drawn from any point in NO would represent the difference of potentials between that point and O, which corresponds with the negative terminal.

118. Kirchhoff's Rules.—The two following deductions from Ohm's law, often referred to as *Kirchhoff's rules*, are useful in calculations relating to complex circuits:—

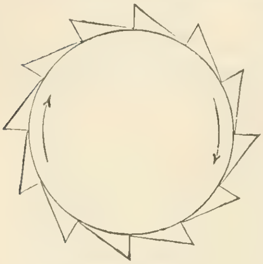


FIG. 108.

1. *If several conductors meet at the same point, the algebraic sum of the strengths of the currents in the several conductors, reckoned positive when the direction of the current is towards the point, is equal to nothing*, or

$$\Sigma c = 0.$$

2. *If two or more conductors form a closed figure, the algebraic sum of the products of the resistances of the several conductors into the strengths of the currents through them, reckoned positive the same way round the figure, is equal to the algebraic sum of the electromotive forces acting round the figure in the same direction*, or

$$\Sigma cr = \Sigma e.$$

The first of these theorems may be regarded as self-evident, it being simply an expression of the fact that under the given conditions there is no accumulation (positive or negative) of electricity at the point considered (Fig. 109).

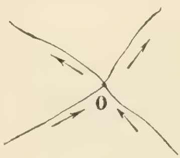


FIG. 109.

The second depends upon the relation already pointed out (**111**), that the difference of potentials, f, between the ends of a conductor which is the seat of electromotive force e, is less than e, when the conductor is traversed by a current in the direction of that which the electromotive force tends to produce, by the product of the strength of current into the resistance, or

$$f = e - cr.$$

Applying this result in succession to each of the conductors forming a closed circuit, we get from any selected starting-point back to the same point again; accordingly, the sum of the differences of potential round the circuit must be zero, or

$$\Sigma f = 0, \text{ whence } \Sigma cr = \Sigma e.$$

An important special case falling under this theorem is when a number of metal wires form a closed circuit (Fig. 110): here e is

equal to nothing for each conductor, and therefore also for the sum, or $\Sigma cr = 0$.

119. Conductors in Multiple Arc.—By aid of the foregoing results it is easy to solve many problems which present themselves in connection with complex combinations of conductors. We will consider first a number of metallic conductors joining the same points A and B; three conductors (Fig. 111) will be enough to illustrate the general case. If r_1, r_2, r_3, are the resistances of the separate conductors, we have already seen (**113**) that the resistance R of the combination is

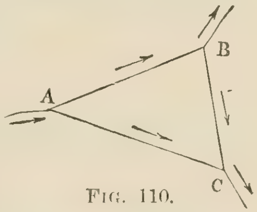


FIG. 110.

$$R = \frac{r_1 r_2 r_3}{r_2 r_3 + r_1 r_3 + r_1 r_2}.$$

The strength of the current in each branch may be expressed in terms of the total current C as follows: putting $A - B$ for the difference of potentials between the points A and B, we have

$$c_1 = \frac{A-B}{r_1}, \qquad c_2 = \frac{A-B}{r_2}, \qquad c_3 = \frac{A-B}{r_3},$$

whence

$$C = \frac{A-B}{r_1} + \frac{A-B}{r_2} + \frac{A-B}{r_3} = c_1\left(\frac{r_1}{r_1} + \frac{r_1}{r_2} + \frac{r_1}{r_3}\right)$$

or

$$c_1 = C\frac{r_2 r_3}{r_1 r_2 + r_2 r_3 + r_3 r_1}.$$

Similarly

$$c_2 = C\frac{r_3 r_1}{r_1 r_2 + r_2 r_3 + r_3 r_1}; \qquad c_3 = C\frac{r_1 r_2}{r_1 r_2 + r_2 r_3 + r_3 r_1}.$$

The case that occurs oftenest in practice is the simple one of two conductors. This is included in the above formula if we suppose r_3 to become infinite, when we get

$$c_1 = C\frac{r_2}{r_1 + r_2}; \qquad c_2 = C\frac{r_1}{r_1 + r_2}.$$

As one among many instances of the application of these formulæ, we may refer to the measurement of currents. It often happens that it is not desirable to allow the whole of the current to pass

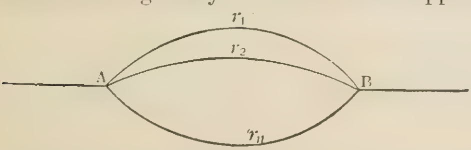


FIG. 111.

through the measuring instrument. In such a case, supposing the resistance of the instrument to be r_1, its terminals may be con-

nected by a wire of resistance r_2, frequently called a *shunt*. Then if the total current is C, and the current indicated by the measuring instrument is c_1, the strength of the total current is given by

$$C = c_1 \frac{r_1 + r_2}{r_2} = c_1 \left(1 + \frac{r_1}{r_2}\right).$$

For example, if $r_1/r_2 = 9$, $C = 10c_1$; if $r_1/r_2 = 99$, $C = 100c_1$.

120. Voltaic Cells or Batteries in Multiple Arc.—Let e_1, e_2, e_3, be the respective electromotive forces of three single voltaic cells or batteries connected as in Fig. 112; r_1, r_2, r_3, the corresponding resistances measured between the points A and B; c_1, c_2, c_3, the currents through the several cells; lastly, r the resistance of a wire connecting A and B, and c the current in it.

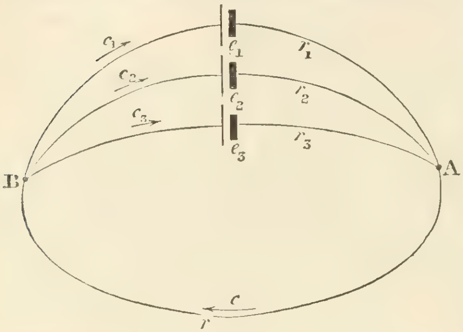


FIG. 112.

By applying Kirchhoff's second rule to the circuit formed by taking the cells, one at a time, together with the connecting wire r, we get the equations

$$cr + c_1 r_1 = e_1, \text{ or } c\frac{r}{r_1} + c_1 = \frac{e_1}{r_1}$$

$$cr + c_2 r_2 = e_2, \text{ or } c\frac{r}{r_2} + c_2 = \frac{e_2}{r_2}$$

$$cr + c_3 r_3 = e_3, \text{ or } c\frac{r}{r_3} + c_3 = \frac{e_3}{r_3}.$$

Then, adding, reducing to a common denominator, and simplifying, we get for the current in the external conductor

$$c = \frac{e_1 r_2 r_3 + e_2 r_1 r_3 + e_3 r_1 r_2}{r_1 r_2 r_3 + r r_2 r_3 + r r_1 r_3 + r r_1 r_2}.$$

For the resistance R of a single cell equivalent to the combination, we have **(113)**

$$R = \frac{r_1 r_2 r_3}{r_2 r_3 + r_1 r_3 + r_1 r_2}.$$

To obtain the electromotive force of the equivalent single cell, we have

$$E = c\,(R + r)$$

which, putting in the values of c and R and reducing, gives

$$E=\frac{e_1 r_2 r_3+e_2 r_1 r_3+e_3 r_1 r_2}{r_2 r_3+r_1 r_3+r_1 r_2}.$$

We may note the following special cases :—

1. All three cells have the same electromotive force, say e, then

$$E=e$$

which shows that the resultant electromotive force of any number of equal cells connected in multiple arc is the same as that of a single cell of the same kind.

2. All the cells have the same resistance; in this case the formula gives

$$E=\frac{e_1+e_2+e_3}{3}$$

or, the resultant electromotive force of any number of cells of equal resistance connected in multiple arc is the arithmetic mean of their separate electromotive forces.

To get the currents in the separate branches, we may start from the equation

$$c_1=(e_1-cr)/r_1$$

and the corresponding expressions for c_2 and c_3. Putting in the value of c already found, we get

$$c_1=\frac{e_1(r_2 r_3+rr_3+rr_2)-e_2 rr_3-e_3 rr_2}{r_1 r_2 r_3+rr_2 r_3+rr_1 r_3+rr_1 r_2}.$$

The values of c_2 and c_3 may be obtained similarly, or they may be written down from the last expression by cyclically changing the suffixes. It is evident that the denominator is not altered thereby.

120.* "Wheatstone's Bridge."—A case of some importance, in consequence of the applications that can be made of it, is that in which six conductors connect four points, as M, N, P, Q, in Figs. 112A or 112B. The arrangement intended is easily under-

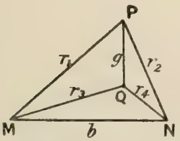


FIG. 112A.

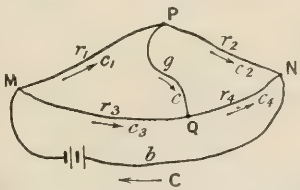


FIG. 112B.

stood if the four points are thought of as situated at the corners of a tetrahedron and the six conductors as lying one along each edge of the tetrahedron. If the conductor connecting one pair of points, say, that connecting M and N, contain a battery, every

branch of the conducting system will, in general, be traversed by a current, the strength of which can be calculated if the electromotive force of the battery and the resistances of the several branches are known.

Let E be the electromotive force, and b the resistance (including that of the battery) of the conductor connecting M and N. Put g for the resistance of the conductor PQ, r_1 and r_2 for the resistances of MP and PN respectively, and r_3 and r_4 for those of MQ and QN respectively. Further, let C stand for the strength of the current from N to M through the battery, c for the current in PQ, and c_1, c_2, c_3, c_4, for the currents in MP, PN, MQ, QN respectively. To find the values of the currents in the various parts of the system, we may proceed as follows. We can at once write down the equations—

$$C = c_1 + c_3 = c_2 + c_4 \quad . \quad . \quad (1) \qquad c_1 r_1 + cg = c_3 r_3 \quad . \quad . \quad . \quad (3)$$
$$c = c_1 - c_2 = c_4 - c_3 \quad . \quad . \quad (2) \qquad c_2 r_2 - cg = c_4 r_4 \quad . \quad . \quad . \quad (4)$$

Combining (1) and (3) we have—

$$c_1 r_1 + cg = (C - c_1) r_3 \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Similarly, (1) and (4) give—

$$c_2 r_2 - cg = (C - c_2) r_4 \quad . \quad . \quad . \quad . \quad . \quad (6)$$

or, by (2)—

$$(c_1 - c) r_2 - cg = (C - c_1 + c) r_4 \quad . \quad . \quad (7)$$

Eliminating c from (5) and (7), we get—

$$\left.\begin{aligned} c_1 &= C\frac{g(r_3 + r_4) + r_3(r_2 + r_4)}{g(r_1 + r_2 + r_3 + r_4) + (r_1 + r_3)(r_2 + r_4)} \\ \text{or} \qquad &= C\frac{g(r_3 + r_4) + r_3(r_2 + r_4)}{D} \end{aligned}\right\} \quad (\text{I.})$$

if, for shortness, we write D for the denominator of this expression.

From this, by properly changing suffixes, we obtain—

$$c_2 = C\frac{g(r_3 + r_4) + r_4(r_1 + r_3)}{D} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{II.})$$

Similarly—

$$c_3 = C\frac{g(r_1 + r_2) + r_1(r_2 + r_4)}{D} \quad . \quad . \quad . \quad . \quad . \quad (\text{III.})$$

$$c_4 = C\frac{g(r_1 + r_2) + r_2(r_1 + r_3)}{D} \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{IV.})$$

Substituting for c_1 and c_2, or for c_3 and c_4, in (2) we get, for the current in PQ, the value—

$$c = C\frac{r_2 r_3 - r_1 r_4}{D} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{V.})$$

Or, without previously finding c_1, c_2, &c., we may get the value of c by eliminating c_1 from (5) and (7).

The difference of potentials between the points M and N is given by $c_1r_1 + c_2r_2$ or by CR, if R stands for the resistance of a single wire which, if substituted for the existing combination of conductors, would leave the current through the battery unaltered. Equating these two values, we have—

$$R = \frac{c_1r_1 + c_2r_2}{C}.$$

From this, by inserting the values of c_1 and c_2 already found (I.) and (II.) and reducing, we get—

$$R = \frac{g(r_1+r_2)(r_3+r_4) + r_1r_2(r_3+r_4) + (r_1+r_2)r_3r_4}{D} \quad \text{(VI.)}$$

The strength of the current through the battery is of course

$$C = \frac{E}{b+R},$$

or, when the value of R is written out in full—

$$C = E\frac{g(r_1+r_2+r_3+r_4) + (r_1+r_2)(r_3+r_4)}{bg(r_1+r_2+r_3+r_4) + b(r_1+r_3)(r_2+r_4) + g(r_1+r_2)(r_3+r_4) + r_1r_2(r_3+r_4) + (r_1+r_2)r_3r_4}. \quad \text{(VII.)}$$

Using this value in (I.), we have—

$$c_1 = E\frac{g(r_3+r_4) + r_3(r_2+r_4)}{bg(r_1+r_2+r_3+r_4) + b(r_1+r_3)(r_2+r_4) + g(r_1+r_2)(r_3+r_4) + r_1r_2(r_3+r_4) + (r_1+r_2)r_3r_4}, \quad \text{(VIII.)}$$

with corresponding expressions for c_2, c_3, c_4, and c, the denominator being the same for all and the numerators those already given.

120. Remarks.**—(1) From equation (V.) of the last section, it appears that there is no current in PQ, or $c = 0$, when the condition $r_1r_4 = r_2r_3$ is satisfied; and, conversely, when there is no current in PQ, we know that $r_1r_4 = r_2r_3$. With this condition, it follows also that $c_1 = c_2$ and $c_3 = c_4$, as also appears directly on comparing the values of these quantities given by (I.) and (II.) and by (III.) and (IV.).

(2) If a battery be inserted in any other branch instead of in NM, the currents in the other branches can be written down from the expressions already obtained by properly changing the suffixes of the symbols. For instance, suppose the battery removed from NM and placed in PQ, so that Q is the positive terminal, and that we want to find the strength of the current in NM, the branch of the network which is not directly connected with PQ: in the case already considered, with the battery in NM, with M as positive terminal, we got for the current from P to Q the expression—

$$c = E\frac{r_2r_3 - r_1r_4}{bg(r_1+r_2+r_3+r_4) + b(r_1+r_3)(r_2+r_4) + g(r_1+r_2)(r_3+r_4) + r_1r_2(r_3+r_4) + (r_1+r_2)r_3r_4},$$

and we have to modify this to suit the new case. We note that each term of the numerator of our expression is the product of a

resistance connected with the positive terminal of the battery into a resistance connected with the negative pole. In the new case r_3 and r_4 are connected with the positive terminal Q, and r_1 and r_2 with the negative terminal P. The corresponding quantities will therefore again be r_2r_3 and r_1r_4, and in order that the current may as before flow from N to M, the difference $r_2r_3 - r_1r_4$ must be positive. Thus the numerator remains unaltered. In the denominator, it is easy to see that the first term remains unchanged. The second is the resistance of the battery, now g, into the sum of the resistances connected with the positive terminal and the sum of those connected with the negative terminal: the corresponding expression in the new case will therefore be $g(r_3+r_4)(r_1+r_2)$. The third term of the denominator is the resistance of the branch not directly connected with the battery, now b, multiplied into the product of the resistances of MPN and MQN, which in the former case connected the terminals of the battery: the corresponding conductors are now QMP and QNP, and their resistances r_3+r_1 and r_4+r_2 respectively. The modified third term is therefore $b(r_3+r_1)(r_4+r_2)$. The fourth and fifth terms consist of the products taken three at a time of the four resistances r_1, r_2, r_3, and r_4, and therefore are not altered. The result is that the whole denominator remains unchanged, for the second and third terms have merely changed places and the rest are not altered.

This leads to an important conclusion which may be stated in general terms, as follows: in a system of six conductors connected in the manner described, if a given electromotive force be applied to any one—call it for distinction number *one*—a current of definite strength will be produced in the conductor—call it number *six*—with which the first-mentioned conductor is not directly connected; if now the electromotive force be removed from conductor number one and applied to number six, the resistance of each branch remaining unchanged, it will produce in number one a current of the same strength as that previously produced in number six. A particular case is when $r_2r_3 = r_1r_4$: then, as we have seen a battery in NM produces no current in PQ: reciprocally, a battery in PQ produces no current in NM.

(3) We have seen that an electromotive force E in the arm NM gives a current c_1, in the arm MP the strength of which is given by (VIII.). If the resistances are such that

$$bg = r_2r_3$$

this current is independent of the value of the resistance r_4 in PN, and reciprocally the current c_4 in PN is independent of the resistance r_1 in MP. Similarly, if

$$bg = r_1r_4$$

the current c_2 is independent of the resistance r_3, and the current c_3 is independent of the resistance r_2.

To prove these relations it will be sufficient to consider the first-mentioned case, as the others can be proved in an exactly similar way. If c_1 is independent of r_4 its value will not be altered if we give to this quantity in turn the extreme values $r_4=0$ and $r_4=\infty$. In the former case, equation (VIII.) becomes

$$c_1 = E\frac{(g+r_2)r_3}{bg(r_1+r_2+r_3)+br_2(r_1+r_3)+gr_3(r_1+r_2)+r_1r_2r_3}.$$

Remembering that $bg=r_2r_3$ gives $g(b+r_3)=(g+r_2)r_3$, this may be written—

$$c_1 = E\frac{(g+r_2)r_3}{(g+r_2)(br_1+br_3+r_1r_3+r_2r_3)}$$

$$= E\frac{r_3}{br_1+(b+r_1+r_2)r_3}.$$

If now we put $r_4=\infty$, we have—

$$c_1 = E\frac{g+r_3}{bg+b(r_1+r_3)+g(r_1+r_2)+r_1r_2+r_1r_3+r_2r_3}$$

$$= E\frac{g+r_3}{(g+r_3)(b+r_1+r_2)+(b+r_2)r_1}$$

or, with the condition $bg=r_2r_3$—

$$c_1 = E\frac{(g+r_3)r_3}{(g+r_3)[(b+r_1+r_2)r_3+br_1]}$$

$$= E\frac{r_3}{br_1+(b+r_1+r_2)r_3}$$

as before.

121. Best Arrangement of Cells.—Suppose that n similar cells, each of resistance r and electromotive force e, are available for sending a current through a conductor of resistance r': the cells may be grouped in various ways, as all in series (Fig. 113),

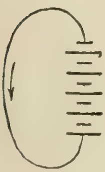


FIG. 113.

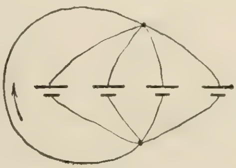


FIG. 114.

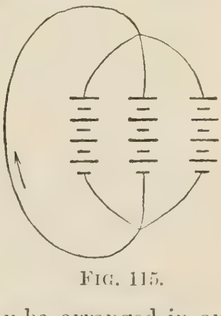


FIG. 115.

or all in multiple arc (Fig. 114); or they may be arranged in an intermediate manner, being divided into two or more series and these again being joined in multiple arc (Fig. 115). In general,

any change in the arrangement of the cells will cause an alteration in the strength of the current in the external conductor, and the question arises, how must they be combined in order to make this current as strong as possible? Let p be the number of cells in each series, and q the number of series, so that $pq = n$. The resistance of each series will be pr, and the resistance of the q series in multiple arc will be $\frac{p}{q}r$. Again, the electromotive force of each series will be pe, and as the series are all equal, this will also be the electromotive force of the whole combination. Consequently the strength of the current will be

$$C = \frac{pe}{pr/q + r'} = \frac{ne}{pr + qr'}.$$

The numerator of the expression for C, in its second form, is constant, and C will therefore be greatest when the denominator is least. But the product of the two terms of the denominator is $pqrr' = nrr'$, a constant quantity, and therefore the denominator has its smallest value when its two terms are equal, or $pr = qr'$, that is, when the resistance of the battery is equal to the external resistance, or, what is the same thing (since the last equation is the same as $p/q = r'/r$), when the number of cells in each series is to the number of series as the external resistance is to the resistance of a single cell. Seeing that p and q must be whole numbers, and $pq = n$, it is not generally possible to satisfy accurately the condition for a maximum current; all that can then be done is to choose the arrangement that comes nearest. If r', the external resistance, is very much greater than r, the resistance of one cell, we must make p/q as great as possible, or $p = n$ and $q = 1$; on the other hand, if r'/r is small, we must make $p = 1$ and $q = n$.

122. Conduction in Two Dimensions.—If, in place of being a thin wire, the conductor is a uniform flat sheet to which electricity is supplied or from which it is withdrawn at a given point, the current, instead of taking place in one direction only, will spread out from the point on all sides, or converge towards it, as the case may be. If we suppose the uniform sheet to extend without limit in all directions, it is evident that the current will spread uniformly, and that the flow will take place along straight lines drawn through the point. If the quantity of electricity given to or removed from such a sheet in a second is Q coulombs, and we draw straight lines through the point, making with each other a common angle α, such that

$$\alpha = 2\pi/Q,$$

the total flow between each pair of consecutive lines will be one ampere. A quantity equal to Q will flow in every second across any circumference having its centre at the *electrode* or point where electricity is supplied or withdrawn, and the flow across equal parts of such a circumference will be equal. The flow per second across *unit length* of a circumference drawn through any point of the sheet may therefore be taken as the measure of the *intensity* or *density* of the current at that point. Thus at any point at a distance r from the electrode, the intensity of the flow is $Q/2\pi r = 1/ar$, and the total flow per second across an arc of length s is $\frac{Qs}{2\pi r} = Q\frac{\theta}{2\pi}$, if we write θ for the angle $\frac{s}{r}$ subtended by the arc s.

It is evident from symmetry that the potential at every point on a circle drawn about the electrode as centre must be the same, or that any such circle must be an *equipotential* line. And since the direction of flow in a conductor is the direction of decreasing potential, the potential must *decrease* with increasing distance from an electrode that supplies electricity to the sheet, that is, from a *source*, and must *increase* with increasing distance from a *sink*, or an electrode that removes electricity from the sheet. Consider (Fig. 116) two concentric equipotential circles of very nearly equal radii, r and $r + dr$, and draw two radii, making a very small angle $d\theta$ with each other. The two circles and the two radii will intercept a small area on the conducting sheet, which, in the limit, may be treated as a rectangle of length dr, in the direction of the current, and of breadth $rd\theta$, at right angles to the current. If z is the thickness of the sheet and ρ its specific resistance, the resistance of this elementary portion may be taken as

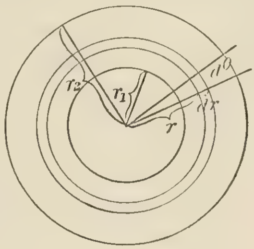


FIG. 116.

$$\frac{\rho dr}{zrd\theta}.$$

The whole circular strip bounded by the two circles contains $2\pi/d\theta$ such areas, which may be considered as conductors connected in multiple arc, and therefore the resistance of the whole strip to the radial flow taking place across it is

$$\frac{\rho dr}{2\pi rz}.$$

To get the resistance of an annular belt contained between circles

of radii r_1 and r_2 differing by a finite amount, we require to integrate this expression between the corresponding limits. Thus the resistance of the belt is

$$R = \frac{\rho}{2\pi z}\int_{r_2}^{r_1}\frac{dr}{r} = \frac{\rho}{2\pi z}\log_e\frac{r_2}{r_1} = 0{\cdot}3665\frac{\rho}{z}\log_{10}\frac{r_2}{r_1}.$$

But (**114**) the difference of potentials between two equipotential circles is equal to the resistance of the intervening portion of the conductor multiplied by the strength of the current: we have therefore for the amount by which the potential at radius r_2 *exceeds* that at radius r_1

$$V_2 - V_1 = -0{\cdot}3665\frac{Q\rho}{z}\log_{10}\frac{r_2}{r_1}, \text{ if the centre is a } \textit{source},$$

$$\text{and } V_2 - V_1 = 0{\cdot}3665\frac{Q\rho}{z}\log_{10}\frac{r_2}{r_1}, \text{ if the centre is a } \textit{sink}.$$

It follows that a series of circles drawn about the same electrode have a constant difference of potentials between each one and the next if their radii are in geometrical progression, and if V_1 be the potential at unit distance from an electrode, the potential at any distance r is given by $V = V_1 - 0{\cdot}3665\frac{Q\rho}{z}\log_{10} r$, the sign of Q being taken positive for a source and negative for a sink.

123. Two Equal Opposite Electrodes—Lines of Flow.—Now consider a source A (Fig. 117) supplying Q coulombs per

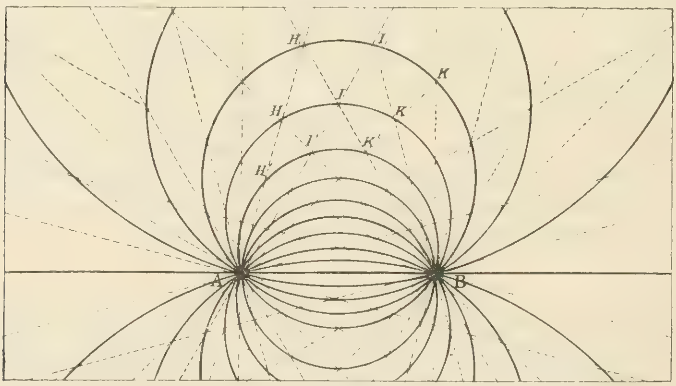


FIG. 117.

second to a conducting sheet, and a sink B removing electricity at the same rate. In this case there is no change in the total quantity of electricity possessed by the sheet, and therefore, with a limited

sheet, if the shortest distance from the electrodes to any point of the boundary is great in comparison with the distance between the electrodes, the state of things does not differ appreciably from what would exist in the corresponding portion of an unlimited sheet, and can be deduced by simply superposing on each other the conditions which we have already seen would apply in the case of a source or a sink respectively existing by itself in an unbounded sheet. Let AH, AI, AK, be drawn through A, making a constant angle $\alpha = \frac{2\pi}{Q}$ with each other; and let BH, BI, BK . . . be drawn similarly through B. Considering the effect of the source A by itself, the flow between AH and AI into the quadrilateral $\text{HH}_1\text{II}'$ across HI′, would be one ampere. Similarly, considering the sink B by itself, the flow due to it between BH and BI would be one ampere out of the same quadrilateral across I′I. The resultant flow due to the co-existence of the source and sink must then be such as to satisfy simultaneously the conditions of a flow of one ampere inwards (with respect to the quadrilateral considered) across HI′ and outwards across I′I. Obviously, also, corresponding conditions must apply to the flow across the boundaries of any of the quadrilaterals into which the whole surface of the conducting sheet is cut up by the mutual intersection of the two sets of rays drawn outwards from A and inwards towards B. Again, consider the flow at one of the points of intersection, say H: the flow due to A alone is along HH_1, and that due to B alone is along HI′. The resultant flow must have an intermediate direction. Similarly at I the resultant flow must have a direction intermediate between II_1 and IK′. Now, trace a curve, such as HIK, through alternate angles of successive quadrilaterals, and consider one of the resulting triangles, such as HI′I: we have already seen that the flow into this triangle across the side HI′ is equal to the outward flow across I′I. Therefore, on the whole, there can be no flow inwards or outwards across the third side, HI, of the triangle. Hence we see that lines drawn like H′I′K′, HIK, $\text{H}_1\text{I}_1\text{K}_1$, in the figure are such that, on the whole, no electricity flows across them; and by taking the common angle between the constructive lines of the figure smaller, so that the flow between consecutive lines due to source or sink alone is less than one ampere, we may determine as many points as we please between the points H, I, K, . . . and thus trace continuous curves such that no electricity flows across them anywhere. But such lines must coincide everywhere with the direction of the current—in other words, they must be *lines of flow* of the combined system of source and sink. We see, then, that the

lines of flow for the case we are considering are continuous curves traced through the alternate intersections of equiangular pencils of rays drawn through the source and sink respectively, and that, if the common angle between the rays corresponds to a flow of one ampere, the flow between consecutive lines of the resultant system, *e.g.* between H′I′K′ and HIK, or HIK and $H_1I_1K_1$, will also be one ampere. From the geometry of the figure it is easy to see that the angles AHB, AIB, AKB, . . . are equal, and therefore that the points H, I, K, . . . are points on a circle passing through A and B,—and similarly for the points on any other line of flow, as H′I′K′, &c.

124. Equipotential Lines.—The system of equipotential lines due to a source and a sink in a conducting sheet can be

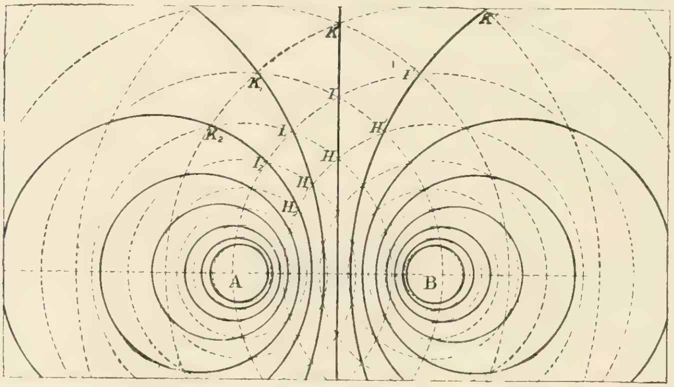


FIG. 118.

obtained by a process of superposition quite analogous to that employed for the lines of flow. In Fig. 118, let A and B again represent a source and a sink, and let the two sets of dotted circles, of which these points are the respective centres, represent systems of equipotential lines which would correspond to the two electrodes if each existed by itself. Further, let the smallest circle round A correspond to potential V, and the smallest circle round B to potential $-V$, and let there be a common difference of one volt between any one circle and the next in each set. Then at the point H in the figure the potential due to the source is $V-6$, the potential due to the sink is $-V+6$, and the resultant potential, being the algebraic sum of these, is $=0$. Similarly, the potential at I and $K=0$. At H_1 the potential due to the source is $V-5$, and that due to the sink is $-V+6$, consequently the resultant

potential is + 1. At I_1, the potentials due to the electrodes separately are $V-6$ and $-V+7$ respectively, and therefore the resultant is again + 1. Similarly at K_1. It follows, then, that the line of zero potential must pass through the points H, I, K,—it is, in fact, the straight line that bisects AB at right angles,—and that the line of potential + 1 must pass through H_1, I_1, K_1. From similar considerations it is evident that the line of potential + 2 must pass through the points H_2, I_2, K_2. In like manner the line of potential − 1 must pass through H', I', K'; the line of potential − 2 through H'', I'', K''; and so on.

If we put r for the distance of any point on the conducting sheet from the source, and r' for the distance of the same point from the sink, it is easy to see that the potential must be the same at all points for which the ratio of the distances r and r' is constant, and that it must be positive for all parts of the sheet for which r/r' is less than unity, and negative for all parts for which this ratio is greater than unity.

The same result is easily obtained algebraically. Let V be the potential which the source, if existing alone, would produce at a given point, and V' the potential which the sink alone would produce at the same point: then, if we put U for the resultant potential due to both source and sink, we have (**122**)

$$V = V_1 - \frac{Q\rho}{2\pi z}\log_e r$$

$$V' = -V_1 + \frac{Q\rho}{2\pi z}\log_e r'$$

and therefore

$$U = V + V' = \frac{Q\rho}{2\pi z}(\log_e r' - \log_e r) = \frac{Q\rho}{2\pi z}\log_e \frac{r'}{r}$$

whence it appears that the potential is proportional to $\log \frac{r'}{r}$, and therefore, as before, it is constant when r'/r is constant.

The curves which fulfil the condition, $r'/r = \text{constant}$, are circles with their centres in the line AB. If P is the radius of one of these circles, and a and b the distances of its centre from A and B respectively, then

$$\frac{r'}{r} = \frac{P}{a} = \frac{b}{P}$$

(Compare **37**).

125. Resistance.—Suppose a portion of a conducting sheet, in which there are a source and a sink of equal strength, bounded by two lines of flow and by two equipotential lines of potentials,

U_1 and U_2 respectively: we can deduce the resistance of this portion of the sheet to the current through it from the formula

$$R = (U_1 - U_2)/C$$

derived from Ohm's Law.

For U_1 we may write $\frac{Q\rho}{2\pi z}\log_e \frac{r_1'}{r_1}$, and $\frac{Q\rho}{2\pi z}\log_e \frac{r_2'}{r_2}$ for U_2, so that the difference

$$U_1 - U_2 = \frac{Q\rho}{2\pi z}\left(\log_e \frac{r_1'}{r_1} - \log_e \frac{r_2'}{r_2}\right) = \frac{Q\rho}{2\pi z}\log_e \frac{r_1' r_2}{r_1 r_2'}.$$

The strength of the current C is to the rate of total flow Q as the angle θ, at which the flow-lines bounding the portion of the sheet

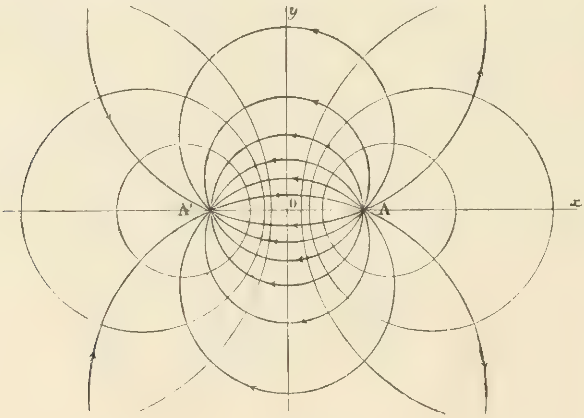


FIG. 119.

considered meet each other at the electrodes, is to 2π. That is $C = Q\frac{\theta}{2\pi}$: hence

$$R = \frac{\rho}{\theta z} \cdot \log_e \frac{r_1' r_2}{r_1 r_2'}.$$

If we have to deal with the whole space between two complete equipotential lines, we have $\theta = 2\pi$ and therefore $C = Q$, and consequently

$$R = \frac{\rho}{2\pi z} \cdot \log_e \frac{r_1' r_2}{r_1 r_2'}.$$

Fig. 119 shows the lines of flow and the equipotential lines for the case we have been discussing combined in one diagram. It will be seen that the two sets of lines intersect at right angles.

126. Resistance between Parallel Cylinders.—In certain cells one electrode consists of the cylindrical vessel itself (copper), while the other is a cylindrical rod (*e.g.* zinc), immersed in the solution contained in the vessel. We can calculate the resistance of such an arrangement as this on the basis of the results of the previous section: the problems are, in fact, converse to one another. For since we may usually neglect the resistance of the metal cylinders in comparison with that of the intervening solution the fall of potential along their lengths will be an insignificant part of the whole fall across from one to the other, and we may treat their surfaces as being equipotential surfaces. If these coincide with two of the equipotential surfaces in Fig. 118, the resistance of the fluid will be that given by the last formula; but if they do not, we may choose the points A and B so that they shall. The data given will be the radii of the cylinders (P_1 and P_2), and the distance between their centres ($C_1C_2 = D$). Let $C_1A = a_1$, $C_1B = b_1$, $C_2A = a_2$, $C_2B = b_2$, then $D = a_2 - a_1 = b_2 - b_1$

$$P_2^2 = a_2\, b_2 = (a_1 + D)\,(b_1 + D)$$
$$P_1^2 = a_1\, b_1$$

whence

$$\frac{P_2^2 - P_1^2 - D^2}{D} = a_1 + b_1 = 2m \text{ (say)}$$

By putting $\dfrac{P_1^2}{a_1}$ for b_1 we obtain

$$a_1 = m - \sqrt{m^2 - P_1^2}$$

and by putting $\dfrac{P_1^2}{b_1} = a_1$ we obtain

$$b_1 = m + \sqrt{m^2 - P_1^2}$$

Hence the expression for the resistance R is

$$R = \frac{\rho}{2\pi z} \log_e \frac{r_1' r_2}{r_1 r_2'} = \frac{\rho}{2\pi z} \log_e \frac{P_1}{P_2} \frac{a_2}{a_1}$$

$$= \frac{\rho}{2\pi z} \log_e \frac{P_1\,(a_1 + D)}{P_2\, a_1} = \frac{\rho}{2\pi z} \log_e \frac{P_2}{P_1} \frac{b_1}{(b_1 + D)}$$

For example: if

$$P_1 = 10,\; P_2 = 16,\; D = 2{\cdot}5$$

then

$$m = 29{\cdot}95,\; a_1 = 1{\cdot}72,\; a_2 = 4{\cdot}22$$
$$b_1 = 58{\cdot}18,\; b_2 = 60{\cdot}68$$

whence the resistance is

$$R = \frac{\rho}{2\pi z} \log_e \frac{10 \times 4{\cdot}22}{16 \times 1{\cdot}72} = {\cdot}06812 \frac{\rho}{z}$$

We may similarly calculate the resistance of the conducting medium between two parallel cylinders that are external to one another; the only change is that a_1 becomes negative: so that writing a_1' for its positive numerical value

$$R = \frac{\rho}{2\pi z} \log_e \frac{P_1 (D - a_1')}{P_2 \, a_1'}$$

If the radii of these cylinders are each equal to P

$$m = -\frac{D}{2}$$

$$a_1' = -a_1 = \frac{D - \sqrt{D^2 - 4P^2}}{2}$$

and

$$R = \frac{\rho}{2\pi z} \log_e \frac{D - a_1'}{a_1'} = \frac{\rho}{2\pi z} \log_e \frac{D + \sqrt{D^2 - 4P^2}}{D - \sqrt{D^2 - 4P^2}}$$

$$= \frac{\rho}{2\pi z} \log_e \left(\frac{P}{a_1'}\right)^2 = \frac{\rho}{\pi z} \log_e \frac{P}{a_1'}$$

If the radius P_2 of one of the cylinders becomes infinite the portion at a finite distance of this cylinder will evidently be represented by the plane of symmetry bisecting the line AB (Fig. 117), and the corresponding resistance will be half that of the total sheet in the case last treated, *i.e.*

$$R = \frac{\rho}{2\pi z} \log_e \frac{P_1}{a_1'}$$

where the value of D determining a_1' is *twice* the distance from C_1 to the plane. If d be this distance $(= D/2)$

$$R = \frac{\rho}{4\pi z} \log_e \frac{d + \sqrt{d^2 - P_1^2}}{d - \sqrt{d^2 - P_1^2}}$$

127. Applications to Analogous Problems.—Since the law of steady flow of currents in a conductor is the same as that which we have previously investigated for the flux of induction in a dielectric, to every problem in connection with the former there is a corresponding problem in electrostatics. Now in the former, when the lines of flow are straight and parallel

$$R = \rho \frac{l}{A}$$

and in the latter, under the same restriction,

$$\text{Capacity} = \frac{KA}{4\pi l}.$$

In each, if the lines are not parallel $\frac{l}{A}$ must be replaced by $\int_o^l \frac{dl}{A}$.

Thus, provided the geometry of the boundaries is the same in both cases the determination of the resistance when the medium is conducting gives us also the value of the capacity if the medium is a dielectric by the following relation

$$4\pi \times \text{Resistance} \times \text{Capacity} = K\rho.$$

For example, the capacity per unit length of the field between two parallel cylindrical conductors of equal radius P and distance D from centre to centre is

$$K \frac{1}{2 \log_e \dfrac{D + \sqrt{D^2 - 4P^2}}{D - \sqrt{D^2 - 4P^2}}}.$$

Again, the capacity per unit length of the field between a plane and a cylinder whose axis is at distance d and parallel to the plane is

$$K \frac{1}{\log_e \dfrac{d + \sqrt{d^2 - P^2}}{d - \sqrt{d^2 - P^2}}}.$$

This case, it should be noticed, is that of a telegraph wire suspended parallel to the earth or to a neighbouring wall.

Again, the flow of heat in a thermally conducting medium also follows the same law. Hence if two cylindrical surfaces are maintained at steady temperatures differing by one degree, and the space between them be filled with matter of thermal conductivity K, the above expressions will determine the heat conducted per second from one to the other by means of the relation

$$\text{Thermal conduction} = \frac{K\rho}{R}.$$

CHAPTER XIII

WORK DONE BY THE ELECTRIC CURRENT

128. Energy of the Circuit.—We have seen (**34, 35**) that, in order to transfer electricity from a point of lower to a point of higher potential, work has to be done *against* electric force, the amount of work being numerically equal to the product of the quantity of electricity transferred into the difference of potentials; and also that, when electricity passes from a higher to a lower potential, a corresponding amount of work is done *by* electric force. Accordingly electric energy is generated when electricity passes from a lower to a higher potential, and is expended when it passes from a higher potential to a lower. In the former case, energy of some other kind must be expended; and in the latter case, energy of some other kind is produced.

Let us consider, from this point of view, the processes that go on in the circuit of a galvanic battery whose terminals are connected by a conducting wire. To make the discussion definite, suppose the electromotive force of the battery to be E, and let b stand for the resistance of the battery itself, and r for that of the connecting wire. Then the circuit is traversed by a current of strength $C = E/(b+r)$, and in t seconds a quantity of electricity, $Q = Ct$, passes round it. By the action of the battery this quantity of electricity is raised in potential to the extent E: consequently a quantity of work, $W = EQ$, is done against electric force, and therefore electric energy equal to W is generated.

It is the function of the battery to supply this energy, and it does so, as will be further explained in Chapter XV., at the expense of the chemical energy of the materials consumed in it. During the passage of the current, chemical action goes on in the battery, such that the products of the action possess less chemical energy than the materials from which they are formed, and the difference is available for the production of electrical energy.

In all cases energy must be continuously supplied to the circuit in order to maintain a current; but according to the nature of the case, the source from which it is derived may vary. Thus, in a

thermo-electric circuit, energy is supplied in the form of heat; in the case of the circuit of a dynamo-electric machine, it is mechanical energy that is supplied. From our present point of view, it may be said that the function of a galvanic battery, of a thermo-electric pile, or of a dynamo-electric machine, is identical: they are all contrivances for effecting the transformation of other kinds of energy into the energy of an electric current, and the differences in their construction and in the details of their action depend on the kind of energy which they respectively transform.

129. Electric Work.—Turning to the other aspect of the matter, we have to consider the electric circuit as giving out energy, or the electric current as doing work. The form in which energy is given out, or the nature of the work done, depends on the nature and conditions of the various parts of the circuit, and much will have to be said on these points in subsequent chapters, but certain general laws as to the amount of energy given out may be considered at this stage.

It follows from the simple consideration that the energy of the circuit does not on the whole either increase or decrease during the passage of a steady current, that the amount of energy given out by it in a given time must be the same as the amount received by it in the same time, though the form may be wholly or partially different. Another consideration which leads to the same conclusion is that the same quantity of electricity passes every section of the circuit in the same time; hence, while the potential of a quantity of electricity, Q, is raised by the amount E, by the action of the battery or other equivalent organ (thermopile, dynamo, &c.), and while, therefore, energy to the amount EQ is received by the circuit, the same quantity of electricity has its potential lowered in an equal degree, energy being thus given out by the circuit to the amount EQ.

130. Available Energy.—We have already seen (**111, 113**), that the difference of potentials, F, between the terminals of a battery is less than the electromotive E by the product of the resistance of the battery, b, into the strength of the current, or $F = E - bC$. Hence the fall of potentials which the current undergoes outside the battery is F, and the energy generated in the part of the circuit external to the battery and given out by it is FQ. This represents the whole of the energy of the battery which is available for doing external work, and it is less than the total energy of the circuit by $(E - F)\ Q = bC^2t$.

The difference between the total energy and the available energy of the circuit is analogous to that between the total work

done by a steam-engine and that available for external purposes. A certain amount of the work done on the piston by the steam is consumed within the engine itself in processes essential to its working and in overcoming unavoidable friction, and it is only the remainder that can be made use of for driving external machinery or other purposes. In like manner, the external or available energy of the electric circuit is the excess of the total energy above that consumed in the battery.

The available energy given out per second is $FQ/t = FC$. This quantity is the *rate of doing work* by the circuit, and is comparable with the horse-power of a steam engine. If the difference of potentials, F, is measured in volts, and the strength of the current C, in amperes, the product, FC, is expressed in watts or volt-amperes, one watt being the rate of doing work represented by 10^7 ergs per second **(39, 79)**.

By analogy with the horse-power of an engine, the rate at which work can be done by a battery may be called the *power* of the battery. It is a quantity which depends in part on the construction of the battery, but in part also on the conditions of its employment, just as the horse-power of an engine is not entirely determined by the characteristics of the engine itself, but also to some extent by the conditions of working, such as the speed, steam-pressure, &c.

131. Joule's Law.—Returning to the case of a battery with its terminals connected by a wire, as supposed in **(128)**, it is found that the temperature begins to rise as soon as the circuit is completed, and if the electromotive force amounts to a few volts, and the total resistance to an ohm or less, the elevation of temperature may become very appreciable, the wire becoming red, or even white hot, if it does not previously fuse or burn. As soon as the temperature of the wire has risen at all above that of surrounding objects, it begins to lose heat by conduction and radiation, and the temperature continues to rise until the rate at which heat is thus given out becomes equal to the rate at which it is generated in the wire by the current. In this case, heat is the only form of energy generated, and during the passage of a quantity of electricity, Q, the amount of heat, H, is such that

$$JH = EQ,$$

where J is the mechanical equivalent of heat. Assuming the strength of the current constant and applying Ohm's law, we may write this

$$JH = ECt = E^2t/R = C^2Rt,$$

where R is put for $b+r$ the total resistance of the circuit. Of the whole amount of heat produced, a part, say H_1, is generated within the battery, and the rest, H_2, in the connecting wire, the relative proportions being such that

$$H_1/H = b/(b+r) \quad \text{and} \quad H_2/H = r/(b+r).$$

The law of the generation of heat in the circuit, which we have here deduced from general considerations relating to electric energy, and have expressed above in three different but equivalent formulæ, was discovered experimentally by Joule about 1843, at a time when the true nature of electromotive force was very imperfectly recognised. It is consequently known as *Joule's law.*

The subjoined diagram (Fig. 120) may be taken as a geometrical representation of Joule's law: MK represents the electromotive force of the battery, MN the battery resistance, NO the external resistance,

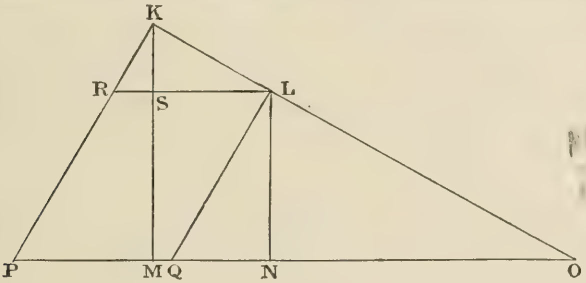


FIG. 120.

and the ratio MK : MO the strength of the current. Then if we draw KP at right angles to KO, the length, MP, represents the energy expended in the circuit per second,—that is to say, the energy converted into heat per second in the whole circuit. To get the heat generated in the external conductor and in the battery respectively, we must draw NL perpendicular to MO, LQ perpendicular to KO, and LSR parallel to OP: then NQ represents the external energy ($=JH_2$), and SR the internal energy ($=JH_1$).

If the units for the electrical quantities in the above formulæ are the volt, the ampere, and the ohm respectively, the energy developed is expressed in joules (**39, 114**). In order to get it expressed in units of heat, gramme-degrees, we have to remember that 1 joule $=10^7$ ergs, and that 1 gramme-degree $=4{\cdot}18 \times 10^7$ ergs. Hence, the heat in gramme-degrees is given by dividing the number of joules by 4·18, or by multiplying it by 0·239. Thus, for the heat produced per second, we have

$$H = 0{\cdot}239\ EC = 0{\cdot}239\ E^2/R = 0{\cdot}239\ C^2R.$$

These formulæ apply either to the heat developed in the complete circuit, or to that developed in any part of it, provided that appropriate values are assigned to the symbols.

The experimental verification of Joule's law is obtained by allowing a measured current to pass for a known time through a wire of known resistance, r, immersed in a calorimeter, and measuring the resulting quantity of heat, h, which, when the experiment is properly performed, is found to satisfy the equation

$$h = 0{\cdot}239\ C^2rt$$

Taking the law itself as established, the same kind of experiment as that just indicated may be employed for determining r, J, or C, when in each case the other quantities are known.

132. Heating of Wires.—If a chain be formed of bits of metal wire—platinum, for instance—alternately thick and thin, and a current be passed through, the thin wires will become red hot, while the stout wires are scarcely heated. In like manner with a chain formed of alternate pieces of platinum and silver wire of equal diameter, the former become red hot while the latter are still dark.

Again, if we take a long thin platinum or iron wire through which a current is passing, and we gradually diminish the length of the wire interposed in the circuit, it will be seen that this becomes more and more heated until it fuses. The strength of the current increases as the length of the wire diminishes. In like manner, if we take a long piece of iron wire and regulate the strength of the current so that the wire is at a dull red heat, and then cool a part of the wire by dipping it in water, the remainder becomes brightly incandescent. By cooling the wire its resistance is lessened (**115**), and therefore the strength of the current is increased; the effect is the same as if part of the wire had been taken out of the circuit.

If there were no loss of heat by radiation, the temperature of a conductor traversed by a continuous current would go on indefinitely increasing. The temperature which it reaches is that for which the gain at each instant is equal to the loss. We may admit Newton's law of cooling as at any rate approximately applicable to this case,—that is to say, the rate of loss by radiation at each instant is proportional to the excess of temperature of the wire over that of the surrounding medium,—then the quantity of heat lost in a second by a wire of length l, of radius r, of emissive power ϵ, for the excess of temperature θ, will be

$\epsilon.2\pi rl.\theta$; and if ρ is the specific resistance of the metal, and C the strength of the current, we have the equation

$$J\epsilon 2\pi rl\theta = \frac{l\rho}{\pi r^2} C^2$$

from which we get

$$\theta = \frac{1}{2\pi^2 J} \frac{\rho}{\epsilon r^3} C^2.$$

In applying this formula, it must be borne in mind that ϵ has only a simple meaning for an isolated bare wire radiating to distant external bodies; and that its value even then depends upon the condition of the surface and upon its radius (increasing as this diminishes) as well as upon the temperature. The conditions are quite changed if the wire is covered with an insulator, and still more if it is coiled on a reel. With copper it is in practice assumed that the current must not exceed six amperes for each square millimetre of cross-section if the wire is bare, and two or three if it is covered.

133. Generalisation of Joule's Law.—We have already stated (**129**) that under proper conditions the energy of the electric circuit can be employed, not only in generating heat, but in doing work of various other kinds. Without inquiring what kinds of work are done by the current between any two given points of the circuit, we may say that if C is the strength of the current in amperes, and e the difference of potentials between the points in volts, the rate of performance of work by the current in this part of the circuit is

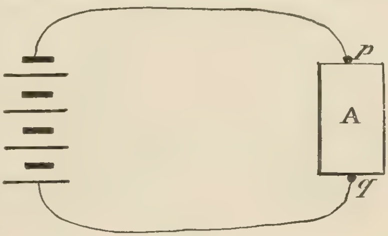


FIG. 121.

$$W = eC \text{ watts.}$$

Suppose, for example, a circuit made up as in Fig. 121, where the current of the battery is sent through some piece of apparatus enclosed in a sealed box, A, entering by the binding-screw, p, and leaving at q. We can measure the difference of potentials between p and q, and the strength of the current C, and thus determine eC. Without its being needful to open the box, or to know anything about the apparatus contained in it, or the effects that the current produces within it, this gives the algebraic sum of all the work done per second within the box. If r is the resistance

of the part of the circuit within the box, there will be a generation of heat, h, per second, such that

$$Jh = C^2r.$$

But there may be work of other kinds as well. Let w_1, w_2, . . . represent the amounts of work of various kinds, other than generation of heat; then

$$\begin{aligned} eC &= w_1 + w_2 + \ . \ . \ . \ + C^2r, \\ &= w + C^2r, \end{aligned}$$

if, for shortness, we put w for $w_1 + w_2 + \ . \ . \ .$, the sum of the work of all kinds except production of heat.

If $w = 0$, the whole of the energy expended between p and q is transformed into heat, and the quantity of this energy may be represented indifferently by C^2r or by eC. This is Joule's law in the special sense. But we may generalise the law so as to make it express the rate of expenditure of energy independently of the form or forms which that energy assumes. In words, the more general form of the law may be stated thus: the rate of expenditure of energy in any part of an electric circuit is equal to the strength of the current multiplied by the amount by which the potential at the point where the current enters exceeds that at the point where it leaves. In symbols we may write it—

$$W = eC = w + C^2r.$$

134. Inverse Electromotive Force.—Suppose the resistance of the complete circuit to be $R = r' + r$ ohms, r' being the resistance of the battery and of everything not included in A, and let the electromotive force of the battery be E, then we have the equation

$$EC = w + C^2(r' + r) = w + C^2R$$

or

$$E = w/C + CR.$$

Now, by Ohm's law, CR must be equal to the algebraic sum of all the electromotive forces of the circuit: let this be denoted by E'. Then

$$E' = E - w/C.$$

That is, when the energy of a current is not wholly expended in generating heat, in order to obtain the effective electromotive force of the circuit, we must *deduct* from the electromotive force of the battery the quotient obtained by dividing the sum of the work of other kinds done per second by the strength of the current. Hence, we conclude that the doing of work, other than the generation of heat, by a current involves the existence of a *negative* electromotive force

$$\epsilon = w/C.$$

If we apply similar reasoning to any part of the circuit instead of to the whole of it we may show that there is a negative electromotive force in that part if non-thermal work is being done; for the equation in **(133)** is equivalent to

$$e = Cr + \frac{w}{C}$$

that is, the drop of potential from end to end of the portion whose resistance is r is greater than that given by Ohm's law by the amount w/C; this last term therefore corresponds to an opposing electromotive force in the part considered.

135. Strength of the Current—Maximum Rate of doing Work.—Seeing that the battery, in the case now under consideration, is not the only seat of electromotive force, we cannot get the strength of the current by considering it alone: we require to take also into account the rate at which work is done. The equation used in the last paragraph—

$$EC = w + C^2R$$

gives for the strength of the current

$$C = \frac{E}{2R} \pm \sqrt{\frac{E^2}{4R^2} - \frac{w}{R}};$$

or

$$C = \tfrac{1}{2}C_0 \pm \sqrt{\frac{\frac{1}{4}C_0E - w}{R}},$$

if we put C_0 for the strength of current E/R that the given battery would maintain in the circuit if all the energy were transformed into heat. This equation indicates two strengths of current for any given value of w: for example if $w = 0$, we may have $C = 0$ or $C = C_0$. It is represented geometrically by the annexed diagram (Fig. 122), in which distances measured to the right denote values of w, and distances measured upwards denote values of C. The electromotive force of the battery is taken as 4 volts, and the resistance of the circuit as 0·04 ohm, so that $C_0 = 100$ amperes. The diagram indicates the double value of the current for a given value of w; for example, $C = 20$ and $C = 80$, both correspond with $w = 64$.

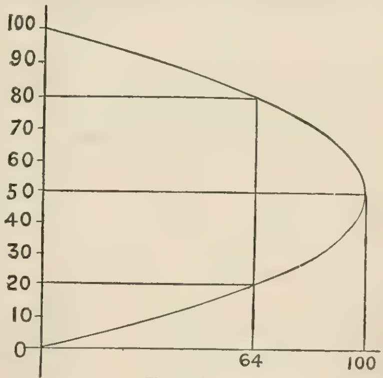


FIG. 122.

Again, the diagram shows very clearly that, however great the current may become, there is a maximum rate of work that cannot be exceeded. This corresponds with $C = \frac{1}{2}C_o$, and, with the values used in the diagram, is equal to 100. The same thing appears from the algebraical formula, in which the quantity under the sign of the square root would assume an impossible value if w were greater than $\frac{1}{4}C_oE$. With this limiting value of w, the quantity under the sign of the square root vanishes, and we get $C = \frac{1}{2}C_o$, as from the diagram.

136. Efficiency.—When a battery of electromotive force E maintains a current of strength C, it gives out energy at the rate EC, of which the amount C^2R per second is converted into heat, while the remainder, $EC - C^2R$, is available for doing work of other kinds. When a battery is employed, say, for driving an electromagnetic motor, or for producing chemical decomposition, the heat generated represents a loss of energy so far as the intended purpose is concerned, and it is only the remainder that is actually used. The ratio of the energy converted into useful work in a given time to the total energy given out by the battery in the same time, or

$$\frac{EC - C^2R}{EC} = \frac{C_o - C}{C_o} = u$$

is called the *efficiency* of the circuit. Since C_o is a constant quantity for the given circuit, it is evident that the value of u increases as C diminishes, and that it attains its maximum value $= 1$ when $C = 0$, and its minimum value $= 0$ when $C = C_o$. Efficiency $= 1$, or perfect efficiency, represents the conversion of all the energy given out by the battery into useful work; the other extreme case, $u = 0$, is when the whole of the energy of the battery is converted into heat. The actual rate of doing useful work is equal to the rate at which energy is given out by the battery multiplied by the efficiency, or

$$w = u \,.\, EC = (C_o - C)\, CR.$$

This equation shows that in the extreme case of perfect efficiency ($C = 0$) the actual amount of work done would vanish, for no energy would be given out by the battery. It also again shows that the maximum rate of doing work is when the inverse electromotive force spoken of in (**134**) is equal to $\frac{1}{2}E$ and therefore sufficient to reduce the strength of the battery current to $\frac{1}{2}C_o$; for the sum of the two factors $C_o - C$ and C in the expression for w is constant, and therefore the product is greatest when the factors are equal, or $C = \frac{1}{2}C_o$. In this case $u = 0{\cdot}5$, and

$$w = 0{\cdot}25EC_o = 0{\cdot}25E^2/R.$$

That is, the maximum rate at which useful work can be done is equal to one quarter the rate at which energy would be given out by the battery if it were all converted into heat.

From the diagram of (**135**) it is easily seen that, when the rate of work is 64 per cent. of the maximum, the efficiency may be 0·8 or 0·2. The greater value corresponds with the lower half of the curve, which represents the conditions that would always be employed in practice.

137. Maximum Power of a Cell.—Suppose a single cell of electromotive force e and resistance ρ, and let the external resistance be negligible, then the maximum rate of work obtainable from the cell is

$$\frac{e^2}{4\rho},$$

which is a constant quantity characteristic of the cell. The greatest value of e is about two volts in cells actually used. It approaches this value in a Grove's or Bunsen's cell, and slightly exceeds it in secondary cells with lead plates. The value of ρ may vary greatly, according to the size and arrangement of the plates, but it happens to be most easily made small in cells of the kinds for which e is great; hence the practical advantage of such cells.

Next suppose n similar cells arranged in q series of p cells each. The electromotive force of the combination will be $E=pe$, and the resistance $R=\frac{p}{q}\rho$; and, if we again suppose the external resistance to be negligible, the maximum attainable rate of work is

$$\frac{E^2}{4R}=\frac{p^2qe^2}{4p\rho}=n\frac{e^2}{4\rho};$$

that is to say, that the maximum rate of work is proportional to the total number of cells employed, but does not depend upon the manner in which they are combined.

CHAPTER XIV

THERMO-ELECTRICITY

138. Peltier's Phenomenon.—The generation of heat depending on resistance being **(131)** proportional to C^2R, is the same whether the current is positive or negative. It is in this sense like the generation of heat by ordinary mechanical friction, which is independent of the direction of the motion, and it may consequently be spoken of as the frictional generation of heat. But it is evident that if there is anywhere a fixed difference of potentials between adjacent parts of a conducting circuit, existing independently of the passage of a current, as, for example, an electromotive force acting at the surface of contact of two different metals, this must give rise to a further production of heat if the current passes from higher to lower potential, and to a disappearance of heat if the current flows in the opposite direction. The sign of any such effect would then be *reversible* with the direction of the current, and the quantity of heat produced in the one case or destroyed in the other, in a given time, would be proportional to the strength of the current.

Conversely, a reversible thermal effect occurring at any part of a circuit must be attributed to the existence at this part of a difference of potentials existing independently of the current. If at a part of the circuit there is a *fall* of potential Π as we pass along the circuit in the direction of the current, the reversible quantity of heat H generated by a current of strength C flowing for t seconds will be given by

$$JH = C\Pi t.$$

Now, an effect of this kind taking place at the surface of contact of two metals was discovered by Peltier in 1834, and is consequently known as the *Peltier effect*. If a current passes from antimony to bismuth, heat is produced at the surface of contact; and if a current passes from bismuth to antimony, heat is destroyed: similarly in the case of any other two metals, heat is generated if a current traverses the surface of contact in one direction, and is

consumed if it passes in the opposite direction, the quantity in a given time being in each case proportional to the strength of the current and to a coefficient (Π in the last equation) depending on the nature of the metals and on their temperature. This coefficient, known as the *Peltier coefficient*, or the coefficient of the Peltier effect, represents for a junction of any two metals the amount of energy converted into or produced from heat per second when a current of unit strength traverses the junction.

Experiments on the Peltier effect present some difficulties, from the fact that they are always complicated by the frictional generation of heat. It is not possible to observe the thermal effects occurring at the mere surface of contact of two metals: in any actual experiment we are obliged to include a finite portion of the circuit, extending from a point, say M, on one side of the junction to a point, say N, on the other side, and to observe the total amount of heat generated in a given time within this part of the circuit. If r is the resistance between the points M and N, the energy converted into frictional heat per unit of time is C^2r, and the energy converted into reversible heat is $C\Pi$. Hence, if H is the quantity of heat produced in t seconds when the current flows in one direction, we have

$$JH = (C\Pi + C^2r)t;$$

and, denoting the heat generated when the current flows in the opposite direction by H', we have

$$JH' = (-C\Pi + C^2r)t.$$

In order to determine the value of the Peltier coefficient for a given case, we may allow a current of constant strength to pass first in one direction, and then, for the same length of time, in the opposite direction: taking the difference of the two quantities of heat produced, we get in this way

$$\Pi = \frac{J(H - H')}{2Ct}$$

To get good results, the frictional or non-reversible term C^2r should be made as small as practicable by making MN, the portion of the circuit over which the experiment extends, short, and therefore r small, and by using a small current.

If an iron wire, AB, be connected by copper wires with the terminals of a battery, on completing the circuit a current will flow from copper to iron at one end, say A, and from iron to copper at

the other end. In this case heat will disappear at A, and will be generated at B; and if the current is sufficiently weak, there may be an actual fall of temperature at A, while the temperature at B rises. This effect may be shown in a striking form by surrounding the joint A with ice-cold water, and placing the joint B in ice. As the current passes, ice is formed at A, and an equal quantity is melted at B.

In a case like that just described, where two opposite junctions have the same temperature, the Peltier coefficient has the same value for both, but has opposite signs; hence the Peltier effects at the two junctions exactly compensate each other, and the quantity of heat generated in the circuit is that which a current of the same strength would produce in a homogeneous conductor of the same resistance.

A similar compensation occurs in a circuit containing any number of metallic junctions all at the same temperature; the sum of the reversible thermal effects vanishes, and Joule's law expresses the total generation of heat.

139. Thomson Effect.—Professor William Thomson (Lord Kelvin) discovered in 1856 that a phenomenon analogous to the Peltier effect occurs in a conductor of one and the same material when a current passes from a hotter to a colder part, or *vice versâ*, whence we must conclude that parts at different temperatures are not at the same potential, even when there is electrical equilibrium. Thus, for instance, if the ends A and B of a copper bar (Fig. 123) are kept at 0°, and an intermediate point, C, is raised to some

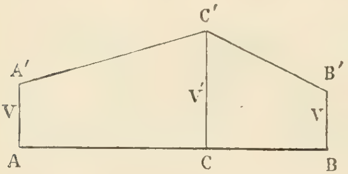


FIG. 123.

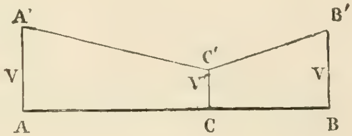


FIG. 124.

higher temperature, T, the potential increases continuously from A to C, and decreases by the same amount from C to B. Consequently, when electricity flows through the wire from A to B, there is an increase of electrical energy as it goes from A to C, and a corresponding quantity of heat is used up, while from C to B there is, to an equal extent, a decrease of electrical energy and a generation of heat. The final result thus shows itself as a transfer of heat from the part A C to the part C B—that is to say, in the direction of the current.

In iron, on the other hand, the effect is reversed: the potential decreases as the temperature rises, so that there is generation of heat between A and C (Fig. 124), and disappearance of an equal quantity between C and B, a result which is equivalent to a transfer of heat in the opposite direction to the current.

To show these effects, the ends A and B of the bar to be experimented upon may be kept at 0°, and the middle C at a higher temperature, such as 100°; then, if everything is symmetrical, the temperature at a point, M, between A and C will be the same as at M′, equidistant from C on the other side. On sending a strong current through the bar in the direction AB, the temperature will be raised throughout by the frictional development of heat, but, as the result of the effect we are considering, the temperature at M′ will be raised more than that at M in the case of copper, silver, zinc, cadmium, or antimony; while with iron, platinum, or bismuth, the temperature at M will become higher than that at M′. In metals of the former class, the Thomson effect is said to be *positive*; in those of the latter class the Thomson effect is *negative*; in lead it is not perceptible.

140. Thermo-Electric Currents.—In a circuit formed of different metals, if all the points where two different metals join are at the same temperature, the differences of potential to which both the Peltier and Thomson effects are due compensate each other all round the circuit, and there is no current. But if the joints are not all at the same temperature, this compensation no longer exists, and the circuit is traversed by a current. The production of currents under such conditions (*thermo-electric currents*), was discovered by Seebeck in 1821, therefore a good while before either of the allied phenomena that we have been discussing were known.

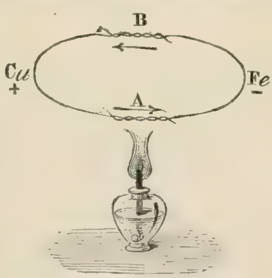


FIG. 125.

Suppose a circuit formed by twisting or soldering together at their ends wires of two different metals, as, for example, iron and copper (Fig. 125): then, if one junction, B, remaining at the ordinary temperature, the other, A, is raised to a higher temperature, a current is produced which goes from copper to iron at the heated junction, and from iron to copper at the cold one.

The metals named in the following list are arranged in such an order that if any two of them are formed into a circuit, as above described, there is a current across the heated junction from the

metal whose name occurs first in the list to that whose name occurs later:

↓	Bismuth.	Manganese.	Gold.	↑
	Nickel.	Silver.	Zinc.	
Hot.	Platinum.	Tin.	Iron.	Cold.
	Palladium.	Lead.	Arsenic.	
	Cobalt.	Copper.	Antimony.	

An actual difference of material is not essential to the production of a thermo-electric current, a difference of physical condition, such as can be brought about by mechanical treatment, being sufficient. For example, if the central portion of a platinum wire is wound into a rather close spiral and the ends of the wire are connected with a sensitive galvanometer, when one end of the spiral is heated (Fig. 126) the galvanometer indicates a current from the untwisted to the twisted part of the wire through the heated part.

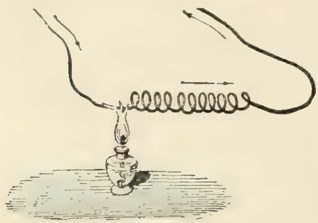


FIG. 126.

Mere dissymmetry of distribution of temperature on opposite sides of a part at which heat is applied does not produce a current if the material on both sides be chemically and physically identical. If the central part, $g\,g$, of a thick copper wire (Fig. 127) is turned

FIG. 127.

down to a much smaller diameter than the rest, no current is produced when the wire is made part of a closed circuit, and heat is applied at one of the points, g.

141. Thermo-Electric Inversion.—The direction of the current is in accordance with the rule just given, only so long as the average temperature of the two junctions is not above or below some definite value, depending on the two metals of which the circuit is formed. Thus with iron and copper, this temperature is about 275° C. If the average temperature of the junctions is above this point, or, what comes to the same thing, if the sum of the two temperatures is more than 550°, the current is from iron to copper at the hotter junction, and from copper to iron at the cooler one. If the sum of the two temperatures is equal to 550°, there is no current in the circuit. Supposing one junction kept permanently

at 0°, and the temperature of the other to be gradually raised, the circuit is traversed by a current which increases in strength, at first comparatively quickly, and afterwards more slowly, till the hotter junction reaches 275°, then decreases, slowly at first, and afterwards more rapidly, till the hot junction is at 550°, when it ceases. If the temperature is raised still higher, a current reappears, but in the opposite direction. If the junction whose temperature remains constant is kept at 100° instead of 0°, the maximum strength of current is again reached when the other junction is at 275°, but the temperature at which inversion of the current takes place is 450° instead of 550°.

142. Law of Successive Temperatures.—*For a given couple, the electromotive force obtained by raising the junctions to the temperatures* t_1 *and* t_2 *is the algebraic sum of the electromotive forces which are obtained by raising the junctions to the temperatures* t_1 *and* θ, *and then to the temperatures* θ *and* t_2.

This law may be expressed in the following manner:—

$$\overset{t_2}{\underset{t_1}{E}} = \overset{\theta}{\underset{t_1}{E}} + \overset{t_2}{\underset{\theta}{E}}.$$

143. Law of Intermediate Metals.—*In a thermo-electric circuit composed of two metals, A and B, with junctions at temperatures* t_1 *and* t_2 *respectively, the electromotive force is not altered if one or both the junctions are opened and one or more other metals are interposed between the metals A and B, provided that all the junctions by which the single junction at temperature* t_1 *may be replaced are kept at* t_1, *and all those by which the junction at temperature* t_2 *may be replaced are kept at* t_2.

If X is the intermediate metal, the law is expressed by the equation

$$\overset{t_2}{\underset{t_1}{E}}(AB) = \overset{t_2}{\underset{t_1}{E}}(AX) + \overset{t_2}{\underset{t_1}{E}}(XB).$$

Several important consequences result from this law.

In a thermo-electric couple it is immaterial whether the two metals are joined directly or are soldered together. If the circuit is cut at any point, and the two ends are connected with the binding screws of an electrometer or of a galvanometer, provided that the binding screws and the various parts in contact are all at the same temperature as the two ends of the circuit, no fresh electromotive force is introduced, and the electromotive forces measured are those of the original circuit.

144. Thermo-Electric Pile or Battery.—Thermo-electric forces are always very feeble, that of the bismuth-antimony pair, which is one of the strongest, and between 0° and 100° is nearly proportional to the difference of temperature of the two junctions, is 0·000057 volt for each degree. As, however, the resistance of couples wholly metallic may be made extremely small, such couples can give currents of considerable strength. Thus the bismuth-copper couple represented in Fig. 128, when heated at the junction B by a spirit lamp, strongly deflects a magnetic needle placed inside the circuit.

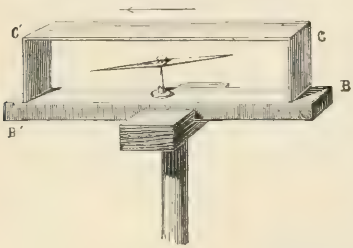


FIG. 128.

Any number of such couples may be associated together in such a way as to form a pile or battery. Melloni's pile, which has been of great service in the investigation of radiant heat, is formed of small bars of antimony, *aa*, arranged alternately with bars of bismuth, *bb*, so that all the even-numbered joints are on one side, and the odd ones on the other (Fig. 129). Several such

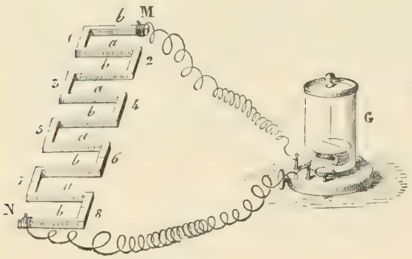


FIG. 129.

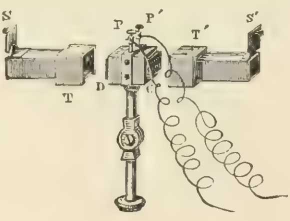


FIG. 130.

piles are connected together, the couples being insulated by mica or asbestos, which is not altered by heat; they thus form a rectangular parallelopipedon (Fig. 130), the faces C and D being those which correspond to the junctions. These faces are covered with lamp-black, and protected by the screens T and T′. The two poles being connected with a galvanometer, a very slight difference of temperature between the two faces is sufficient to produce an appreciable deflection.

Clamond's battery, which is occasionally used in laboratories, is made up of an alloy of zinc and antimony, alternating with an alloy of zinc, nickel, and copper. The metals are arranged in

series so as to form a sort of cylinder (Fig. 131); the even-numbered junctions are inside and are heated by a gas-flame, while the odd-numbered junctions are on the outside, and are arranged so that they present a large cooling surface to the air.

We may cite as an example a Clamond battery consisting of 120 couples, having an electromotive force of 8 volts and a resistance of 3·2 ohms, which, therefore, at the maximum, has a power of 5 watts. The couples may be grouped so as to satisfy in each case the conditions of the maximum (**136**). To produce this current 180 litres of coal-gas are consumed per hour.

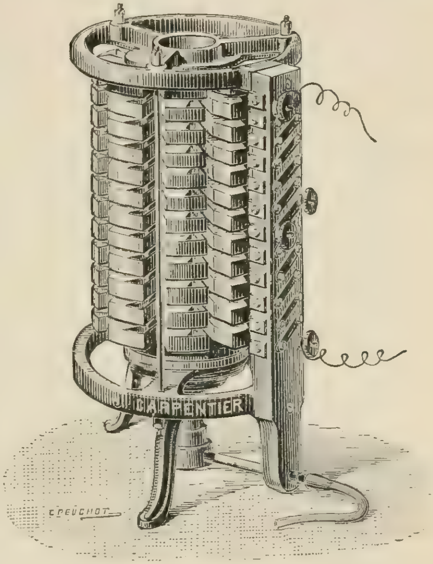


FIG. 131.

Assuming that a litre of gas gives 5200 gramme-degrees of heat, the heat expended is 260 gramme-degrees per second, which corresponds to a power of 1084 watts. The efficiency is about $\frac{1}{200}$.

145. Theory of Thermo-Electrical Phenomena.—The existence of thermo-electric currents shows clearly that the resultant electromotive force in a metallic circuit is a function of the temperature. In order to investigate the matter, let us again take the case of a circuit formed by joining at the ends a piece of iron wire and a piece of copper wire. It is known by experiment (**138**) that heat disappears when an electric current traverses the junction of the two metals in the direction from copper to iron, and that heat is produced by a current traversing the junction from iron to copper. Hence we infer that, at ordinary temperatures, the potential of iron is higher than that of copper to an extent measured by the value of the Peltier coefficient for the two metals, that is to say, by the energy required to transfer a unit of electricity from copper to iron. It is further known (**140**) that if, one of the junctions remaining at the temperature of the air, the other is gradually heated (Fig. 125), the circuit is traversed

by a current whose direction is from iron to copper at the cold junction, and from copper to iron at the heated one. The direction of the current may also be stated as being from hot to cold in the iron wire, and from cold to hot in the copper.

In seeking for the origin of the electromotive force in a thermo-electric circuit, we must bear in mind (**129**) that the electromotive force is equal numerically to the work done during the passage of a unit quantity of electricity round the circuit, and that this amount of work must be equal to the energy simultaneously taken in by the circuit. A simple thermo-electric circuit consists of two homogeneous wires and two junctions, and any energy taken in by it must be taken in at one or more of these four parts.

In the case we have supposed, the direction of the current, being that of increasing temperature in the copper and that of decreasing temperature in the iron, is from lower to higher potential in both (**139**), and therefore there is a *gain* of energy in both wires as the current passes. Energy is also *gained* at the heated joint, where the current flows from copper to iron, that is, from lower to higher potential, but there is *loss* of energy at the other joint, where the direction of the current is from iron to copper. The net gain is therefore the amount by which the sum of the energy taken in, in the continuous wires and the hot junction taken together, exceeds that given out simultaneously at the cold junction, and the electromotive force is numerically equal to this net gain of energy per unit of electricity traversing the circuit.

In order to express this relation by means of an equation, we may use the symbol σ as the coefficient of the Thomson effect; that is to say, when a current of strength C flows for t seconds in a given metal, from a part where the temperature is T to a part where the temperature is higher by the very small amount dT, the energy taken in is equal to the quantity of electricity Ct multiplied by the small difference of temperature dT and by the coefficient σ depending on the nature of the metal, or

$$Ct\,\sigma\,dT.$$

If T_1 be the temperature of the colder, and T_2 that of the hotter junction, we may suppose an infinite number of points taken one after another along each wire, so that the temperature at each point exceeds that at the preceding point by dT. Then the energy gained by the flow of a current along each elementary length of wire intervening between two consecutive points will be represented by $Ct\sigma dT$, if the direction of the current is the direc-

tion of increasing temperature (or dT positive), and by $-Ct\sigma dT$, if the current flows in the direction of decreasing temperature (dT negative). The total gain of energy due to the whole wire will be expressed by the sum of all such terms from one end to the other. The factor Ct is common to all, but the value of σ for a given metal depends in general on the temperature; hence we cannot calculate the required sum by adding together all the intervals of temperature dT, so as to get the total difference $T_2 - T_1$, and multiplying this by $Ct\sigma$, but we must, before adding, multiply the temperature-difference dT corresponding to each successive element of the wire by the value of σ applicable to it, that is, we must take the integral $\int_{T_1}^{T_2} \sigma dT$, and multiply this by Ct.

Distinguishing the values of σ corresponding to copper and iron respectively as σ_{Cu} and σ_{Fe}, and remembering that the direction of the current is from cold to hot in the copper, and from hot to cold in the iron, we have, for the total gain of energy due to the continuous wires,

$$Ct\left(\int_{T_1}^{T_2} \sigma_{Cu} dT - \int_{T_1}^{T_2} \sigma_{Fe} dT\right).$$

We have next to consider the effect of the junctions. We have already (**138**) seen, that the coefficient of the Peltier effect for the junction of two metals measures their permanent difference of potentials. Putting Π_1 for the value of this coefficient in the case of iron and copper at the temperature T_1 of the cooler junction, and Π_2 for the value of the same coefficient at the temperature T_2 of the hot junction, the gain of energy at the latter may be written $Ct\Pi_2$, and the loss at the former $Ct\Pi_1$. Hence the net gain of energy at the two junctions taken together is

$$Ct(\Pi_2 - \Pi_1).$$

If we denote the electromotive force of the circuit by E, the work done by the current in time t, or the energy given out, is ECt. Writing this as equal to the whole amount of energy taken in, we get the equation,

$$ECt = Ct\int_{T_1}^{T_2} \sigma_{Cu} dT - Ct\int_{T_1}^{T_2} \sigma_{Fe} dT + Ct\Pi_2 - Ct\Pi_1.$$

The quantity of electricity traversing the circuit $Ct = Q$, is a factor of every term of this equation. If we divide throughout by this, or, what comes to the same thing, if we suppose the quantity

of electricity to be unity, we get, as the expression for the electromotive force,

$$E = \int_{T_1}^{T_2} (\sigma_{Cu} - \sigma_{Fe}) dT + \Pi_2 - \Pi_1.$$

146. Application of Thermo-Dynamics.—The only form in which energy is supplied to a thermo-electric circuit is that of heat, but the work done by it may, in part at least, take any form we please. For example, by including in the circuit an electro-magnetic motor, a greater or less amount of mechanical energy may be obtained. If, therefore, we leave out of account the intermediate processes and attend only to final results, we may say that a thermo-electric circuit employed to maintain the motion of an electro-magnetic motor is equivalent to a steam-engine, or any similar contrivance, the function of which is to convert heat into mechanical energy.

It is thus natural to inquire how far the general conditions which have been shown to apply to the transformation of heat into mechanical energy in the steam-engine, or other forms of heat-engine, are applicable to the corresponding transformation by means of a thermo-electric circuit. The most important of these principles is that, in the case of a reversible engine,—that is to say, an engine such that an inversion of the direction of the motion is accompanied by an exact reversal of the dynamic and thermal processes going on in it,—the sum of the quantities of heat taken in by the engine, each of them divided by the temperature, reckoned from absolute zero, at which it is taken in, is equal to the sum of the quantities of heat given out, each of them divided by the temperature at which it is given out. The application of this conclusion to the case of a thermo-electric circuit leads to very important consequences.

But before this application can be made, certain conditions must be discussed. A thermo-electric circuit cannot in general be regarded as reversible. Part of the energy of the circuit, namely, the amount C^2R per second, R being the resistance of the circuit, is converted into heat in a non-reversible manner, that is to say, the production of heat is independent of the direction of the current and the application of heat to the circuit cannot be made to reproduce the equivalent quantity of electrical energy. The evolution and absorption of heat in consequence of the Peltier and Thomson effects are, however, reversible; for, being each of them proportional to the strength of the current, they are reversed when the current is reversed. The non-reversible evolution of heat

being thus proportional to the *square* of the current-strength, and the reversible effects being proportional to the *first power*, it follows that, if the strength of the current is sufficiently small, the former will be very small as compared with the latter, and may, without sensible error, be neglected. To fulfil this condition, we have only to suppose that the temperatures T_1 and T_2 are very nearly the same: let them be T and $T+dT$ respectively, differing from each other by the indefinitely small amount dT. In this case the electromotive force, and therefore the strength of the current, will be very small.[1]

Going back to the equation at the end of (**145**), and assuming the difference of temperature to be very small, we may write it

$$dE = (\sigma_{Cu} - \sigma_{Fe})dT + d\Pi.$$

Where $d\Pi$ stands for the increment of the Peltier coefficient corresponding to an increment of temperature dT. Under the conditions here supposed, the processes taking place in the circuit are all reversible; we may therefore apply to them the thermo-dynamic conclusion referred to above.

The heat absorbed in the continuous parts of the circuit divided by the temperature at which it is received, gives $(\sigma_{Cu} - \sigma_{Fe})\dfrac{dT}{JT}$ per unit of electricity, if we put J for the mechanical equivalent of heat. To this is to be added the heat absorbed at the hotter junction divided by the temperature of the junction; this may be written $\dfrac{\Pi + d\Pi}{J(T+dT)}$. The only part of the circuit at which heat is given out is the cooler junction. Dividing the heat here evolved by the temperature of the junction, we get $\dfrac{\Pi}{JT}$. Then, equating the value applicable to the heat taken in and that corresponding to the heat given out, we have

$$\frac{(\sigma_{Cu} - \sigma_{Fe})dT}{T} + \frac{\Pi + d\Pi}{T+dT} = \frac{\Pi}{T},$$

whence, by reduction,

$$(\sigma_{Cu} - \sigma_{Fe})\,dT = \frac{\Pi}{T}\,dT - d\Pi.$$

[1] Without making the difference of temperatures infinitely small, the current might be made as small as we please by a sufficient increase of resistance. But by writing EC instead of C^2R to express the non-reversible production of heat, we see that, so long as E is finite, this is a small quantity of the first, not of the second, order, and consequently would not vanish in comparison.

Putting this value on the right-hand side of the equation for dE above, we get

$$dE = \frac{\Pi}{T} dT.$$

If the junctions have the temperatures T_1 and T_2 respectively, the electromotive force of the circuit is represented by

$$E = \int_{T_1}^{T_2} \frac{\Pi}{T} dT.$$

147. Values of the Peltier and Thomson Coefficients.—It has been pointed out already (**138**) that the direct experimental measurement of the coefficient of the Peltier effect presents a good deal of difficulty, but the last equation but one of (**146**) contains the principle of another method of measuring this quantity. Writing this equation in the form

$$\Pi = T \frac{dE}{dT}$$

we get the Peltier coefficient expressed in terms of the rate of change of the electromotive force of the circuit with temperature.

In this formula dE stands for the infinitesimal electromotive force of a circuit whose junctions are at the absolute temperatures T and $T + dT$ respectively; but, by the law of successive temperatures (**142**), this is the same as the increment of electromotive force in a circuit with one junction kept at a fixed temperature when the other is raised from temperature T to $T + dT$.

Now it is found by experiment that the connection between electromotive force and temperature, for a circuit formed of a given pair of metals, can, in a large number of cases and between wide limits of temperature, be expressed by the formula

$$E = a(T_2 - T_1) + b(T_2^2 - T_1^2),$$

where a and b are constant coefficients whose values depend on the particular metals employed. They can be determined experimentally in each case by measuring the electromotive forces corresponding to various values of the temperatures T_1 and T_2. If, in this formula, T_1 stands for the temperature of one junction which is kept constant, while T_2 varies, we get, by differentiating with respect to T_2,

$$\frac{dE}{dT} = a + 2bT_2,$$

and therefore, for the value of Π at temperature T_2,

$$\Pi = T_2 \frac{dE}{dT} = aT_2 + 2bT_2^2.$$

For copper and iron,

$$a = 3187, \text{ and } b = - 2\cdot91.$$

With these numbers the value of Π can be calculated for any given temperature: the following table gives the results for a few temperatures:—

Temperature from absolute zero .	$= T$	273°	373°	473°	547·6°	573°
Temperature centigrade	$= t$	0	100	200	274·6	300
Π (ergs per unit of electricity)		436,000	379,000	205,000	0	−85,000

The value of $\sigma_{Cu} - \sigma_{Fe}$ can also be expressed by means of the same coefficients. Referring to (**146**), we see that

$$\sigma_{Cu} - \sigma_{Fe} = \frac{\Pi}{T} - \frac{d\Pi}{dT}.$$

Consequently

$$\frac{\sigma_{Cu} - \sigma_{Fe}}{T} = -\left(\frac{1}{T}\frac{d\Pi}{dT} - \frac{\Pi}{T^2}\right) = -\frac{d\left(\frac{\Pi}{T}\right)}{dT}.$$

But (see above)

$$\frac{\Pi}{T} = \frac{dE}{dT},$$

therefore,

$$\sigma_{Cu} - \sigma_{Fe} = -T\frac{d\frac{dE}{dT}}{dT} = -T\frac{d^2E}{dT^2}.$$

Writing, as before, the electromotive force in the form

$$E = a(T - T_1) + b(T^2 - T_1^2),$$

where T_1 is the fixed temperature of the cooler junction, we have

$$\frac{dE}{dT} = a + 2bT$$

and

$$\frac{d^2E}{dT^2} = 2b.$$

Therefore, finally,

$$\sigma_{Cu} - \sigma_{Fe} = -2bT.$$

This gives at 0° C. ($T = 273$)

$$\sigma_{Cu} - \sigma_{Fe} = 1589,$$

and at 300° ($T = 573$)

$$\sigma_{Cu} - \sigma_{Fe} = 3335.$$

In the same way the difference between the values of the coefficients of the Thomson effect can be calculated for any two metals when the electromotive force of a thermo-electric circuit formed of these metals has been determined for various intervals of temperature.

It is found that the passage of an electric current through lead, between parts at different temperatures, is not accompanied by a reversible production or destruction of heat to an extent capable of being detected by experiment. Hence the coefficient σ for lead must be either accurately or nearly $= 0$.

Suppose, then, that a thermo-electric circuit is made up of lead and another metal, say iron, and that the electromotive force of the circuit has been measured for various ranges of temperature. The results enable us to determine the values of the coefficients a and b, applicable to this circuit, in the equation—

$$E = a(T - T_1) + b(T^2 - T_1^2)$$

and therefore to calculate

$$\sigma_{Pb} - \sigma_{Fe} = -2bT,$$

or, putting

$$\sigma_{Pb} = 0,$$

$$\sigma_{Fe} = 2bT.$$

In this way the value of σ_{Fe} has been found to be

at 0° C. . . .	-1330
at 300° C. . . .	-2791

Combining these values with those given above for $\sigma_{Cu} - \sigma_{Fe}$, we get for σ_{Cu},

at 0° C. . . .	$+259$
at 300° C. . . .	544

If we distinguish by suffixes the values of the coefficients a and b, in the case of circuits made up of three given metals, 1, 2, and 3, when 1 and 2, 2 and 3, or 1 and 3 are taken together, the law of intermediate metals (**139**) gives the following relations—

$$a_{1,3} = a_{1,2} + a_{2,3}$$
$$b_{1,3} = b_{1,2} + b_{2,3},$$

by which, when the values applicable to two of the pairs have been determined, that corresponding to the third pair can be found.

148. Neutral Point.—As already stated (**141**), when the mean temperature of the junctions of a copper-iron circuit is about 275° C. ($= 548°$ from absolute zero) the current ceases. With circuits formed of other metals, the same thing happens at

other temperatures. The formula of (**147**) for the electromotive force of a circuit, when written thus—

$$E = (T_2 - T_1)\left(a + 2b \cdot \frac{T_2 + T_1}{2}\right)$$

shows that E must vanish whenever the mean temperature of the junctions satisfies the condition.

$$\tfrac{1}{2}(T_2 + T_1) = -\frac{a}{2b}.$$

The temperature thus determined is called the *neutral temperature* for the given pair of metals. If we denote it by the symbol T_n, we get for the coefficent a the value

$$a = -2bT_n.$$

Putting this into the formula for electromotive force, we may write this as containing the product of two differences, namely,

$$E = 2b(T_2 - T_1)\left(\frac{T_2 + T_1}{2} - T_n\right),$$

which shows that it must vanish when either difference vanishes. The first case is when both junctions have the same temperature; the second is when the mean temperature is equal to the neutral temperature. The meaning of the neutral temperature for a given pair of metals becomes more apparent when we consider the value of the corresponding Peltier coefficient. As we have seen, this is expressed by the formula

$$\Pi = aT + 2bT^2.$$

Expressing a in terms of the neutral temperature, this becomes

$$= -2bTT_n + 2bT^2 = 2bT(T - T_n).$$

It follows that the value of the Peltier coefficient for a given pair of metals which are at the neutral temperature is zero, or that no reversible thermal effect accompanies the passage of a current from one to the other at this temperature.

When the temperature of one junction is kept constant, the electromotive force reaches a maximum when the second junction is at the neutral point; and if one junction is kept at the neutral point, the current is in one direction or the other round the circuit, according as the temperature of the second junction is below or above the neutral point.

149. Geometrical Representation of the Phenomena.—Two methods may be used to represent thermo-electrical phenomena geometrically.

Suppose a couple formed of two metals, A and X, one of the

junctions being kept at 0°, and the other at any temperature, t. Take the values of t as abscissæ (Fig. 132), and the corresponding electromotive forces as ordinates; a curve, AX, is obtained, which is, in fact, a parabola with a vertical axis.

If the cold junction, instead of being at 0°, is at the temperature t_1, the same curve still gives the electromotive forces, provided we measure the ordinates from the line MM′ drawn parallel to the axis of abscissæ through the point M of the curve, which corresponds to the temperature t_1. This is a consequence of the law of successive temperatures (**142**).

The point M′, where the curve cuts the parallel MM′ to the axis of abscissæ, gives the temperature at which the electromotive force changes its sign, and gives, consequently, the temperature

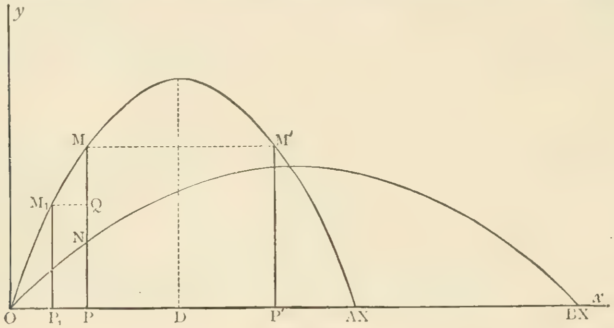


FIG. 132.

of inversion (**141, 148**). This temperature varies, as we see, with that of the cold junction.

The temperature t_m of the maximum electromotive force, on the contrary, is fixed, and is represented on the diagram by the point D. It will be seen that it is the mean of the temperatures of the cold junction and of the temperature of inversion.

In like manner, let BX be the curve corresponding to the couple formed by the metals B and X; the difference MN of the two ordinates represents, for the temperatures 0° and P of the two junctions, the electromotive force of the couple formed by the two metals A and B. We have, in fact, from the law of intermediate metals— $$E(AX) = E(AB) + E(BX).$$

150. Thermo-Electric Height.—Another method is to plot temperatures as abscissæ, and as ordinates the values of

$$\frac{dE}{dT} = \frac{\Pi}{T},$$
$$= a + 2bT,$$

which are known as *thermo-electric heights.* The line AB in Fig. 133 represents part of a curve so drawn. The ordinate of the point where it intersects the vertical axis represents the value of a; the tangent of the angle which it makes with the axis of temperature gives the value of $2b$. The area of the vertical strip, PMM′P′, being the product of the vertical height into the base, represents the value of dE, namely, the thermo-electric height $\frac{\Pi}{T}$ multiplied by

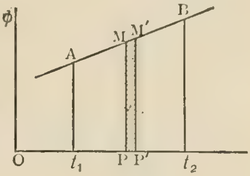


FIG. 133.

the infinitesimal difference of temperature dT of the junctions. The sum of all such areas between the lines t_1A and t_2B is the area of the figure t_1ABt_2, and is expressed algebraically as $\int_{t_1}^{t_2}\frac{\Pi}{T}\,dT$, and therefore represents the electromotive force of a circuit, formed of the two metals for which the figure is drawn, with junctions at the temperatures t_1 and t_2 respectively.

In Fig. 134 the area PMM′P′ represents in like manner the electromotive force of a circuit of two metals, say A and X, with junctions at the temperatures denoted by the points P and P′. Similarly the area PNN′P′ represents the electromotive force for the same range of temperature of a circuit formed of the metals B and X.

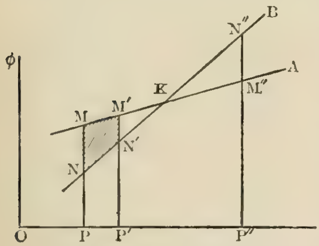


FIG. 134.

From the law of intermediate metals it follows that the difference of these areas, or NMM′N′, represents the electromotive force —still for the same range of temperature—of a circuit formed of the metals A and B. The abscissa of the point K, where the lines MM′ and NN′ intersect, represents the neutral point for the metals A and B, and the area NMM′N′ increases with rise of temperature of the hotter junction until this point is reached. An area like N″KM″ to the right of the point K is to be reckoned negative, and represents an electromotive force acting the opposite way round the circuit to one represented by an area to the left of K. If the temperature of one junction is below, and that of the other is above the neutral point, the electromotive force of the circuit is represented by the difference of the areas to the left and right of the point K respectively, and vanishes when the temperatures

are such that these areas are equal, that is, as the figure indicates, when the temperatures of the two junctions are equidistant from the neutral point.

151. Thermo-Electric Diagram.—Fig. 135 gives a thermo-electric diagram, according to Professor Tait, for the principal metals. Lead is the metal taken as standard, since this does not show the Thomson effect. The abscissæ represent temperatures in centigrade degrees, and the ordinates the thermo-electric heights in microvolts (or millionths of a volt) per degree.

All the curves, excepting those for iron and nickel at high temperatures, are straight lines. In order to obtain the electromotive force of a couple AB working between the temperatures t_1 and t_2,

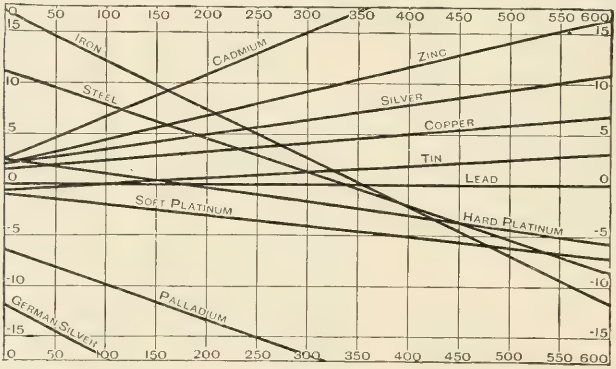


FIG. 135.

the area to be calculated is that of a trapezium, such as MNM′N′ (Fig. 134); it is sufficient to multiply half the sum of the two parallel sides by their distance apart. This distance is the difference $t_2 - t_1$ of the two temperatures; half the sum of the parallel sides is equal to the portion intercepted by the lines A and B on the ordinate which corresponds to the mean $\frac{1}{2}(t_2 + t_1)$ of the temperatures of the two junctions.

For instance, let one of the junctions of an iron-copper couple be at 50°, and the other at 250°; the diagram gives about 7 microvolts for the difference of the ordinates corresponding to the mean temperature, 150°: the electromotive force will then be $7 \times 200 = 1400$ microvolts or $1{\cdot}4 \times 10^{-3}$ volts.

152. As the area N′M′MN (Fig. 136) represents the electromotive force in volts, it also represents in joules the work done by a coulomb traversing the circuit. In the case of the figure the

current flows in the direction N′M′MN; in other words, the metal B is positive in reference to A.

The outline of the area N′M′MN may be considered as representing the cycle traversed by the electricity, and if we assume that the temperatures are measured from absolute zero, the area comprised between one of the bounding lines of the figure, the axis $O\phi$, and two lines parallel to the axis of temperature, represents the work depending on the corresponding element of the cycle.

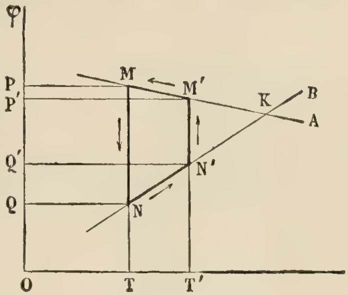


Fig. 136.

Thus NN′ represents the passage of a unit of electricity from the lower temperature T to the higher temperature T', along the conductor B, and the area NN′Q′Q the energy taken in in this part of the circuit in consequence of the Thomson effect.

The line N′M′ denotes the passage of electricity from the conductor B to the conductor A at the temperature T', and the area N′M′P′Q′ the energy corresponding to the Peltier effect.

The line M′M denotes the passage of electricity along the conductor A from the temperature T' to the temperature T, and the area M′MPP′ represents the energy corresponding to the Thomson effect in the conductor A.

Finally, the line MN expresses the passage of electricity from the metal A to the metal B through the cold junction at the temperature T, and the area MNQP the energy corresponding to the Peltier effect.

The diagram is so drawn that gain of energy corresponds to the passage of a current from a lower to a higher point, and loss of energy to the passage from higher to lower. Consequently the areas QNN′Q′, Q′N′M′P′, and P′M′MP represent energy gained during the passage of a coulomb of electricity round the circuit, and the area PMNQ represents energy given out. The net gain of energy per coulomb, or the electromotive force in volts, is represented by the difference of these areas, namely, NN′M′M.

152.* Circuit of Three Metals in Series.—When a circuit consists of three metals in series, the junctions being all at different temperatures, the thermo-electric electromotive force is given by the sum of the terms arising from the various wires and junctions. If the three metals are denoted as in Fig. 136A by the letters a, b, and c, and the junctions

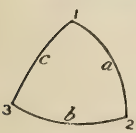


Fig. 136A.

by the figures 1, 2, and 3, the total electromotive force equals

$$E = \underset{ab}{\pi_2} + \underset{bc}{\pi_3} + \underset{ca}{\pi_1} + \int_1^2 \sigma_a d\tau + \int_2^3 \sigma_b d\tau + \int_3^1 \sigma_c d\tau$$

where $\underset{ab}{\pi_2}$ is the Peltier coefficient between the metals a and b at the temperature 2 and similarly for like terms; and $\int_1^2 \sigma_a d\tau$ is the contribution from the wire a, whose Thomson coefficient is σ_a, the integration being between the extreme temperatures of this wire, and similarly for like terms.

This may be written (by simultaneously subtracting and adding several terms)—

$$E = \underset{ab}{\pi_2} - \underset{ab}{\pi_1} + \underset{bc}{\pi_3} - \underset{bc}{\pi_1} + \left\{ \underset{ab}{\pi_1} + \underset{bc}{\pi_1} + \underset{ca}{\pi_1} \right\}$$

$$+ \int_1^2 (\sigma_a - \sigma_b) d\tau + \int_3^1 (\sigma_c - \sigma_b) d\tau$$

$$+ \left\{ \int_2^3 \sigma_b d\tau + \int_3^1 \sigma_b d\tau + \int_1^2 \sigma_b d\tau \right\}$$

But $\underset{ab}{\pi_1} + \underset{bc}{\pi_1} + \underset{ca}{\pi_1}$ is the total electromotive force when the three junctions are all at the same temperature and is therefore zero; and the other expression in brackets is obviously equivalent to $\int_1^1 \sigma_b d\tau$ and is therefore also zero. The remaining terms can be grouped into two pairs: one of which is the value of the e.m.f. for a circuit of the two metals a and b, the junctions being at temperatures 2 and 1; and the other is the e.m.f. for a circuit consisting of b and c, the junctions being at temperatures 3 and 1. This result may be written

$$\underset{abc}{E} = \underset{ab}{E_1^2} + \underset{bc}{E_1^3}.$$

Obviously two other expressions for the same quantity can be written down from analogy

$$\underset{abc}{E} = \underset{bc}{E_2^3} + \underset{ca}{E_2^1}$$

$$= \underset{ca}{E_3^1} + \underset{ab}{E_3^2}.$$

These results are readily obtainable in an alternative way. Let an auxiliary conductor b' of the same metal as b connect any point on b with a point on c which is as near as may be to the junction 1 (Fig. 136B), so that the minute part of c which is intercepted may be considered as being all at one temperature and therefore non-contributory to the total effect (**143**). Then the electromotive force sought for is the sum of the electromotive forces in the two circuits to which the wire b' is common. The value for the right-hand circuit is that for two metals a and b with their junctions at temperatures 2 and 1 $(\underset{ab}{E_1^2})$; and similarly for the left-hand circuit the value is $\underset{bc}{E_1^3}$. The addition of these two terms gives the same result as before. The alternative expressions may be obtained in succession by aid of auxiliary wires joining points on a and c respectively to the immediate neighbourhood of the opposite junctions.

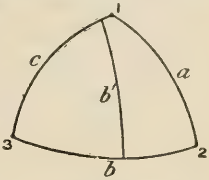


FIG. 136B.

It will be easily recognised that the terms which are added and subtracted in the calculation given at the beginning of this section represent the effect of introducing the auxiliary conductor b', which in the result cancels out, since its effect as belonging to the circuit abb' is exactly equal and opposite to that which it has as a part of the circuit $b'bc$.

153. Measurement of Temperatures by Thermo-Electric Couples.—As the electromotive force of a thermo-electric couple is, for a fixed temperature of the cold junction, a function of the temperature of the hot junction, it can be used to determine this latter temperature.

A simple arrangement for this purpose consists in twisting together, as at O (Fig. 137), the ends of two wires, AB, forming a couple, and connecting the other two ends to a galvanometer, G, all the junctions from A and from B being at the same temperature. It is necessary to know once for all the curve of the electromotive forces as a function of the temperature, and to be certain that the couple is comparable with itself, and is not altered by being reheated. Care must also be taken to work below the point of inversion.

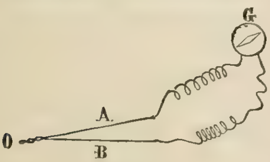


FIG. 137.

A couple of pure platinum and an alloy of rhodium with platinum is well adapted for measuring temperatures up to 1200°, and

towards this limit indicates temperatures to within 10 or 20 degrees, which in most cases is sufficient.

For temperatures below 100°, by the arrangement known as Becquerel's *thermo-electric needle* (Fig. 138), the curve can be dis-

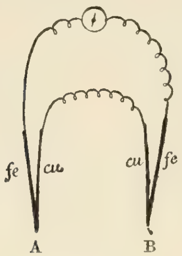


FIG. 138.

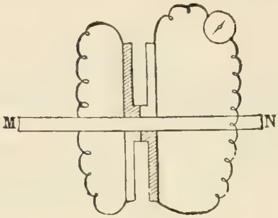


FIG. 139.

pensed with. Two identical couples are joined in series in opposition to each other. The junction A being placed at the point whose temperature is to be determined, the junction B is placed in a bath, the temperature of which can be varied until there is no deflection of the galvanometer. The temperature of the bath is then the temperature to be measured.

For very small differences of temperature, it is advantageous to use a bismuth-antimony couple (Fig. 139). So long as the temperature of 100° is not exceeded, the current is proportional to the difference of temperature of the two junctions. By arranging two couples in series, as shown in Fig. 139, we have Peltier's thermo-electric clip, by which the temperature of any given body may be taken, such, for instance, as that at any part of a bar, MN. This apparatus was used by Peltier in investigating the so-called "Peltier effect" at the junction of two metals.

Melloni's pile, described above (Fig. 130), constitutes a differential thermometer of extreme sensitiveness. When the two faces are at the temperature of the surrounding air, and heat is allowed to radiate against one end, for instance, if the hand is held in front of it, a difference of temperature is established between them, which is proportional to the radiation, and therefore a current is produced whose intensity is proportional to this radiation.

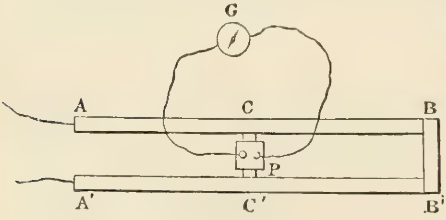


FIG. 140.

As a further example, may be cited the application of the pile by M. Leroux to measuring the Thomson effect (**139**). If we suppose two similar bars, AB, A′B′ (Fig. 140), placed parallel to each other and joined by a cross-piece, BB′, at the heated end, a thermo-electric pile, placed with its faces in contact with two symmetrical points, C and C′, will indicate a difference of temperature between these points when a strong current is passed through the bars from A to A′ or in the opposite direction.

CHAPTER XV

CHEMICAL ACTION OF THE CURRENT

154. Electrolysis.—If the interpolar wire of a battery is cut, and the two ends are then placed in a liquid (Fig. 141), so that the circuit is completed by a column of liquid, two cases may present themselves. The liquid may act like air, as a perfect insulator, and then no current passes; or the current passes, and then, except in the case of mercury or any melted metal, the liquid is decomposed. A non-metallic liquid never acts like a simple conductor; it never allows any quantity of electricity to pass without a correlated decomposition.

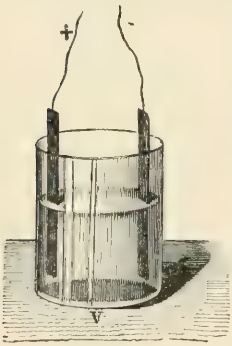


FIG. 141.

This phenomenon is called *electrolysis;* the term *electrolyte* is applied to the liquid which undergoes decomposition; and the conductors by which the current enters and leaves the liquid are called *electrodes;* that in connection with the positive terminal is called the *positive electrode*, or sometimes the *anode*, and that connected with the negative terminal is the *negative electrode* or *kathode.*

The only bodies which are susceptible of electrolysis are apparently salts liquefied either by solution or by fusion. Perfectly pure liquids, such as water, alcohol, ether, bisulphide of carbon, &c., are not electrolytes. We understand by the term salt a compound formed of a metal united either to an element such as Cl, Br, S, or to a compound radical such as SO_4, NO_3, . . . &c. The primary decomposition which takes place under the action of the current appears always to consist in the separation of the metal from the simple or compound radical with which it was combined.

The constituents into which a body is decomposed never appear in the mass of the liquid itself, but only at the electrodes: the metal at the negative and the radical at the positive electrode.

Thus if we use as electrodes two platinum plates, and immerse them in solution of sulphate of copper, copper is deposited on the negative electrode, with its characteristic colour, while at the positive electrode oxygen is given off in the form of gas, and sulphuric acid, SO_4H_2, remains in solution. This may be supposed to result from the action of the liberated SO_4 on water, thus:—

$$SO_4 + H_2O = SO_4H_2 + O.$$

When the electrode is not unalterable, the body which is given off may give rise to chemical actions, which are called *secondary actions.* Thus, in the decomposition of sulphate of copper, if a plate of copper is taken as positive electrode instead of a platinum plate, no oxygen is liberated, but the radical SO_4 unites with copper and forms a quantity of sulphate of copper exactly equal to that which has been decomposed; the quantity of copper sulphate in solution remains constant, and in each unit of time the positive electrode loses just as much copper as is deposited on the negative electrode.

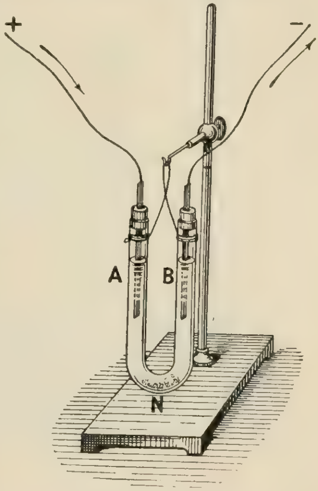


FIG. 142.

Again, secondary actions may take place between the immediate products of electrolysis and the solvent. Thus with an alkaline salt such as potassium sulphate, K_2SO_4, the decomposition may be supposed to take place in the same way as with copper sulphate; but the potassium set free at the negative electrode, being in contact with water, at once decomposes water, with the formation of potassium hydrate and liberation of hydrogen in the form of gas; thus:—

$$2H_2O + K_2 = 2KHO + H_2.$$

The result is that hydrogen is liberated at the negative electrode, and an equivalent quantity of oxygen at the positive; at the same time, sulphuric acid is found in solution at the positive electrode, and potassic hydrate at the other. This can be shown by means of a U tube (Fig. 142) containing a solution of potassium sulphate, coloured with infusion of red cabbage, and provided with platinum electrodes, A and B. The liquid, which is at first violet-coloured, becomes red at A where the oxygen is given off, while at B, where

hydrogen is liberated, it becomes green; these changes of colour indicating the presence of free acid and base respectively.

When a solution of common salt is electrolysed, the direct action is similar, but the final result is more complicated. The sodium liberated at the negative electrode decomposes water, with the formation of sodium hydrate, NaHO, and liberation of hydrogen; at the positive electrode, chlorine is liberated, partly in the form of gas, while another portion decomposes water with formation of hydrochloric acid and liberation of oxygen, and another portion still is converted into hypochlorous acid and other oxygen compounds of chlorine.

155. Decomposition of Water, of Potass, &c.—Water furnished the first instance of decomposition by the electric current. The experiment may be made with the apparatus represented in Fig. 143. This is a glass vessel, V, in the base of which are two holes, in which are cemented two wires or strips of platinum, A and B, which act as electrodes. The vessel V contains acidulated water, and two graduated glass tubes, C and D, also filled with water, are inverted over the electrodes. When a battery of three or four cells is connected with the binding screws, P and P′, the decomposition commences; the gases are liberated at the electrodes, oxygen at the positive, and hydrogen at the negative, and it is found that the volume of the latter is twice that of the former. The addition of acid to the water is essential, as perfectly pure water does not conduct the current, and, therefore, is not decomposed. It is probable that the acid is decomposed into hydrogen and the radical SO_4, and that this acts on water, reforming sulphuric acid and liberating oxygen.

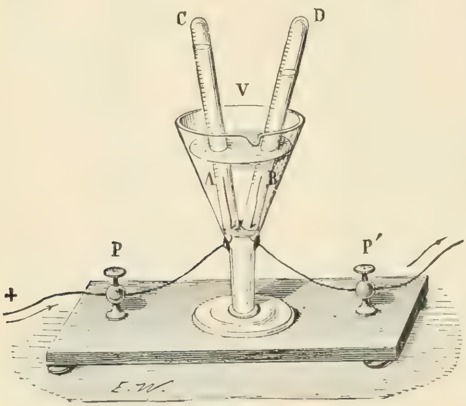


FIG. 143.

The celebrated experiment of Davy, by which he first discovered potassium, sodium, &c., may be made in the following manner. A small piece of moist caustic potash, M (Fig. 144), is placed on a platinum plate which forms the positive electrode, the negative

electrode is formed by a small quantity of mercury placed in a hollow made in the upper side of the potash. When a pretty strong current is passed, oxygen is liberated on the positive plate, and the mercury swells up and becomes pasty, owing to its dissolving the potassium as it is liberated. If the resulting amalgam of mercury and potassium is heated out of contact with air, the mercury may be distilled off and potassium left behind. This decomposition may be regarded as a decomposition of 2KHO into K_2 and 2HO, which latter further breaks up into water and oxygen: $2HO = H_2O + O$.

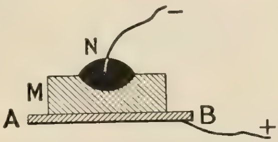


FIG. 144.

156. Electro-Chemical Equivalents.—The quantity of an electrolyte decomposed by a current depends on the nature of the electrolyte and on the quantity of electricity which passes. Thus, if the electrolyte is water, 1 gramme is decomposed by every 10,704 coulombs of electricity that are passed through it. In this case the products of decomposition are $\frac{1}{9}$ gramme of hydrogen and $\frac{8}{9}$ gramme of oxygen. To liberate 1 gramme of hydrogen requires nine times as much electricity, or 96,340 coulombs, and it was shown by Faraday that, in order to liberate the same quantity of hydrogen by the decomposition of any other substance, say hydrochloric acid, or a solution of caustic potash, the same quantity of electricity is required. He further showed that if, instead of comparing *equal* quantities of different substances, we compare quantities which are chemically equivalent to each other, the same quantity of electricity is required in every case. This result is known as *Faraday's law.*

The quantity of any substance which is liberated or decomposed, or, in general, which undergoes chemical change, during the passage of unit quantity of electricity, is called the *electro-chemical equivalent* of that substance. Thus the electro-chemical equivalent of hydrogen is $\frac{1}{96340} = 0{\cdot}00001038$ grammes per coulomb, that of oxygen is $7{\cdot}98 \times 0{\cdot}00001038 = 0{\cdot}00008283$, and that of water is $8{\cdot}98 \times 0{\cdot}00001038 = 0{\cdot}00009321$. Using this term, we may enunciate Faraday's law by saying that *the electro-chemical equivalents of all substances are in the same proportion as their ordinary chemical equivalents.* In fact, to get the electro-chemical equivalent of any substance, we have only to multiply the chemical equivalent of that substance by the constant factor 0·00001038. The following table gives the electro-chemical equi-

valents of a few important substances in addition to those named above :—

	Chemical Equivalents.	Electro-chemical Equivalents.
Silver	107·7	0·001118 grammes per coulomb
Copper (in cupric salts) .	$\frac{63\cdot 2}{2}$	·000328 " "
Zinc	$\frac{64\cdot 9}{2}$	·000337 " "
Potassium	39	·000405 " "
Sodium	23	·000239 " "
Chlorine	35·4	·000367 " "

Thus, whatever be the electro-negative or acid radical in a salt, or the time taken in the operation, a coulomb liberates always 0·000328 gr. of copper, or 0·001118 gr. of silver; it decomposes 0·00009321 gr. of water, and liberates 0·00001038 gr. of hydrogen, which at 0° and 76 cm. occupies a volume of 0·1153 cubic centimetre. And as the same quantity of electricity, that is, the same number of coulombs, traverses simultaneously all parts of the circuit, all chemical actions which take place simultaneously in the circuit take place in equivalent proportions. Hence, in order to ascertain the amount of chemical action which takes place in a given time in the battery, it is only necessary to insert a *voltameter* at any part of the circuit (that is to say, an apparatus in which electrolysis is caused by the current, and which is so arranged that the products can be collected and their quantity determined), and to measure, as the case may be, the quantity of hydrogen liberated, or of copper or silver deposited, during the time in question. The action taking place simultaneously in each cell of the battery is then chemically comparable with that observed in the voltameter. This is on the understanding that the cells of the battery are connected in a single series: if they are arranged in two or more parallel series, only one-half, or some smaller fraction, of the total current traverses each cell, and the resulting amount of chemical action is proportionately smaller.

157. Grotthus's Hypothesis.—In order to explain how it is that the products of decomposition always appear at the electrodes, Grotthus assumed that the elements of any electrolyte—water, for instance—are arranged in a line between the two electrodes, and that while an equivalent of oxygen and one of hydrogen are liberated at the same instant at the ends, water is

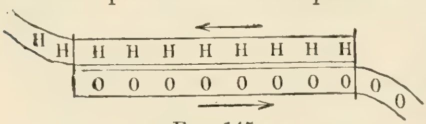


FIG. 145.

being reformed all along the line, the oxygen of each molecule combining simultaneously with the hydrogen of the following one. This conception may be represented materially by two strips of paper marked O and H respectively at equidistant intervals; if the two bands are placed side by side and drawn in opposite directions, the symbols O will be seen separately on one side, and the symbols H on the other, while H and O are in juxtaposition, representing combination in the state of water, in the intermediate space. It will be understood that each letter H stands for two atoms of hydrogen, since this is the number which is equivalent to each oxygen atom.

In order to understand why an electrolyte never acts like a simple metallic conductor, and at the same time to explain Faraday's law, it is sufficient to suppose that the atoms of the electrolyte are the sole carriers of electricity—for instance, in the case of water, that oxygen carries negative and hydrogen positive electricity—provided that each atom always carries a charge proportional to its chemical valency. Thus, if an atom of hydrogen always possesses a certain absolute charge, then Faraday's law can be accounted for if an oxygen atom always carries a charge equal in magnitude (but opposite in sign) to that which is on the *two* hydrogen atoms with which it can chemically combine.

When the constituents of an electrolyte are regarded as carriers of electricity they are called *ions;* that which travels *down with* the current is called the *kation*, that which travels *up against* the current, the *anion*.

When a current is passed through copper sulphate solution between copper electrodes, it is observed that the blue colour of the solution becomes deeper in the neighbourhood of the anode and paler near the kathode. The total amount of salt dissolved remains constant, because for each gramme of copper which is dissolved from the anode a gramme is deposited on the kathode. If platinum electrodes are used, the solution will, on the whole, become weaker, but in this case it is observed that more is removed from the neighbourhood of the kathode than from near the anode, and consequently the strength of the solution decreases faster at the former. The phenomenon is best seen if the electrodes are placed with their planes horizontal and the kathode uppermost, for it is then not disturbed by convective currents arising from differences in concentration. To account for this effect, Hittorf supposed that the ions moved with different velocities toward the two electrodes, in the case of copper sulphate the anion (SO_4) travelling the faster.

The following diagram (Fig. 146) in which the velocities are taken in the ratio of two to one will serve to illustrate this explanation.

The upper portion represents the uniformly concentrated fluid at the commencement; in the lower portion a second stage is represented in which each of the positively charged ions has travelled an eighth of the whole distance between the electrodes, while each of the negatively charged ions has travelled a quarter. The result is that, while three molecules have been decomposed, the left-hand compartment retains three undecomposed molecules, while the right-hand compartment only retains two. Thus while both have lost in concentration, the latter has lost most; and the loss in the two cases is proportional to the velocities of the ions.

This, however, can only represent the initial march of events, for it will be seen that only one positive ion is represented as having reached the kathode and given up its charge, while two negative ions have reached the anode and been discharged; the solution would in consequence have become positively charged; and since the charge on each ion is very large compared with its mass, any such accumulation of ions of one sign in the fluid would, even if it took place to a very small extent, lead to the fluid becoming strongly charged, but this does not occur.

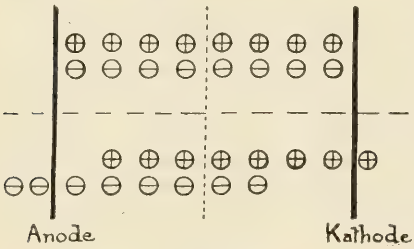


FIG. 146.

A compensating action must therefore be set up *in the neighbourhood of the electrodes* which will oppose this accumulation, and this is to be found in the changes of concentration that take place. For, as soon as a difference of concentration is produced, diffusion tends to annul it; but so long as it persists, the local electric force to which the motion of the ions is due is modified (the fall of concentration acts, in fact, as in a concentration cell, **165**), and it can be shown that, taking these two circumstances into account, the solution can automatically adjust itself, so that while the actual transport through any cross-section at a distance from the electrodes is that given above, the sharing of the transport of electricity between the positive and negative particles at the boundary between each electrode and the solution shall be the same on both sides of the boundary—a condition imposed by the mode of transport in the metal itself.

According to Hittorf's theory the ratio of the velocities may be arrived at by examining the composition of the liquid in the two compartments of an electrolytic cell, one containing the anode and the other the kathode; and he made measurements towards this end.

The first experimental proof of the theory was given by Sir O. Lodge, who measured the actual velocities in the following way: Sodium chloride was dissolved in agar-agar jelly, and placed in a horizontal glass tube; the jelly also contained a trace of phenol-phthallein with enough caustic soda to bring out its colour. The ends of the tube were immersed in two vessels containing dilute sulphuric acid, and an electric current was passed. The hydrogen atoms of the sulphuric acid pass down with the current, and when they reach the jelly displace the sodium ions which are travelling in the same direction, thus forming hydrochloric acid which decolourises the phenol-phthallein. The rate of advance of the decolouration gives, therefore, the velocity of the hydrogen ion; this came out as about ·0026 cm. per second under a potential gradient of one volt per cm.

158. Kohlrausch's Calculation of the Velocities from Conductivity.—Since on the above theory the current is carried in an electrolyte by the moving ions, the conductivity of a solution must depend upon the velocity with which they travel: the relation between these quantities was first investigated by Kohlrausch.

Take a solution containing n gramme-equivalents [a gramme-equivalent is a number of grammes equal to the chemical equivalent] of an electrolyte per cubic centimetre. Let U and Y be the velocities of the ions, which we shall suppose to be univalent. Then they will move past each other with relative velocity $(U+Y)$, and the charge carried per second through any unit area at right angles to the flow, provided that all the ions present take part in the transfer, will be $\frac{n(U+Y)}{\eta}$ where η is the electrochemical equivalent of hydrogen; but again, the current through unit area equals $K\times$ fall of potential along unit length, that is $-K\frac{dV}{dx}$ where K is the conductivity; therefore

$$U+Y=-\frac{K\eta}{n}\frac{dV}{dx};$$

the relative velocity is therefore proportional to the potential gradient.

Let u and y be the velocities produced by a potential gradient of 1 volt per centimetre—*i.e.* 10^8 C.G.S. units—then

$$u + y = \frac{K}{n} \times 1 \cdot 038 \times 10^4.$$

Kohlrausch made an elaborate series of measurements of conductivity from which he showed that K/n, which he called the *molecular conductivity*, was more and more independent of the strength of the solution the more dilute the solution became, and had then its greatest value. If we know $(u+y)$ for any strength, and also u/y (which can be obtained from Hittorf's measurements) we can calculate both u and y. For very dilute solutions the values so obtained agree with those obtained by Lodge's method. Moreover, Kohlrausch showed that if we take different electrolytes in which one of the ions is the same, the velocity of this ion is always the same for infinitely dilute solutions. These results led him to assign a specific velocity to each ion; the values in a few cases are given in the following table:—

	Velocity in cm. per second due to 1 volt per cm. u.		Velocity in cm. per second due to 1 volt per cm. y.
K	66×10^{-5}	Cl	69×10^{-5}
Na	45 ,,	I	69 ,,
Li	36 ,,	NO_3	64 ,,
NH_4	66 ,,	OH	182 ,,
H	320 ,,	$C_2H_3O_2$	36 ,,
Ag	57 ,,	$C_3H_5O_2$	33 ,,

Conversely, if the specific velocities of any two ions are known, it is possible to calculate the conductivity of infinitely dilute solutions of the salt formed by their combination.

We have stated above that the molecular conductivity ceases to be independent of the concentration when the solution ceases to be very dilute. This might arise from two causes: (1) the velocities of the ions may not be so great in the stronger solution. It should be observed that the velocities represent the limiting values acquired under the action of the unit electric force. If we suppose that the motion is resisted by the viscosity of the fluid in the same way as the motion under constant force of a small body in a fluid is resisted, this limiting velocity will be inversely proportional to the viscosity of the fluid, and this will be different for solutions of different strength. (2) It is possible that all the electrolyte present may not take part in the conduction: the

result obtained is the same as though each ion were able to take part in the process during only a fraction of the time. There is evidence that the work done in the body of an electrolyte when a current passes is sensibly zero, for FitzGerald and Trouton have shown that Ohm's law is satisfied for electrolytes as well as for solid conductors, and this could not be if work were required to separate the ions before their relative motion could take place. A theory which has been developed by Arrhenius, Ostwald, and others, supposes that a certain proportion of the ions in a solution are free from one another, or at any rate potentially so; and that it is only this fraction of the whole amount of salt present which takes part in electrical conduction. This theory is known as the dissociation theory; one form of it had previously been put forward by Williamson, from a chemical standpoint, to explain the velocity of etherification; and other facts are known which give support to it. Thus osmotic pressure (like ordinary gaseous pressure) for different substances, depends at constant temperature only upon the total number of molecules present, if the various solutions compared with one another are very dilute; when a solution of any one substance is made more concentrated the osmotic pressure does not increase proportionally to the salt added; the behaviour is what we should expect if, in very dilute solutions, the salt split up into its ions, thus making the number of molecules as large as possible, but if in more concentrated solutions this dissociation were not so complete. And when the variation of osmotic pressure with concentration is compared with the corresponding variation of molecular conductivity, the two exactly correspond after allowance has been made for the influence of change of viscosity, which is relatively small. Thus there is a considerable mass of evidence in favour of this theory; however, it is at present only on its trial, and other theories have been proposed. In order, therefore, not to suggest any particular theory, and since, moreover, the state of dissociation which is supposed to occur is different from the ordinary dissociation of a gas with rise of temperature, the term *ionisation* is used to denote the peculiar quality which part of an electrolyte possesses, enabling it to be effective in connection with electric conduction.

159. Work of Electrolysis.—A coulomb always decomposes equivalent quantities of different electrolytes. On the other hand, this decomposition requires very different quantities of heat. It follows from this that the work done by a coulomb must differ greatly according to the electrolyte through which it passes. If it be assumed that the work of electricity, like that of water, is

represented by the product of a quantity into a fall of potential (**34, 35, 128**), the conclusion follows that there must be a difference of potential between the two electrodes of a voltameter, and that this difference is such that, when expressed in volts, it is numerically equal to the number of joules of work required to effect the decomposition of one electro-chemical equivalent of the electrolyte.

If a is the mass of a substance, expressed in grammes, decomposed by the passage of one coulomb of electricity, that is, the electro-chemical equivalent of the substance (**156**), and h the quantity of heat per gramme resulting from the recombination of the products of decomposition, the electrical work per coulomb is Jah (**91**); if P denotes the difference in potential in volts of the two electrodes, we have $P = Jah = 4{\cdot}18 \times 0{\cdot}00001038M$ (M being the equivalent mass referred to hydrogen as unity) $h = 0{\cdot}0000434Mh$. For water, $Mh = 34500$, so that $P = 1{\cdot}50$ volts for the difference of potential between the electrodes of a water voltameter.

160. Polarisation of the Electrodes.—How does this difference of potential arise? Let us take as an example a water voltameter: at the beginning of the experiment, before the passage of the current, the two platinum plates are at the same potential. When, however, the current passes, the substances liberated at the electrodes change the nature of the surface and set up a difference of potential between them, which goes on increasing until it attains a constant value, from which point onwards the decomposition is a normal one. This phenomenon is known as the *polarisation of the electrodes*, and the difference of potential between the electrodes, which determines the fall in the direction of the current, is called the *electromotive force of polarisation*.

The quantity of electricity necessary to bring about a given state of polarisation depends on the nature and the dimensions of the plates. The quantity required to produce unit difference of potentials represents their *capacity of polarisation*. By taking two electrodes of very unequal surface, it may happen that a quantity of electricity which is sufficient to completely polarise the smaller, only produces a very slight modification in the larger one; the final difference of potential is, however, always the same. Experiment shows, moreover, that the capacity of polarisation, in the case of unequal electrodes, is the same, whatever be the direction of the current, and therefore the sign of the electrode.

In the case of copper sulphate solution and copper electrodes no back electromotive force exists. This might be expected from the

nature of the chemical action that takes place: copper is dissolved from one plate and an equal quantity of copper is deposited on the other, but no change is produced thereby in either surface.

161. Polarisation Currents.—The difference of potential between the polarised electrodes may be measured by the electrometer. If the battery is disconnected from the electrodes and a galvanometer (an instrument which indicates the passage of a current by the deflection of a magnetic needle, the construction of which is described in Chapter XXVIII.) is connected with them in its place, it is found that a current traverses the galvanometer from the positive electrode to the negative, and consequently passes through the liquid from the negative electrode to the positive, or in the *opposite* direction to the battery current. This result occurs whether the electrodes are left in the original liquid, or whether they are transferred to acidulated water contained in another vessel.

This current, which is due to the electromotive force of polarisation, is called a *polarisation current;* it rapidly diminishes, and ceases when the substances formed on the electrodes have completely combined again. The quantity of electricity which corresponds to their recombination is manifestly equal to that which had caused their separation. We shall afterwards see (**164**) that important applications of the polarisation current have been made. It may be added that the production of a polarisation current in the manner described above is a very delicate means of investigating whether a liquid has been decomposed by a current, and therefore whether or not it is an electrolyte.

162. Chemical Work in Batteries.—In a properly arranged battery there is no chemical action so long as the circuit is open; when it is closed the chemical action produced is solely that which results from the passage of the current in conformity with Faraday's law. As the quantity of electricity which traverses any section of a circuit is everywhere the same, the chemical action in each element of the battery is equal, equivalent for equivalent, to that which would be produced in an external voltameter.

In Volta's couple the current traverses acidulated water from the zinc to the copper plate; the hydrogen which goes along with the current is disengaged in the form of gas at the negative plate, while the acid group, SO_4, attacks the zinc plate and forms zinc sulphate. For each coulomb that traverses the circuit, 0·00034 gramme of zinc (**156**) is dissolved and converted into zinc sulphate.

In Daniell's cell the zinc plate dissolves and gives zinc sulphate

in consequence of the electrolysis of dilute sulphuric acid; the copper plate is covered with copper owing to the electrolysis of copper sulphate. Hydrogen and the group SO_4 pass through the porous diaphragm in opposite directions, and reform sulphuric acid. The formation of zinc sulphate and decomposition of copper sulphate take place in equivalent proportions, in accordance with the equation

$$Zn + CuSO_4 = Cu + ZnSO_4.$$

In Grove's element the action is the same as regards the dilute acid and the zinc; in consequence of secondary actions it is a little more complicated for the nitric acid and the platinum. At first there is reduction of nitric acid to nitrous acid, probably as expressed by the chemical equation

$$Zn + H_2SO_4 + HNO_3 = ZnSO_4 + H_2O + HNO_2.$$

Afterwards, the nitrous acid is still further reduced, red fumes are given off, and, from this stage, the electromotive force decreases.

In like manner we may explain the reactions which take place in other couples. In all couples in which zinc is used as the positive metal, the passage of one coulomb corresponds to the same expenditure of zinc in every cell—that is, 0·00034 gramme.

163. Energy of Batteries.—The only difference which exists between the chemical work of a voltaic couple and that of a voltameter is, that the first necessarily corresponds to actions which, considered as a whole, are *exothermic*—that is to say, produce heat. The energy of the current is derived from these reactions. A voltaic battery is, in short, an agent by which chemical energy is transformed into electrical energy.

This is very clearly shown by the following experiments, due to Favre and Silbermann. The heat which was produced by the solution of a given weight of zinc in dilute acid was measured in a calorimeter. A small voltaic couple of zinc and platinum was then introduced into the calorimeter, and the circuit closed by a wire, when it was found that the same quantity of heat was produced for the same quantity of zinc dissolved. The couple was next placed in the calorimeter while the interpolar wire was outside, and it was then found that for the same quantity of zinc dissolved a smaller quantity of heat was developed in the calorimeter, and the quantity was smaller in proportion as the resistance of the wire increased. If the couple was made to do chemical work, such as the decomposition of water in a voltameter, the heat produced fell short of that corresponding to the quantity of

zinc consumed by an amount equivalent to the chemical work done.

If the whole of the chemical energy is converted into electrical energy, it follows from a calculation analogous to that of (**159**), that if q is the quantity of heat resulting from the whole of the reactions which correspond to a gramme-equivalent, the expression for the electromotive force E of the couple will be

$$E = 0{\cdot}0000434q.$$

Thus in Daniell's cell the chemical work consists in the conversion of metallic zinc into zinc sulphate, and the formation of copper from copper sulphate. This process causes a production of 25,300 gramme-degrees per gramme-equivalent of the metals taking part in the action. We find, therefore, that

$$E = 25{,}300 \times 0{\cdot}0000434 = 1{\cdot}1 \text{ volts},$$

which agrees very closely with the results of direct measurement.

The agreement, however, is not equally good in all cases. Generally the chemical energy exceeds the electric energy, but in some instances it falls short. In the first case, more heat is developed in the cell during the passage of the current than is due, according to Joule's law, to the strength of the current and the resistance; in the second case, the production of heat in the cell is less than that calculated from Joule's law.

The difference is due to thermal action in the cell similar to the absorption or disengagement of heat at a metallic junction when a current passes (**138**). It can be shown that there is a necessary connection between such thermal actions, reversible with the direction of the current, and the variation of electromotive force with temperature. This can be shown by supposing the cell to go through a thermodynamically reversible cycle of operations. Imagine a unit quantity of electricity passed through it at absolute temperature T, under such conditions that the heat developed according to Joule's law is negligible, and let H be the quantity of heat, *positive or negative*, that must be supplied simultaneously to keep the temperature constant. The work done by the cell during this operation is E. Now lower the temperature of the cell slightly and let a unit quantity of electricity pass through it in the opposite direction, heat being withdrawn *or supplied* at the same time, as may be required, to keep the temperature constant. The work spent on the cell during this operation is $E - \frac{\delta E}{\delta T} dT$. Finally restore the original temperature. During the complete cycle, heat H has been given to the cell and the electrical energy yielded by

exceeds that spent upon it by $\frac{\delta E}{\delta T}dT$; that is, electrical energy to this amount has been gained. Applying the thermodynamical expression for the efficiency of a reversible cycle, we thus get

$$\frac{\frac{\delta E}{\delta T}dT}{H}=\frac{dT}{T}, \text{ or } H=T\frac{\delta E}{\delta T},$$

which shows that the rate of change of electromotive force with temperature is of the same sign as H. Also the electromotive force of the cell, or the work done by it during the passage of unit quantity of electricity, is equal to the sum of Jah, the amount by which its internal energy is decreased, and of the heat H required to keep its temperature constant during the passage of this quantity. That is

$$E=Jah+H=Jah+T\frac{\delta E}{\delta T}.$$

In the case of a Daniell's cell, as shown above, the energy of chemical action accounts very closely for the electromotive force. It is in accordance with this result that the ratio $\delta E/\delta T$ is sensibly zero for a Daniell's cell.

In any case, the work measured in joules which a cell can do for each coulomb of electricity is numerically equal to its electromotive force expressed in volts. This work is expended in heat in the circuit, and in other actions produced by the current, such as electrolysis. Hence it is not difficult to understand why a Daniell, which can only give 1·1 joules per coulomb, cannot effect the decomposition of water in a voltameter, which requires 1·50 (**161**), while this decomposition can be effected with a single Grove, which gives 1·8. It will thus be still more readily effected by two Daniells, which give 2·2 joules per coulomb.

164. Secondary Batteries—Accumulators.—We have seen that two polarised electrodes joined by a conducting wire produce a current opposite to that which had brought about the polarisation. The secondary current thus obtained gradually becomes weaker, and soon disappears, unless the polarisation is kept up by an extraneous cause.

This is done in what is known as *Grove's gas battery*, a single cell of which is represented in Fig. 147. It consists of a kind of voltameter in which platinum plates acting as electrodes occupy the whole length of the tubes which are to be filled with gas. The

tubes contain dilute sulphuric acid, and the current of a battery is passed until the tubes are filled with oxygen and hydrogen respectively. Contact with the battery is then broken, and the electrodes are connected either directly or through a galvanometer; the polarisation current is then formed; the water gradually rises in the tubes, and the current only ceases when all the gas in one or both tubes has disappeared.

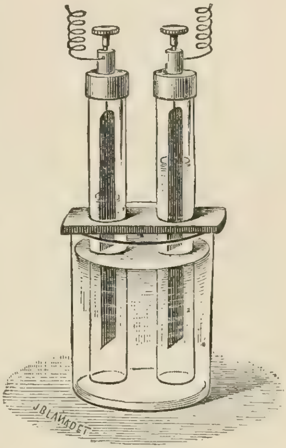


FIG. 147.

Planté has shown that by using lead plates secondary couples of great capacity can be obtained. Such couples are called *accumulators*. They are formed of two lead plates placed parallel and near to each other, and immersed in dilute acid. While being charged, the plate which serves as positive electrode is transformed to a greater or less depth into an oxide of lead, while the negative plate is reduced if it had been previously oxidised. The disengagement of free hydrogen on the negative plate is an indication that the reduction is complete. The plate which formed the positive electrode during the charge becomes the positive electrode during the discharge. This is called the *positive plate* of the accumulator. The discharge takes place in two periods: at first the electromotive force is pretty constant at about two volts; it then begins to fall off, and continues to do so very rapidly. It is best only to utilise the first portion of the charge, and to recharge before the potential begins to sink materially.

Planté showed that accumulators *form* by usage—that is to say, that up to a certain point their capacity is greater, the more frequently they have been charged and discharged. By repeated oxidation and deoxidation the lead acquires a spongy structure, and gradually a larger mass of metal takes part in the reaction. The *formation* is accelerated by immersing the fresh plate for a day or two in nitric acid diluted with its own volume of water.

With well-formed accumulators from 10,000 to 20,000 coulombs can in practice be stored for each kilogramme of lead. Planté obtained still higher numbers, 40,000, or even 60,000. As the discharge usually takes place under an electromotive force of about two volts, the available energy in the ordinary case is 20,000 to 40,000 joules per kilogramme of lead. Of course this energy is

expended in a longer or shorter time, according to the strength of the current.

In practice the capacity of an accumulator is frequently expressed in *ampere-hours*, an ampere-hour being the quantity of electricity which passes through the circuit in an hour, when the strength of the current is 1 ampere.

$$1 \text{ ampere-hour} = 3600 \text{ coulombs.}$$

Thus, in practice, from 5 to 10 ampere-hours may be stored for each kilogramme of lead.

Experience shows that, except for very slight losses, which, of course, increase with the time which elapses between the charge and the discharge, accumulators in being discharged restore almost all the electricity which had been given to them during the charge. It must, however, be remarked that the quantity of energy restored is always necessarily less, since the charge is effected at a higher potential than the discharge. If E is the electromotive force, R the resistance of the accumulator, and C the strength of the current during the charge, E', R', and C' the corresponding values in the discharge, and if Q and Q' are the quantities of electricity which come into play in the two cases, the efficiency will be given by the fraction

$$u = \frac{Q'(E' - C'R')}{Q(E + CR)}$$

which is always less than unity, even if we suppose $Q = Q'$ and $E = E'$.

165. Concentration Cells.—In order to obtain a galvanic cell it is not necessary that the electrodes shall be in any way different; it is sufficient if they are immersed in solutions of different strength. The electromotive force of such a cell is not great, however. For example: in the following cell—silver, decinormal silver nitrate solution, centinormal silver nitrate solution, silver—the electromotive force is ·055 volt. The current in this case flows through the cell from the dilute solution to the more concentrated, while the concentrations of the solutions tend to become equal.

It has been shown experimentally that the electromotive force of all such cells is nearly the same for the same ratio of concentrations if the metal is univalent; and in other cases is inversely proportional to the valency. Thus, if the metal is copper and the salt a copper salt, then for the same ratio of concentrations the electromotive force is about half the above. Similarly, in an ordinary cell, the electromotive force is modified to a slight extent if the strengths of the solutions are changed. Thus in a Daniell's cell its value is increased by concentrating the copper sulphate solution, but decreased by increasing the concentration of the zinc sulphate.

A theoretical account of the action of these so-called *concentration-cells* has been developed, chiefly by Nernst, and, although some of the suppositions on which it is based cannot be said to have received direct experimental verification, the agreement with experiment in many cases of the results to which it leads must be admitted to afford strong indirect support to the hypotheses assumed.

It is assumed that the transference of electricity through an electrolyte is due solely to its convection by charged ions, the anions carrying negative electricity towards the anode, and the kations carrying positive towards the kathode. We have seen (**156**) that the electrolytic separation of one gramme-equivalent of hydrogen, that is, of 1 gramme, requires the passage of 96,340 coulombs;[1] this then, according to the law of electro-chemical equivalents, is the quantity required to separate a chemically equivalent amount of any other substance, or it is the quantity conveyed by one gramme-equivalent proportion of univalent ions of whatever kind: we will denote this quantity of electricity by e in the following discussion: in (**158**) it is called $1/\eta$. The strength of the current through any cross-section of an electrolyte is reckoned as the sum of the positive electricity conveyed through it in unit time toward the kathode by the kations, and of the negative electricity conveyed through it toward the anode in the same time by the anions.

All compound substances when dissolved in a liquid are supposed to be broken up more or less completely into ions, and the property of conducting electricity and undergoing electrolysis is supposed to depend essentially on the fact of ionisation. This action increases the number of separate particles contained in a given volume of the solution and consequently affects all those properties of it that depend upon this number. If n' is the number of molecules of a salt dissolved in unit volume of an aqueous solution, and if a is the fraction of these that undergo ionisation, each ionised molecule yielding h ions, each unit volume of the liquid now contains $n = n'[1 + a(h - 1)]$ separate particles, that is, $n'(1 - a)$ unchanged molecules and $hn'a$ ions. In very dilute solutions of electrolytes a is approximately equal to unity; h is a small whole number: its smallest value is 2 for such compounds as HCl or $AgNO_3$, 3 for such compounds as K_2SO_4 or $CuCl_2$.

It is well known that a saline solution of which the concentra-

[1] There is reason to suppose that the mass of an atom of hydrogen is about $8{\cdot}3 \times 10^{-25}$ grm. Since 1 coulomb is $\frac{1}{10}$ of an absolute electromagnetic unit of electricity, the charge of 1 grm. of hydrogen in the ionic condition is $9{\cdot}6 \times 10^3$ absolute units, and the charge of 1 ion $= 9{\cdot}6 \times 8{\cdot}3 \times 10^{-22}$ $= 8{\cdot}0 \times 10^{-21}$.

tion is different in different parts becomes more and more uniform throughout, if left to itself, as the result of spontaneous diffusion of the salt. This phenomenon may be otherwise described by saying that, if a layer of pure water be poured over the surface of an aqueous solution of a salt, the salt expands from the volume it originally occupied so as to extend throughout the whole mass of liquid. This expansion is analogous to the expansion of a gas into an exhausted vessel, and is explained in a similar way by supposing the particles of the liquid and of the dissolved substance to be moving about among each other in all directions and thus to exert a pressure on any surface upon which they impinge. Just as, with a mixture of two gases, the total pressure is the sum of the partial pressures exerted by the two constituents separately, so with a solution, the total pressure is partly due to the particles of the solvent and partly to those of the dissolved substance. Again, just as the pressure of a gas on a given surface is not directly perceptible so long as it is the same on both sides, so in the case we have supposed of an aqueous solution with a layer of pure water above it, the water particles are present in equal numbers throughout the whole space (neglecting any small change of volume that may result from the solution of the salt) and therefore the pressure due to them is uniform, but the particles of salt, or the ions resulting from them, are absent from the upper part of the liquid, and therefore there is nothing to counterbalance the pressure resulting from those in the solution below. The resultant upward pressure which causes the diffusion of the salt is called *osmotic pressure* and is recognised in a great variety of phenomena.

Between the uniform solution forming the lower part of our liquid and the pure water above, there will be a horizontal transition-stratum, within which the strength of the solution decreases from below upwards. Let p be the osmotic pressure at any point and consider a vertical cylindrical portion of the liquid, of cross-section A, extending right through the transition-stratum. If n be the number of particles in unit volume of the liquid, the number contained within an elementary length dx of the cylinder is $nAdx$. On the lower surface of this element there is an upward force Ap due to osmotic pressure, and on the upper surface a downward pressure $A(p+dp)$, the pressure decreasing from below upwards at the rate $\frac{dp}{dx}$; consequently there is a resultant upward pressure Adp, which, being shared among $nAdx$ particles gives a force $\frac{1}{n}\frac{dp}{dx}$ acting on each. To fix ideas, we will suppose the dis-

solved substance to be fully ionised, each molecule yielding two univalent ions ($a=1$, $h=2$): then $n=$ twice the number of molecules of salt dissolved in unit volume of the solution. In general the two kinds of ions, anions and kations, move through the liquid with different degrees of facility: let y' be the steady velocity of an anion through the liquid which unit force can maintain, and u' that of a kation; then, under the actual conditions, the anions will stream upwards with velocity $\frac{y'}{n}\frac{dp}{dx}$, and the kations with velocity $\frac{u'}{n}\frac{dp}{dx}$. Suppose that y' is greater than u', then the negatively charged anions move more quickly than the positively charged kations and will begin to separate from them at the rate $(y'-u')\frac{1}{n}\frac{dp}{dx}$. But as soon as the oppositely charged ions are unequally distributed a difference of potential is set up between the different strata of the liquid which goes on increasing as separation proceeds. Let $\frac{dV}{dx}$ be the consequent rate of decrease of potential upwards at a particular level at a given instant: this will correspond to an upward force on the kations and a downward force on the anions, thus tending to counteract the separation of the two kinds of ions. As each gramme-ion carries an electric charge e, positive or negative, the force per gramme-ion on the (positive) kations will be $e\frac{dV}{dx}$, which, if it acted alone, would cause an upward velocity $u'e\frac{dV}{dx}$, and on the (negative) anions there will be a force which, if it acted alone, would produce a downward velocity $y'e\frac{dV}{dx}$. The algebraic difference of these, or $(y'+u')e\frac{dV}{dx}$ represents the downward velocity of the anions relatively to the kations which would result from the electric forces if they acted alone; in other words, it represents the rate at which these forces tend to undo the separation of the ions resulting from the osmotic pressure, and it goes on increasing as long as the rate of separation exceeds the opposite action due to the resulting gradient of potential. The condition of equilibrium is that the resultant relative velocity due to the conjoined action of the gradient of osmotic pressure and the gradient of potential shall be zero, and is expressed by the equation

$$(y'+u')e\frac{dV}{dx}=(y'-u')\frac{1}{n}\frac{dp}{dx}.$$

This gives, as the ultimate value of the gradient of potential

$$\frac{dV}{dx}=\frac{y'-u'}{y'+u'}\cdot\frac{1}{en}\frac{dp}{dx},$$

or, in terms of the velocities y and u, which unit potential-gradient would cause in the kations and anions respectively in the absence of any other force,

$$\frac{dV}{dx}=\frac{y-u}{y+u}\cdot\frac{1}{en}\frac{dp}{dx},$$

since $y'e=y$ and $u'e=u$. The establishment of this permanent condition, though of necessity a gradual process, seems to be completed almost instantaneously.

It has been shown by Arrhenius and others that, in many respects, the properties of substances in the state of dilute solution are parallel to those of gases. We have already compared generally osmotic pressure to the pressure of a gas, but according to Arrhenius and van't Hoff osmotic pressure obeys the same numerical laws as gaseous pressure and the characteristic equation of perfect gases, namely—

$$pv=zRT,$$

applies to dilute solutions if p be interpreted as the osmotic pressure due to the presence of z independently moving particles, ions and un-ionised molecules (if any), in a volume v of solution at absolute temperature T, and R[1] be the so-called gas-constant $8{\cdot}290\times10^7$ ergs, or $\frac{8{\cdot}29\times10^7}{4{\cdot}18\times10^7}=1{\cdot}98$ gramme-degrees per degree.

Assuming this relation, we have

$$p=RT\frac{z}{v}=RTn$$

where n, as above, is the number of gramme-ions per cubic centimetre of the solution, and, in the case we have supposed (every molecule of the dissolved salt being split up into two univalent ions), $z=2n'v$; that is, twice the number of gramme-molecules of

[1] R may be defined physically as the external work done by 1 gramme-molecule of a gas when it is heated 1 degree under constant pressure. To calculate its numerical value we may take any nearly perfect gas for which we have the necessary data: taking 1 gramme-molecule of hydrogen (2 grammes) at 0° C., and atmospheric pressure, $p=76\times13{\cdot}596\times981$ dynes per cm.^{2}; $v=1{\cdot}116\times10^4$ cm.^{3}; $T=273$: hence

$$R=8{\cdot}290\times10^7 \text{ dyne-centim.}=8{\cdot}29\times10^7 \text{ ergs.}$$

salt dissolved in the v cubic centimetres of solution. With T constant, this gives $\frac{dp}{dx}=RT\frac{dn}{dx}$, where $\frac{dn}{dx}$ is the rate of decrease of concentration upward, and using this, in the expression found above for the final value of the gradient of potential, we have

$$\frac{dV}{dx}=\frac{y-u}{y+u}\frac{RT}{en}\frac{dn}{dx}.$$

Integrating this between the values V_2 and n_2 corresponding to the dilute liquid above the transition-stratum and V_1 and n_1 corresponding to the less dilute liquid below, we get

$$V_1-V_2=\frac{y-u}{y+u}\frac{RT}{e}\log\frac{n_1}{n_2},$$

as the difference of potentials between the two parts of the liquid.

Now insert in each part a metallic electrode, both being of the same metal, and we have a complete concentration cell; but, in order to determine its electromotive force we must take account of possible differences of potential at the liquid-metal surfaces and add these algebraically to that due to the difference of concentration in the liquid. So as to introduce as few complications as possible, suppose the electrodes to be of the same metal as that contained in the solution; *e.g.* silver in solution of silver-nitrate. At the metal surface there is in general a passage of metallic ions (kations) from liquid to metal or from metal to liquid, but no passage of anions; and in the metal we may suppose the concentration of the ions to be very great, say N per unit volume. Hence in adapting the above formula to the case of the metal surfaces we may put $y=0$, and N for the concentration in the metal, and so write $\frac{0-u}{0+u}\frac{RT}{e}\log\frac{N}{n}$, or $-\frac{RT}{e}\log\frac{N}{n}$ for the excess of the potential of the metal over that of the liquid in contact with it. Hence, for the electromotive force of the cell—that is, for the excess of the potential of the metal in the stronger solution over that of the metal in the weaker—we may write

$$E=-\frac{RT}{e}\log\frac{N}{n_1}+\frac{y-u}{y+u}\frac{RT}{e}\log\frac{n_1}{n_2}+\frac{RT}{e}\log\frac{N}{n_2}$$
$$=\frac{RT}{e}\cdot\frac{2y}{y+u}\log\frac{n_1}{n_2}.$$

Hence the metal in the stronger solution (concentration, n^{I})

is at a higher potential than that in the weaker solution (n_2) so that the electromotive force acts through the liquid from the latter to the former, and if the electrodes are joined by a wire the electrode in the stronger solution will act as the positive pole of the combination.

As an example of the application of the formula to an actual case, we may take that of a cell with two strengths of nitrate of silver solution between silver electrodes, such as that referred to at the beginning of this section as having an electromotive force of 0·055 volt. Taking the value $y=64$ (for NO_3) and $u=57$ (for Ag) given in the table in (**158**), the calculation becomes, for 15° C.,

$$E=\frac{8{\cdot}29\times 288\times 10^7}{9{\cdot}63\times 10^4}\times\frac{2\times 64}{64+57}\times 2{\cdot}303\ \log_{10}\frac{0{\cdot}1}{0{\cdot}01}$$

$$=6{\cdot}04\times 10^5 \text{ ergs per coulomb}$$

$$=\frac{6{\cdot}04\times 10^5}{10^7}\text{ joule per coulomb}$$

$$=0{\cdot}0604 \text{ volt.}$$

166. Electro-Capillary Phenomena.—The polarisation of a surface modifies those properties which, like the surface-tension, depend upon the condition of the surface.

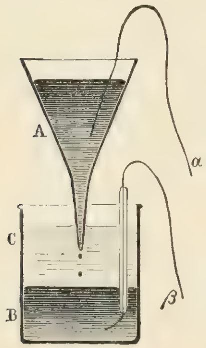


Fig. 148.

Suppose two masses of mercury separated by acidulated water (Fig. 148), one, A, contained in a funnel with so fine a point that the mercury is in equilibrium owing to the action of the meniscus, while the other, B, forms a large surface at the bottom of the vessel C. If a platinum wire, α, dipping in the top vessel, is connected with the negative terminal of a battery, and an insulated platinum wire, β, dipping with its bared end in the mercury in the lower vessel, is connected with a point on a wire joining the terminals, the potential of the mercury A becomes less than that of the mercury B, and the difference of potentials becomes greater and greater as the resistance of the part of the battery-circuit, between the points to which the wires α and β are attached, is increased. As this takes place, the surface-tension of the mercury in the drawn-out point increases, and the mercury recedes farther and farther from the point, until

the difference of potentials has reached a value of about 0·9 volt: as the potential difference is still further increased the mercury returns toward the point, and may pass beyond its initial position.

If we mark the position of the mercury when the conducting wires $\alpha\beta$ are directly connected, and the two masses of mercury are therefore at the same potential, the pressure which must be exerted on the surface of the mercury, to bring the meniscus to

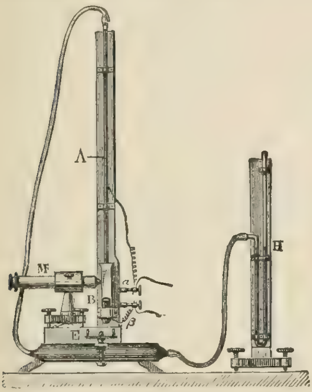


FIG. 149.

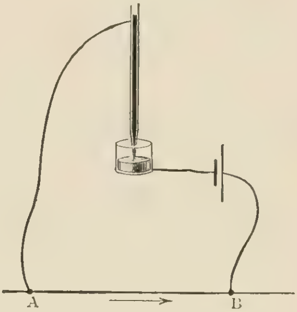


FIG. 150.

the mark, might serve as a measure of the surface-tension, and at the same time of the difference of potential.

Lippmann's capillary electrometer (Fig. 149) is based on this principle. The mercury, A, is contained in a long tube, terminated by a finely drawn out point. By means of a caoutchouc bag the desired pressure can be exerted at the top of the tube. The pressure can be measured by means of the manometer H, and the position of the index can be read off by the microscope M. From the pressure, the difference of potential is deduced by means of a table constructed once for all.

This apparatus is very sensitive, and serves to measure differences of potential from 0 to 0·9 volts. But it is of special use in what are called *zero methods*. The difference of potential to be measured is introduced into the circuit with the electrometer, and along with it a known difference of potential which can be varied at will, like that obtained between two points of a wire, AB (Fig. 150), traversed by a constant current (**111, 117**); the position of one of these points is varied until the mercury comes back to its mark; the two portions of mercury are then at the same potential, and the electromotive force to be measured is equal to the fall of potential between the two points, A and B.

167. Drop Electrodes.—If the height of the mercury in the funnel (Fig. 148) is just so small that there is no spontaneous flow by the point, this commences as soon as the two wires, $\alpha\beta$, are connected, and at the same time the circuit is traversed by a current proceeding through the wire from the mercury B to the mercury A.

If a quadrant electrometer be connected to the leads, α and β, no deflection of its needle takes place, because although there may be a difference of potential between each mercury electrode and the solution, yet this is the same for both. If we assume that there is a drop of potential in passing from the mercury to the solution, it follows that in the immediate neighbourhood of the electrode there must be a field of electric force $\left(f = -\frac{dV}{dx}\right)$, and the boundaries of this field must be charged—the mercury positively and the neighbouring layer of electrolyte negatively. When such an electrode is first dipped into the liquid, this state of things needs to be established before the full potential difference is set up. Suppose now that pressure be applied to the mercury in the funnel so as to cause it to escape from the nozzle in a fine jet; in this way a *fresh* surface of mercury is constantly being formed, the charge which had accumulated being carried away by the falling drop; and in consequence the difference of potential at this dropping electrode becomes very much smaller. If the jet be so arranged that it breaks into drops just below the surface of the liquid, the discharging action of the drops can be made practically complete (comp. **85**). The result is that if an electrometer be now connected a deflection will take place, owing to the difference of potential at the fixed mercury electrode. The value of the potential difference found in this manner between mercury and normal potassium chloride solution saturated with mercurous chloride is ·56 volt, and this is taken as a standard. The potential differences between other metals and solutions are usually obtained indirectly by constructing a cell in which one electrode and solution are those given above, while the other electrode and solution are the ones under test. The electromotive force of this cell is found by any electrometer method, and this is equal to the sum of all the potential differences in the cell since no current is flowing. The value found in this way includes, however, the potential difference between the two solutions.

The following table gives some of the values so obtained for the potential difference between a metal and a solution of its sulphate containing one gramme-molecule per litre. The positive sign indicates

that the metal is at a higher potential than the solution; and the negative sign the reverse :—

Metal.	Difference of Potential.
Zinc	$-\cdot 524$
Cadmium	$-\cdot 162$
Iron	$-\cdot 093$
Copper	$+\cdot 515$
Silver	$+\cdot 974$

167.* Pressure Cells.—Another interesting type of cell is that in which the electrodes dip into a single solution, but are at different hydrostatic pressures. This can be realised in the case of mercury in the following way. The mercury electrodes are contained in vertical glass tubes with parchment bases, and are placed with their lower ends in a solution of a mercurous salt. If the columns of mercury are of different vertical lengths the pressures on the bases will be different. Under these circumstances if a wire connect the upper portions of the two columns a current flows. During the passage of this current mercury dissolves from the electrode at high pressure and is precipitated at the other; in other words, it flows in such a direction as to tend to equalise the two columns. The current is thus accompanied by an expenditure of gravitational energy, and the idea readily suggests itself to look to this as the source of the electrical energy. During the passage of a quantity Q of electricity, the electrical energy generated by the cell is EQ, if E is the electromotive force. The amount of gravitational energy expended at the same time may be written mgh, where m is the mass of mercury transferred from one column to the other during the passage of the quantity Q of electricity, h the difference of the heights of the two mercury columns, and g the intensity of gravity. If gravity is the source of the electrical energy of the cell, these two quantities must be equal, or

$$EQ = mgh.$$

Now $m/Q = \eta_{Hg}$ is the electro-chemical equivalent of mercury, hence we get

$$E = \eta_{Hg} gh,$$

or, numerically

$$E = 200 \times 0{\cdot}0001038 \times 981h - 20{\cdot}4h,$$

in absolute units. Here $0{\cdot}0001038$ is the electro-chemical equivalent of hydrogen: seeing that we have here to do with mercurous salts,

the electro-chemical equivalent of mercury is 200 times this quantity. To get the result in volts, we must divide by 10^8.

Measurements of the electromotive force of such cells have been made by Des Coudres. For the greatest difference of level used by him, namely, 113 cm., the above calculation gives

$$20{\cdot}4 \times 10^{-8} \times 113 = 23 \times 10^{-6} \text{ volts};$$

experiment gave 21×10^{-6} volts.

MAGNETISM

CHAPTER XVI

GENERAL DESCRIPTION OF PHENOMENA

168. Natural Magnets.—In many parts of the world an ore of iron occurs, fragments of which are found to have the power of lifting small masses of iron, such as filings or small nails. This ore, which has the same composition as smithy scales, $Fe_3 O_4$, is first recorded as having been found in the Greek province Magnesia; pieces of it which exhibit the property mentioned were hence called *magnets*—in vernacular English, *loadstones* or *lodestones.* The characteristic property is not shown equally by all parts of the surface: if a magnet is covered with iron filings and then lifted out, the filings adhere to two, or perhaps more, parts of the surface while other parts remain quite clear. The parts where the magnetic property seems thus to be specially concentrated are spoken of as the *poles* of the magnet.

169. Artificial Magnets.—If one of the poles of a magnet is placed on a strip of steel and carried along it from end to end a few times, always in the same direction, the steel is found to have acquired magnetic properties, which may be even more marked than those of the original magnet, while the latter is as strongly magnetised as before. The special properties are thus communicated, but not transferred, from one body to another.

Artificial magnets can be made of any required form, and are thus much more convenient for experimental purposes and the study of magnetic phenomena than natural magnets which have accidental and often irregular shapes.

FIG. 151.

Fig. 151 shows tufts of iron filings adhering to the ends of a steel-bar magnet, the middle parts remaining clear.

170. Directive Property—Magnetic Axis.—If a magnet is supported so that it can turn freely about its centre of mass, it

assumes a definite position of stable equilibrium. When in this position it can be turned through any angle about a certain line, called its *magnetic axis*, without the equilibrium being disturbed; but, if it is turned through two right angles about a line perpendicular to the magnetic axis, so that this is exactly reversed in direction, it is found to be in a position of unstable equilibrium. Strictly speaking, the property just mentioned belongs to a particular direction in a magnet rather than to a particular straight line, but nevertheless the name magnetic axis is commonly given to that line through the centre of mass of a magnet, or through its centre of geometric symmetry, about which it can be turned without disturbing its condition of stable equilibrium.

If a magnet is of elongated form, as a narrow straight strip of steel or a steel wire, its magnetic axis usually, but not necessarily, coincides at least approximately with its length. In all cases the points where the axis meets the surface lie within the regions where the power of attracting small bits of iron is specially exhibited.

The direction of the magnetic axis may be found experimentally by the following process. Support the magnet in a paper stirrup suspended by a few fibres of unspun silk, or by any other very flexible line, and let it come to rest. If the magnet is decidedly longer in one direction than at right angles thereto, it is usually convenient to place it with its longest dimension horizontal. When it has come to rest, note the direction assumed, relatively to fixed objects, by any well-marked line in it. Then turn the magnet over, suspend it the other side up, and note the direction taken by the same line as before. The magnetic axis now lies in one of the two vertical planes which bisect the angles between the two positions of the chosen line. In order to discriminate between these two bisecting planes, the direction of one of them (selected at pleasure) must be observed. The magnet is then turned round through 180° about a horizontal axis contained in this plane, and is then set free to move. If it remains practically at rest the axis lies in the selected plane; if, however, the magnet turns through 180° about a vertical axis the other bisecting plane is the one sought. Let the intersection of this plane with the surface of the magnet, and also with that of the table over which the magnet is suspended, be marked, and suspend the magnet a third time so that what was the vertical dimension in the two previous cases is now horizontal: the magnetic axis is the line in which the fixed vertical plane already found intersects the plane section of the magnet already marked upon it.

171. Action of the Earth—Magnetic Meridian. — The direction assumed by the axis of a freely suspended magnet varies with geographical position, but at the same place it remains practically fixed for considerable periods. In England, for example, it changes by only about one degree in eight or ten years. The line of intersection of the surface of the earth by the vertical plane containing the axis of a magnet in equilibrium, when freely suspended at a distance from all other magnets or electric currents, is called the *magnetic meridian*, and the vertical plane is the *plane of the magnetic meridian.* Over a great part of the world the magnetic meridian makes a comparatively small angle with the geographical meridian : in London at the present time rather less than 16 degrees to the W. of N. and E. of S. This angle is known as the *magnetic variation* or *declination.*

The fact that the position of equilibrium of a magnet varies from place to place on the earth's surface, is a proof that the

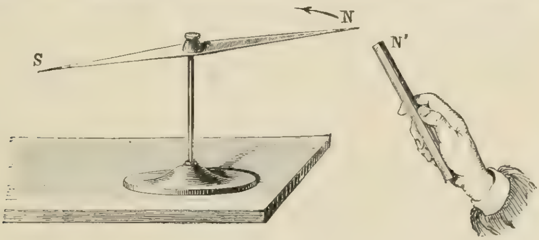


FIG. 152.

directive action of a magnet, or its tendency to assume a definite direction, is not an inherent property of a magnet as such, but depends on some mutual action between it and the earth.

172. North and South Poles—Mutual Action of Two Magnets.—If a magnet is balanced on a pivot, or suspended in a horizontal plane by a silk fibre, one and the same end always turns in a northerly direction when it is left to itself. This indicates an essential difference between the two poles or ends of the magnetic axis. These are accordingly distinguished as the *north pole*, or north-pointing pole, and the *south pole*, or south-pointing pole, respectively, and as the action of a magnet depends in many cases on the difference between its two poles, it is convenient to have these permanently marked with the letters N and S, or otherwise.

When the north pole of one magnet is brought near the north pole of another, they are found to repel each other (Fig. 152); similarly when the south pole of one magnet is brought near the

south pole of a second: but the north pole of one attracts the south pole of the other, and *vice versâ*. Hence we have this fundamental rule, like the corresponding one for electrified bodies—*two like magnetic poles repel one another, and two unlike poles attract one another.*

By the help of a *magnetic needle*, provided near the middle with an agate or metal cap whereby it can be poised on a vertical point so as to move freely in a horizontal plane (Fig. 152), and by the application of this rule, it is easy to detect whether a body is magnetised, and to distinguish its poles in the case where, from the weight or shape of the object, the directive action of the earth upon it could not readily be tried. It is generally best to present the body to be tested end on towards the centre of the needle from the east or west, and to observe whether the needle moves when the body is turned end for end.

173. Magnetisation by Influence—Magnetic Substances.—When a piece of iron is under the attraction of a magnet, it is itself a magnet for the time being. Thus, if a bar magnet is laid on a table so that one pole projects over the edge, a nail held up by the projecting pole can usually hold another nail, and this perhaps a third. If the bits of iron used for such an experiment are not too heavy, a long string of bits may be supported, each one becoming, for the time being, a magnet and holding on to the next. If the pole of a magnet is dipped in iron filings, the filings adhere to it, not as a single layer in contact with the surface, but as coherent tufts, which lose all coherence and fall to pieces as soon as they are removed from the magnet.

Each bit of iron concerned in such an experiment is said to be *magnetised by influence:* it becomes a complete magnet with a north and a south pole. If the primary pole used is a north pole, the nearest part of a piece of iron acted on by it becomes a south pole, and the more distant part a north pole. Thus, magnetic influence always acts in such a way that the neighbouring part of a piece of iron is attracted by the influencing magnet.

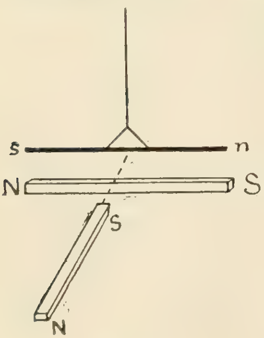


Fig. 153.

Of the innumerable experiments that might be cited to prove or illustrate the magnetic condition of a bit of iron when in the neighbourhood of a magnet, the following may suffice. Let a soft iron rod, *sn* (Fig. 153), be suspended at its middle by a silk fibre

so as to be balanced horizontally. When at rest it will not, unless it is already more or less magnetised, be disturbed if a bar magnet is brought near from one side and fixed in the same horizontal plane with its axis at right angles to the rod and in line with its centre. But if now a second bar magnet is placed parallel to the rod and gradually brought near it from below, the rod becomes magnetised by influence, and is deflected just as a permanent magnetic needle in the same place would be by the first magnet; but the deflection is reversed if the second magnet is lowered, turned end for end, and again brought up towards the rod.

Hence we may conclude that magnetisation by influence is a condition of the attraction of iron by a magnet, and that the attraction of apparently non-magnetised iron is really a case of the mutual action of unlike magnetic poles. Silver, lead, wood, &c., which are not susceptible of magnetic influence, except to a minute degree, are not sensibly acted on by a magnet.

Besides iron and its varieties (cast-iron, steel, &c.), the metals cobalt and nickel readily undergo magnetisation, and are consequently attracted by a magnet. These are therefore called *magnetic substances*.

A piece of pure iron, cobalt, or nickel is magnetised only so long as it is under the influence of a magnet; but, as indeed is implied by the existence of permanent steel magnets, a piece of steel when once magnetised retains its magnetisation to a greater or less degree even when entirely removed from other magnets.

This difference is often expressed by saying that hard steel possesses a great amount of *coercive force*, whereas soft iron, cobalt, and nickel possess little or none. Coercive force may be defined as the measure of resistance of a substance to change of magnetic condition. We shall see later (Chapter XIX.) how this and other magnetic properties of different materials can be accurately measured and made the subject of numerical comparison.

174. Result of Breaking a Magnetised Bar.—When a thin steel rod—a knitting needle, for instance—is magnetised, the magnetic properties are apparently confined very nearly to the extreme ends. At the middle, and for a good distance on either side of it, no magnetic attraction or repulsion is exerted. But if the rod is broken in half, each fragment is found to be a complete magnet with a north and south pole: not only do the poles of the original magnet retain their properties, but the two new ends, formed at a part where the unbroken bar seemed completely neutral, exhibit the properties of magnetic poles. The process may be repeated again and again, with similar results, as long

as the fragments obtained are large enough to admit of examination (Fig. 154).

We infer that, however far the subdivision were carried, the results would be the same, in fact, that the ultimate molecules of a magnet are themselves magnetic, and that the observed properties of the magnet are resultant effects due to the molecular magnets of which it is built up.

We infer, further, that it is impossible to get a detached north pole or south pole not associated with a pole of the opposite kind in the same piece of iron or steel.

175. Equality of the Two Poles of a Magnet.—The tendency of a magnet to assume a definite position of equilibrium may be interpreted as a tendency of the north pole to move, under the action of the earth, as far as possible in one direction (towards "magnetic north"), and of the south pole to move as far as possible in the opposite direction (towards "magnetic south"). Now these two tendencies exactly balance each other, so that

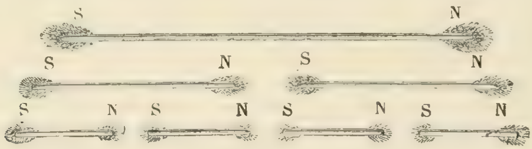


FIG. 154.

there is no force tending to produce translation of the magnet as a whole.

The experimental proof of this fact may be obtained as follows:—If a magnet be made to float on water by means of a cork, it turns so as to set itself with its axis in the plane of the magnetic meridian, but, in the absence of other magnets, it does not move bodily in any particular direction. Again, if a magnet is hung from a fixed point by a flexible thread, when the magnet is at rest the thread is vertical. Either of these results proves the absence of any horizontal resultant force acting on the magnet. That there is no vertical resultant follows from the fact that the weight of a piece of steel is not altered by magnetisation.

176. Magnetic Field.—The term *magnetic field* is conveniently applied to any region where a magnet is subject to directive action. Every point of a magnetic field is characterised by a definite direction and intensity. The direction at a given point may be defined as that in which the axis of a freely movable small magnet, placed with its centre at that point, will set itself; the intensity

is proportional to the moment of the couple tending to restore the small magnet to its position of equilibrium when its axis is displaced therefrom through a given angle.

We shall have later (**241–243**) to consider in detail how these characteristics can be accurately determined in some important cases: we give here a method that is often available for obtaining rapidly a general view of the nature of a magnetic field. We have already mentioned that a piece of iron placed near a magnet—that is to say, in the magnetic field due to it—is magnetised for the time being. If the piece of iron is very small, say, an iron filing, and rests on a smooth surface, the friction arising from its small weight interferes but little with its tendency to set itself with its axis along the direction of the field, and it thus approaches the condition of a freely movable magnet. If a sheet of glass or smooth cardboard is laid above a bar magnet, and iron filings are sprinkled uniformly over it, the filings arrange themselves in regular curves (Fig. 155) each of which runs from one point of the outline of the magnet to the symmetrically situated point on the other side of the centre. If the glass or cardboard is gently tapped, the friction of the particles of iron is momentarily overcome and their magnetic arrangement is facilitated. In this way the magnetic field is visibly mapped out. Figs. 156 and 157, which, like Fig. 155, are from photographs of actual figures made with filings, give further examples of magnetic

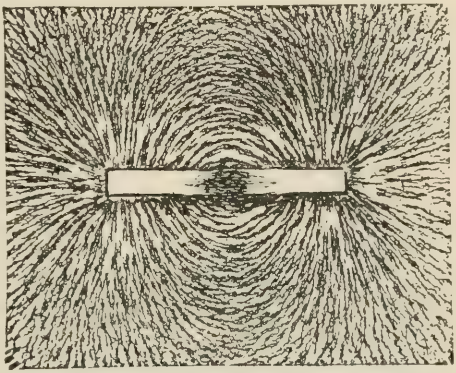


Fig. 155.

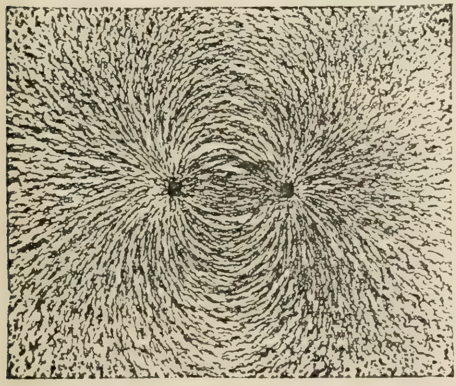


Fig. 156.

fields: Fig. 156 represents the field given by two equal opposite poles; Fig. 157 that given by two equal similar poles.

The directive action of the earth is found to be sensibly the same both in direction and intensity for considerable distances: for example, it does not vary appreciably within the dimensions of an ordinary building; hence a map of the lines representing any such part of the earth's field as we are concerned with in any experiment, would be made up of parallel and equidistant straight lines. It is in accordance with this that a magnet acted on by the earth alone is not subject to any force of translation.

But in cases where the curvature of the lines of the field is perceptible within a distance comparable with the dimensions of a magnet, a magnet is subject, not only to a directive action urging it to set its axis as a tangent to the lines, but also to a translatory force acting transversely from the convex to the concave side of the lines. This is illustrated in the above experiments with iron filings by the effect of repeated tapping of the cardboard, which is to make the curves contract by moving inwards towards the concave side. The mechanical explanation is the same as that of the pressure towards the centre of curvature which arises from longitudinal tension applied to a curved thread.

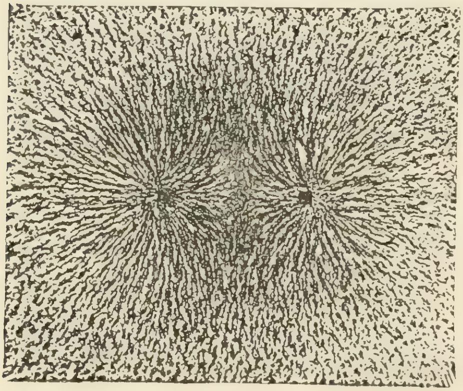


FIG. 157.

Again, if the lines of field converge or diverge, a magnet is subject to a force urging it towards the part of the field where the lines are nearest together. This effect is well illustrated by the action of the familiar floating magnetic toys.

177. Magnetic and Diamagnetic Bodies.—In the preceding paragraphs we have given an account of the more obvious and easily observable magnetic phenomena. When careful experiments are made it is found that the class of magnetic substances includes not only magnetic oxide of iron, Fe_3O_4, which constitutes loadstone, and the metals iron, cobalt, and nickel, but many other substances, such especially as salts and compounds of these metals whether solid or in solution. Indeed, when a more and more

intense field and delicate means of observation are employed, it is found that there is probably no body, whether solid, liquid, or gaseous, which is not susceptible in some degree to the action of magnetism, and does not acquire by influence magnetic polarity. But in this respect bodies fall into two distinct categories: those of one class are acted on in the magnetic field like iron, their axis of magnetisation is along the lines of field (Fe, Fig. 158); those of the other class acquire an inverse magnetisation, that is, parallel, but in the opposite direction to the lines of field (Bi, Fig. 158). The term *magnetic* is applied to the former, and the term *diamagnetic* to the latter class of bodies. Diamagnetic properties are only shown to a very slight extent; the most strongly diamagnetic body is bismuth.

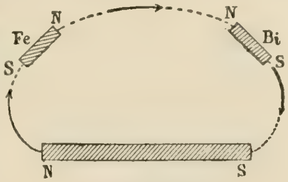


FIG. 158.

Magnetism is thus seen to be a general property of bodies, but it is extremely remarkable that while three substances—iron, cobalt, and nickel—possess it in a very high degree, it may be considered as in comparison non-existent in the others.

CHAPTER XVII

LAWS OF MAGNETIC FIELD

178. Law of Force.—We have already (**174**) given reason for supposing that the observed action of a magnet is a resultant effect due to the combined action of the constituent molecules, and the fact that this action is found to depend, not only on the magnetic condition of the material of the magnet, but also on its shape and dimensions is in accordance with this view of its constitution. As with electricity, so with magnetism, it is only in a few quite special cases, or when the dimensions of the acting bodies are so small in comparison with the distance between them that the shape of the bodies becomes unimportant, that the laws of the mechanical forces exerted become susceptible of simple mathematical statement.

One of the simplest cases is that presented by long thin magnets, such as may be made of steel wire, where the effective poles are confined to a very small space at each end. From experiments with such magnets, letting one pole of one act on one pole of the other, the second pole of each being so far away as not to come into account, Coulomb deduced the law that the force between two magnetic poles varies inversely as the square of the distance.

With magnets in general no precise points can be specified which have the properties of poles in the sense of centres of force: but it is common to conceive of real magnets as being built up of longitudinal elements, each with equal opposite poles at its two ends, and to regard the centre of mass of the elementary poles of each kind as being the pole of corresponding kind of the whole magnet. The straight line passing through these ideal poles is then defined as the "magnetic axis" of the magnet. The "strength," m, of each pole is supposed to be the sum of the strengths of the elementary poles of the same kind, and the product of the strength of either pole into the distance between them is defined as the "magnetic moment": $M = ml$.

But as neither of the factors m nor l can in general be measured

experimentally, even approximately, whereas the moment M of a magnet is accessible to precise experimental measurement, and as all the measurable effects of magnets can be stated in terms of their magnetic moments, it seems more rational to base the quantitative discussion of magnetic phenomena on magnetic moments and not on magnetic poles.

179. Directive Couple.—We have already pointed out that the action between the earth and a magnet is such as to urge the magnet to set with its axis in a definite direction, but that there is no force urging the magnet to move bodily; the action is thus represented by what in mechanics is called "a couple." The moment of this couple can be measured by suspending the magnet, with its axis horizontal, over a graduated circle—a non-magnetic pointer, if needful, being attached to the magnet—by means of an elastic thread of glass or quartz, or a hard-drawn silver or copper wire, attached to a torsion-head. Starting with the magnet in the meridian and the thread without twist, let the upper end be turned through an angle α: the magnet will follow in the same direction through some angle, β, and the thread will be subject to a twist, $\alpha-\beta$. The elasticity of the thread tends to make the lower end turn through the same angle as the upper end and thus exerts on the magnet a couple of moment $T(\alpha-\beta)$ tending to increase the deflection from the meridian (T being a constant for a given suspending thread and representing the couple required to keep it twisted through unit angle). This couple due to the thread balances the magnetic couple tending to restore the magnet to the magnetic meridian. On observing the value of the deflection β which corresponds to various values of the couple $T(\alpha-\beta)$, it is found (so long as β does not exceed a right angle) that the quotient $T(\alpha-\beta)/\sin\beta$ is sensibly constant. The numerical value of the quotient is found to depend upon the magnet used; it also, as well as the position of equilibrium when the thread is without twist, depends upon the place on the earth's surface where the experiment is made. We conclude, therefore, that it may be expressed as the product of two factors: one depending upon the magnet, which we shall call its *magnetic moment* and denote by the letter M, the other depending upon the earth, for which we shall use the symbol H. Hence with appropriate units the condition of equilibrium may be expressed by the equation

$$T(\alpha-\beta)=MH\sin\beta.$$

The factor H in this expression may be defined as the *horizontal intensity of the earth's magnetic field*. In this connection it

should be noted that in most parts of the world the direction of equilibrium of a completely free magnet is when the axis is inclined to the horizon. In the United Kingdom this direction is more nearly vertical than horizontal, being in London at the present time inclined at about 67° to the horizon. A couple tending to make a magnet set in this direction may be resolved into a component acting in a vertical plane, and a component acting in a horizontal plane. In the case of a magnet which can turn about a vertical axis only, the vertical component is counteracted by the support of the magnet and by gravity, and the horizontal component (or more generally the component parallel to the plane of motion) is alone effective.

The expression $MH \sin \beta$, forming the right-hand side of the last equation, denotes a couple depending on magnetic conditions which tends to bring the magnet back to its natural direction of stable equilibrium: we may therefore speak of it as the *directive couple.*

The above expression shows that when the axis of the magnet is in the plane of the magnetic meridian ($\beta = 0$), the moment of the directive couple vanishes, and that it attains the maximum value MH when the angle β is a right angle. The magnet, in fact, behaves as though equal parallel forces, one acting towards magnetic north and the other towards magnetic south, were applied at fixed points within it. If the line joining the points of application of such a pair of forces coincided with the direction of the field, the magnet would be in equilibrium, which would be stable if the point of application of the northerly force were at the north.

We shall have later to consider what is to be understood by a field of unit intensity, and how the intensity of a given field can be ascertained.

180. Comparison of Magnetic Moments.—If two magnets of moments M_1 and M_2 are suspended in succession by the same elastic thread, and α_1 and α_2 are the respective angles through which the upper end must be turned to deflect them through the same angle β, the above equation gives for the ratio of the moments

$$\frac{M_2}{M_1} = \frac{\alpha_2 - \beta}{\alpha_1 - \beta}.$$

181. Combination of Magnets.—It is found by experiment that the external action of a magnet upon distant magnets depends upon its moment and the direction of its axis, characters which can be conveniently expressed geometrically by a straight line drawn in

the direction of the axis (usually reckoned positive in the direction from the south end to the north), and of length proportional to the moment. Experiment also shows that, if two magnets are fastened together, the effect of the combination at any point whose distance is great in comparison with the dimensions of the magnets or their distance apart, is the same as that of a magnet whose representative line, drawn according to the rule just stated, is derived from the representative lines of the two separate magnets by the ordinary rule for the composition of vectors. Thus if OA and AQ (Fig. 159) represent respectively the moments M_1 and M_2 of two magnets and the directions of their axes, θ being the angle between them, OQ will represent the moment M and direction of the axis of the single equivalent magnet. Or, expressing the same thing in symbols,

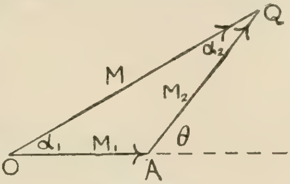


FIG. 159.

$$M^2 = M_1{}^2 + M_2{}^2 + 2\,M_1 M_2 \cos\theta.$$

If the combination is freely suspended with the axes of the constituent magnets horizontal, it will set itself with the line OQ in the magnetic meridian. For the angles α_1 and α_2 between the direction of this resultant axis and those of the two components, the geometry of the figure gives

$$\sin\alpha_1 = \frac{M_2 \sin\theta}{M} \qquad \sin\alpha_2 = \frac{M_1 \sin\theta}{M}$$

$$\cos\alpha_1 = \frac{M_1 + M_2\cos\theta}{M} \qquad \cos\alpha_2 = \frac{M_2 + M_1\cos\theta}{M}.$$

Conversely, the action of a given magnet at a distant point can be replaced by the combined action of two component magnets whose axes make a given angle with each other.

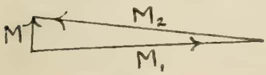


FIG. 160.

An important case of the combination of two magnets is when the moments of the separate magnets are sensibly equal, and their axes nearly in opposite directions. The moment of the combination is then very small, and the axis nearly at right angles to that of each component (Fig. 160). Such a system is said to be *astatic*.

182. Comparison of Magnetic Fields.—Just as the moments of two magnets can be compared (**180**) by noting the torsion of a given elastic thread required to deflect them equally from the position of equilibrium in the same magnetic field, so the torsion

required to deflect a given magnet equally in fields of different intensity gives a comparison of the fields. Thus, let H_1 and H_2 be the intensities of the two fields to be compared, resolved if needful in the plane of motion of the magnet, and α_1 and α_2 the angles through which the upper end of the torsion thread must be turned to deflect the same magnet through the same angle β when suspended in the respective fields, we get for the ratio of the intensities of these

$$\frac{H_2}{H_1}=\frac{\alpha_2-\beta}{\alpha_1-\beta}.$$

It should be noted that in this experiment, if the fields to be compared are inclined to the plane of motion of the magnet, the effective component in each case is the component acting parallel to the plane of motion, and that it is these components that are compared.

Another method whereby the same comparison can be made depends on the fact that, when a magnet is displaced from its position of equilibrium, the moment of the restoring couple acting upon it is proportional to the sine of the angle of displacement—that is, for small displacements, proportional to the angle itself. The result of this is that, when a magnet suspended by a silk fibre without perceptible torsion-elasticity is slightly displaced from its position of rest, and then left to itself, it performs isochronous vibrations about its position of rest, the period of which, with the same units as in (**179**), is

$$\tau=2\pi\sqrt{\frac{I}{MH}},$$

where I is the moment of inertia of the magnet and of anything that moves with it.

Hence if τ_1 and τ_2 are the periods of vibration of the same magnet in fields whose intensities are H_1 and H_2 respectively, we have

$$\frac{H_2}{H_1}=\frac{\tau_1^2}{\tau_2^2}.$$

Magnetic fields can be superposed and compounded, or resolved, like other "vector" quantities—that is, quantities characterised by magnitude and direction. Thus the field at a given point due to an artificial magnet may be compounded with that of the earth, or the fields due to two or more magnets may be compounded together. The direction of the resultant field can be ascertained experimentally by observing the direction assumed by the axis of a small magnet freely suspended at the point in question.

For instance, let H represent the intensity of the earth's field at a given point, and H' the intensity at the same point of the field due to a fixed magnet: then if θ is the angle which the axis of a small magnet at this point makes with the direction of equilibrium due to the earth alone, and similarly θ' the angle which it makes with the direction due to the fixed magnet alone, the small magnet is subject to the two directive couples

$$MH \sin \theta \quad \text{and} \quad MH' \sin \theta' \text{ respectively,}$$

and if it is freely movable it will set itself so that these are equal or

$$H \sin \theta = H' \sin \theta'.$$

If the fixed magnet is placed with its axis at right angles to the magnetic meridian, and so that the line of its axis produced passes through the point where the small magnet is suspended, we have $\theta + \theta' = \frac{\pi}{2}$, or $\sin \theta' = \cos \theta$, and consequently

$$\frac{H'}{H} = \tan \theta.$$

In this experiment θ is the angle through which the suspended magnet is deflected from its natural position, and can be readily observed by means of a graduated circle, or by the method of reflection (**84**).

A suspended magnet employed, as in the above case, to indicate the direction of the resultant magnetic field at a given point may be conveniently referred to as a *magnetic needle*. We shall frequently use this term. The axis of a magnetic needle will in general be supposed to be confined to motion in a horizontal plane.

183. Further Comparison of Magnetic Moments.—The principle of the experiment referred to in the last paragraph affords an additional method of comparing the moments of two magnets. Let the magnets to be compared be employed in succession as the fixed magnet in such an experiment, and let θ_1 and θ_2 be the deflections they respectively produce. Then, if H'_1 and H'_2 are the corresponding fields, we have

$$H'_1 = H \tan \theta_1, \quad \text{and} \quad H'_2 = H \tan \theta_2.$$

Now it is found further, by experiments to be described shortly (**186**), that the fields produced by two magnets at points at the same distance measured along the axis from their centres are proportional to their magnetic moments as compared by the method

of torsion already referred to (**180**). Hence if the two magnets are placed successively with their centres at the same point, we get for the ratio of their moments

$$\frac{M_1}{M_2} = \frac{\tan\theta_1}{\tan\theta_2}$$

184. Field due to a Small Magnet.—The law of variation of the magnetic field of a magnet with distance and direction was first systematically investigated by Gauss, who showed experimentally that at distances which are great as compared with the dimensions of the magnet, the intensity of the field is approximately proportional to the *inverse cube* of the distance, and that the approximation becomes closer the greater the distance, or, what comes to the same thing, the smaller the magnet employed. The intensity also varies with direction as referred to the axis of the magnet, in a manner to be discussed presently. For the present we shall consider two cases, namely (1), that of points in the line of the axis produced; (2), that of points in the plane bisecting the axis at right angles.

185. Direction.—If a magnet is placed with its axis along the magnetic meridian, the position of equilibrium of a magnetic needle, whose point of suspension is in the line of the axis, is not altered by the presence of the magnet, but the couple required to displace it through a given angle is *increased* if the magnet is in its natural position (north end towards magnetic north), but is *decreased* if the magnet is in the reversed position (north end towards magnetic south). In the latter case, the position of equilibrium of the needle may be reversed if the magnet is too near. This proves that the direction of the field of a magnet at any point on its axis produced is along the axis, and acts in the sense of a line drawn from the south pole to the north pole of the magnet.

If a magnet is placed as before, and a magnetic needle is suspended so that the line joining its centre with that of the magnet is at right angles to the meridian, the position of equilibrium of the needle is again unaltered, but the couple required to displace it is *decreased* when the magnet is in the natural position and *increased* when it is in the reversed position. Hence at any point in the plane which bisects the axis of a magnet at right angles the field due to the magnet is parallel to the axis, and acts in the sense of a line drawn from the north pole to the south pole of the magnet.

186. Intensity.—By placing a magnet at right angles to the

magnetic meridian at various distances from a magnetic needle, and so that the line of its axis passed through the centre of the needle, Gauss found that the tangent of the deflection of the needle from its position of equilibrium could be represented by the formula

$$\tan \theta = \frac{A}{r^3} + \frac{B}{r^5} + \frac{C}{r^7} + \ldots .$$

where r is the distance between the centre of the magnet and that of the needle and A, B, C . . . . are constants. In proportion as r becomes greater, or the dimensions of magnet and needle become smaller, the several terms of this series become unimportant as compared with the first. Hence, in the case of great distances or very small magnets, the expression may be reduced to the first term.

We have already seen (**182**) that, in such a case as that here referred to, the tangent of the angle of deflection of the magnetic needle gives the ratio of the field at the point of suspension due to the magnet to that due to the earth. Hence, putting h for the former and H for the latter, and supposing the magnet small, we have

$$h = H \tan \theta = H \frac{A}{r^3}.$$

When a magnet with its axis at right angles to the magnetic meridian is placed so that the line joining its centre with that of a magnetic needle coincides with the meridian, the needle is deflected in the opposite direction to that in the case last considered, but otherwise the result follows the same general laws. Distinguishing by accents the symbols that apply to this case, we may therefore write

$$\tan \theta' = \frac{A'}{r^3} + \frac{B'}{r^5} + \ldots .$$

or, with the same limitations as before,

$$h' = H \tan \theta' = H \frac{A'}{r^3}.$$

These results may be stated otherwise by saying that the products $r^3 \tan\theta$ and $r^3 \tan\theta'$ approach to constant values, here represented by A and A' respectively, as the distance r is taken greater and greater, or as the dimensions of the magnets are diminished. There is a practical difficulty about determining these values by making the distance r very great, namely, that the deflection then becomes too small for accurate observation. The

difficulty can be avoided by making observations at two or more distances, and eliminating the constants B, C, &c., the distances then require to be only moderately great.

Thus, if θ_1 is the deflection when the magnet is at a distance r_1 and θ_2 the deflection with distance r_2, we have the two equations

$$\tan\theta_1 = \frac{A}{r_1^3} + \frac{B}{r_1^5} \text{ and } \tan\theta_2 = \frac{A}{r_2^3} + \frac{B}{r_2^5},$$

whence, by eliminating B, we get

$$A = \frac{r_1^5 \tan\theta_1 - r_2^5 \tan\theta_2}{r_1^2 - r_2^2}.$$

With one and the same magnet, experiment gives the relation

$$A = 2A', \text{ whence } h = 2h';$$

while, for different magnets, A and A' are proportional to the moments of the magnets as measured by the method of torsion (**180**) or otherwise.

Putting M for the moment of the magnet employed and k for a constant depending on the units of measurement, we may accordingly sum up the whole discussion by writing

$$h = -2h' = k\frac{M}{r^3},$$

or in words: *the intensity of the field due to a magnet is, at a sufficient distance from the centre, twice as great at a point on the axis as at an equal distance in the plane bisecting the axis at right angles, and is further proportional to the magnetic moment of the magnet and inversely proportional to the cube of the distance.*

187. Unit Magnetic Moment and Unit Magnetic Field. —Any measurements referred to in preceding sections of this chapter involve comparisons only of magnetic moments and magnetic fields, and whatever quantitative statements have been made would be equally valid whatever system of units were adopted for these magnitudes. We have now to consider how absolute units founded on the C.G.S. system can be defined.

It has been stated already (**179**) that the moment of the couple required to displace a magnet from its natural position of equilibrium depends jointly upon two factors: one, which we have called the magnetic moment of the magnet, characteristic of the magnet; and the other, which we have called the intensity of the earth's magnetic field, depending on the place of experiment and the magnetic properties of the earth.

Denoting these factors by M and H respectively, we may say that the couple required to hold a magnet of moment M with its axis at right angles to the direction of a magnetic field of intensity H is *proportional* to the product MH, and if we adopt appropriate units for the two factors, we may say that the couple is numerically

equal to this product. Taking this as one of the conditions which the unit magnetic moment and unit of field-intensity must satisfy, we get the following definition :—

> *Unit magnetic moment is the moment of a magnet such that a couple of moment unity is required to hold it with its axis at right angles to the direction of a magnetic field of unit strength ;*

and the converse definition :—

> *A magnetic field of unit intensity is such that a couple of unit moment is required to hold a magnet of unit magnetic moment with its axis at right angles to the direction of the field.*

It will be noted that these two definitions are reciprocally dependent on each other, and that therefore some further criterion is required to enable us to define either of the two quantities in question without reference to the other. This is supplied by the experimental law of the mutual action of two small magnets which was discussed in the last section. If in the expression

$$h = k\frac{M}{r^3}$$

there given h and M are expressed in arbitrary measure, such a value must be given to the factor k as will satisfy the equation. Reciprocally, if an arbitrary value is assigned to k, the units to which h and M are referred must be such as to satisfy the equation. The units we shall adopt are those determined by giving to the factor k the value 2. With this condition the relations already pointed out enable us to adopt the following independent definition :—

> *If two small magnets of equal moment are placed with their axes in the same straight line and their centres one centimetre apart, each has unit magnetic moment if a couple of moment two dyne-centimetres is required to hold the axis of either of them at right angles to the common initial direction* (Fig. 161);

FIG. 161.

or the equivalent definition :—

> *If two small magnets of equal moment are placed with their centres one centimetre apart and their axes parallel and at right angles to the line joining their centres, each of them has unit magnetic moment if a couple of moment one dyne-centimetre is required to hold the axis of either of them at right angles to the common initial direction* (Fig. 162).

FIG. 162.

With regard to either form of this definition it is to be observed that neither magnet is supposed to be acted on by any other magnetic field than that of the other magnet, and that, the unit of distance being one centimetre, the dimensions of the magnets must be supposed small in comparison with this length. The definition as it stands is therefore directly applicable only to magnets of elementary dimensions, such as the magnetic molecules of which actual magnets may be regarded as being built up. In this respect the definition is comparable to that of the electrostatic unit (**15**) which presupposes electrified bodies whose dimensions are negligible in comparison with one centimetre.

From the definition of unit magnetic moment that of *unit magnetic field* follows at once and need not be further discussed.

Adopting these units the couple on a magnet in a uniform field H equals $MH \sin\theta$, as written in (**179**). It may also be pointed out that the units above defined are identical with those which are arrived at by applying to the action of magnetic poles the law of the inverse square of distance, as in the case of quantities of electricity, and defining the unit magnetic pole as that which exerts in air a force of one dyne on an equal pole at a distance of one centimetre.

188. Field at any Point due to a Small Magnet.—To determine the strength of the magnetic field at any point P due to a small magnet with its centre at O and its axis in a direction making an angle α with OP, we may replace the actual magnet of moment, say M, by two component magnets, one of moment $M_1 = M\cos\alpha$ with its axis along OP, and the other of moment $M_2 = M\sin\alpha$ with its axis at right angles to OP: then the field at P will be the field compounded of h_1 the field due to the first component magnet and h_2 due to the second. By (**187**) we have, putting r for the distance OP—

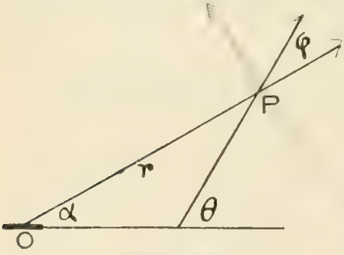


FIG. 163.

$$h_1 = 2\frac{M_1}{r^3} = 2\frac{M\cos\alpha}{r^3} \text{ acting along OP, and}$$

$$h_2 = \frac{M_2}{r^3} = \frac{M\sin\alpha}{r^3} \text{ acting at right angles to OP.}$$

The resultant field is then

$$h = \sqrt{h_1^2 + h_2^2} = \frac{M}{r^3}\sqrt{4\cos^2\alpha + \sin^2\alpha} = \frac{M}{r^3}\sqrt{3\cos^2\alpha + 1}.$$

Seeing that the components h_1 and h_2 act respectively along OP and at right angles to it, the angle ϕ which the resultant makes with OP is given by

$$\tan\phi = \frac{h_2}{h_1} = \frac{1}{2}\tan\alpha.$$

Putting θ for the angle between h and the axis, the only direction directly given, the figure gives at once

$$\theta = \alpha + \phi,$$

whence

$$\tan\theta = \frac{\tan\alpha + \tan\phi}{1 - \tan\alpha\tan\phi} = \frac{3\tan\alpha}{2 - \tan^2\alpha} = \frac{3\sin\alpha\cos\alpha}{3\cos^2\alpha - 1}.$$

Also

$$\cos\theta = \frac{1}{\sqrt{1+\tan^2\theta}} = \frac{3\cos^2\alpha - 1}{\sqrt{3\cos^2\alpha + 1}};$$

$$\sin\theta = \cos\theta \, . \, \tan\theta = \frac{3\sin\alpha\cos\alpha}{\sqrt{3\cos^2\alpha + 1}}.$$

These values of $\cos\theta$ and $\sin\theta$ give us at once the components of h parallel and perpendicular to the axis of the magnet. Calling these h_x and h_y respectively, we have

$$h_x = h\cos\theta = \frac{M}{r^3}(3\cos^2\alpha - 1),$$

$$h_y = h\sin\theta = \frac{M}{r^3} \, . \, 3\sin\alpha \, . \, \cos\alpha.$$

For the special cases where $\alpha = 0°$ and $\alpha = 90°$, which are those referred to in **(186)** and are known as Gauss's *first* and *second principal positions* respectively, the component h_y vanishes. These cases will be referred to again in connection with magnetic measurements **(243)**. The component h_x vanishes when $\cos\alpha = \frac{1}{\sqrt{3}}$, or $\alpha = 54° \, 45'$ nearly.

CHAPTER XVIII

MAGNETIC POTENTIAL

189. Magnetic Potential.—It will facilitate the application of the results, obtained in the last chapter for the intensity and direction of the field due to a very small magnet, to determine the effect of actual magnets, if we note that the expressions obtained can all be derived from a single quantity by finding its rate of decrease in the direction of the required component. The quantity in question may be written

$$\frac{M \cos \alpha}{r^2};$$

and in order to derive from it the value of any required component of the field all that is necessary is to find its differential coefficient with respect to the direction of the component, and to invert the sign. This is the process whereby the component in any direction of an electric field can be deduced from the electric potential. We shall therefore, by analogy, call the above quantity the *magnetic potential* due to a small magnet with moment M at a point distant r centimetres from the magnet in a direction making an angle α with the magnetic axis. Using for shortness the symbol V for magnetic potential, and denoting as before the several components that have been found, we leave it to the reader to verify the following results:

$$h_1 = -\frac{dV}{dr} = 2\frac{M}{r^3}\cos \alpha, \qquad h_2 = -\frac{dV}{rd\alpha} = \frac{M}{r^3}\sin \alpha,$$

$$h_x = -\frac{dV}{dx} = \frac{M}{r^3}(3\cos^2 \alpha - 1), \qquad h_y = -\frac{dV}{dy} = 3\frac{M}{r^3}\sin \alpha \cos \alpha.$$

If there are, in the neighbourhood of a point where the magnetic field is to be determined, two or more small magnets, the component intensity in any direction is the algebraic sum of the components in that direction due to the magnets taken severally. Hence the general expression for the potential at a point may be written

$$\Sigma\frac{M \cos \alpha}{r^2},$$

or, in the case of a continuous distribution of magnetised matter, such as we suppose to constitute an actual magnet,

$$\int \frac{dM \cos \alpha}{r^2},$$

where dM is the magnetic moment of an element of volume of the matter, and the integration is to extend throughout the mass.

190. Intensity of Magnetisation.— The ratio A of the magnetic moment of a magnet to its volume is defined as the mean *intensity of magnetisation* of the magnet—

$$A = \frac{M}{v}.$$

In general, if equal similar and similarly oriented parts are taken at different positions in a magnet this ratio varies from one to another. The intensity of magnetisation at a given point may be taken as the limiting value of this ratio when a smaller and smaller volume is taken so as always to include the point in question: that is, the intensity of magnetisation at a point is given by

$$A = \frac{dM}{dv}.$$

For the whole magnet the moment is the sum of the moments of its smallest parts resolved parallel to the resultant axis of the magnet, or

$$M = \int A \cos \theta \, dv, \ = \int A a \cos \theta \, ds$$

where the integration extends throughout the volume, a is the cross-sectional area of the element, ds its length, and θ is the angle which the direction of magnetisation of the element makes with the resultant axis.

191. Value of Magnetic Potential in Special Cases.—We now proceed to determine the potential in a few typical cases.

(i.) *Longitudinally Magnetised Thin Rod.*—Let a be the area of cross-section of a longitudinally magnetised rod or wire (that is, a rod or wire magnetised so that the magnetic axis of any short piece is in the direction of its length), the volume of an element of length ds is ads, and its moment $Aads$, if, as we shall assume, the

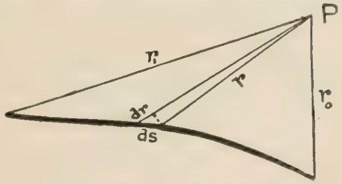


FIG. 164.

magnetisation is uniform for the whole cross-section. The resulting potential at a point P (Fig. 164) at distance r from the element

considered, and in a direction making an angle α with the axis, is therefore

$$dV = \frac{Aa ds \cos \alpha}{r^2}.$$

But $ds\cos\alpha = dr$, consequently the potential due to the whole rod is

$$V = \int Aa \frac{dr}{r^2}$$

the integration being taken from one end to the other. Now

$$Aa \frac{dr}{r^2} = -d\left(\frac{Aa}{r}\right) + \frac{d(Aa)}{r};$$

consequently, using suffixes $_o$ and $_1$ to refer to the two ends of the rod, we have

$$V = \int_o^1 Aa \frac{dr}{r^2} = \frac{A_o a_o}{r_o} - \frac{A_1 a_1}{r_1} + \int_o^1 \frac{d(Aa)}{r}.$$

The product of intensity of magnetisation into cross-section, which, when it is uniform, gives the magnetic moment per unit of length of the rod, is conveniently called the *magnetic strength* and represented by a single symbol. Using the symbol S for this quantity we may write

$$V = \frac{S_o}{r_o} - \frac{S_1}{r_1} + \int_o^1 \frac{dS}{r}.$$

If the magnetic strength is uniform, the third term of this expression, under the sign of integration, vanishes, and $S_o = S_1$. Therefore, omitting suffixes, we have then

$$V = \frac{S}{r_o} - \frac{S}{r_1},$$

and it is important to note that in this case the potential depends only on the magnetic strength and on the positions of the two ends of the rod; the intermediate parts contribute nothing to the potential. It will be recognised that this expression is identical with that for the electrical potential at a point due to equal opposite electric quantities at distances r_o and r_1 respectively from the point. We might, therefore, by analogy interpret S as representing a "quantity of magnetism" distributed over the ends of the rod with surface-density $A = \frac{S}{a}$, the magnetism being positive at one end and negative at the other.

For the intensity of the magnetic field in the case we are considering we have to differentiate V with respect to r_o and r_1

separately, to reverse signs, and compound by the parallelogram law the resulting values $\frac{S}{r_0^2}$ and $-\frac{S}{r_1^2}$. The result is again the same as though there were equal opposite electric charges of numerical value S in the positions of the two ends of the rod. The ends then appear as centres of magnetic force, and this is the sense often attached to the term "magnetic pole," though, as we have said, it is only in quite special cases that the magnetic action of a magnet can be treated as being concentrated at definite points. The case we are considering is one in which the idea of poles in the sense mentioned is admissible: adopting it we see that S and $-S$, which we have already interpreted as quantities of magnetism, may also be called the *strengths of the respective poles.* This agrees with the definition of magnetic moment often given as the product of the strength of one pole into the distance between the poles.

If the two ends of a uniformly magnetised rod coincide, that is, if the rod forms a closed ring, we have $r_0 = r_1$, and the potential due to it is everywhere zero. It follows that the field is also zero, that is to say, the ring is without any action on external magnets as long as it is complete.

In general the intensity of magnetisation of a cylindrical or prismatic magnet, though it may be sensibly uniform over some considerable portion of the length, decreases from the middle towards each end; or as we pass from end to end in the same direction S, increases at first, then remains nearly constant, and finally decreases. In this case the term representing the integral of dS/r in the general expression for the potential due to a rod cannot be neglected, but consists of a positive part and a negative part corresponding respectively to the increase and decrease of magnetic strength. The corresponding portions of the rod act as though they were covered with positive or negative (north or south) magnetism to a surface-density proportional to the rate of variation of S. Whether one end or the other behaves as the positive end depends on the direction of magnetisation which can be expressed by giving the appropriate sign to S.

(ii.) *Transversely Magnetised Plate*, or *Magnetic Shell.*—We will consider the plate as made up by the juxtaposition side by side of a great number of elementary prisms of length e equal to the thickness of the plate and of cross-section da, where da is an element of the area of the surface of the plate turned towards the point P where the potential is required. We suppose the elementary prisms to be magnetised longitudinally, so that their north ends are in one face of the plate and their south ends in the other.

Putting A for the intensity of magnetisation, the moment of a prism is represented by $Aeda$, and the potential due to it by $Ae\,\dfrac{da\cos\alpha}{r^2}$, r being the distance to the point P, and α the angle between r and the magnetic axis of the elementary prism under consideration, that is, the angle between r and the normal to the shell. If we call Ae the magnetic strength of the shell (it represents, if uniform, the magnetic moment per unit of area), and denote it by the single symbol Φ, and put $d\omega$ for $\dfrac{da\cos\alpha}{r^2}$, which represents the solid angle subtended at P by the elementary area da, we get for the potential due to the element in question

$$dV = \Phi d\omega.$$

Supposing the magnetic strength constant, the total potential due to the shell is

$$V = \int \Phi d\omega = \Phi\omega$$

where ω is the solid angle subtended by the periphery of the whole shell. We may reckon the potential positive if only the *north* polar face is visible from the point; negative if only the *south* polar face is visible.

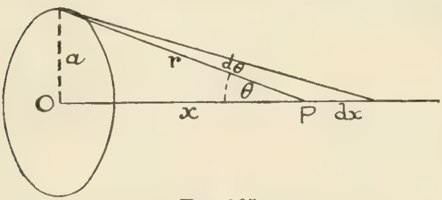


FIG. 165.

In the case of a shell with circular contour, it is easy to calculate the potential and field for points which lie on the axis.

Let Φ be the strength of the shell, a its radius, x the axial distance of a point P from its centre O, θ the plane angle which the radius of the circle subtends at P; the solid angle which the shell subtends at P is $2\pi(1-\cos\theta)$, and the potential at P is therefore

$$V = 2\pi(1-\cos\theta)\,\Phi.$$

Considerations of symmetry show that the field at P is directed along the axis; its value is therefore

$$-\frac{dV}{dx} = 2\pi\Phi\frac{d(\cos\theta)}{dx} = -2\pi\Phi\sin\theta\frac{d\theta}{dx}$$

$$= -2\pi\Phi\frac{\sin\theta}{r}\,r\frac{d\theta}{dx} = 2\pi\Phi\frac{\sin^2\theta}{r}$$

$$= 2\pi\Phi\frac{a^2}{r^3}$$

If the shell is curved so that, supposing it transparent, one part

would be visible through another, opposite faces will be turned towards the point by these parts, and they must therefore, in calculating the solid angle, be treated as cancelling each other.

It follows that the potential at an *external* point due to a closed shell is *zero*. At an *internal* point the potential due to such a shell is $\pm 4\pi\Phi$, the sign depending on whether the positive or negative face is inwards.

There is no field either inside or outside the shell, since the potential has a constant value in each case. If the north polar face is outside, the potential inside is $-4\pi\Phi$ and outside it is zero: the space-rate of change of the potential in the substance of the shell is $\frac{4\pi\Phi}{e}$ or $4\pi A$, the direction of decrease being from the outer to the inner surface. We have no right, however, to assert that this is a measure of the magnetic field in the substance of the shell, for our definition of magnetic field was in terms of the turning-moment on a magnet placed *in air*. It will be shown later (**198**) that it represents only a part of the total field within the magnet. Lord Kelvin has called it the magnetic force according to the polar definition: we shall call it simply the *polar field* of the magnet.

(iii.) *Uniformly Magnetised Circular Cylinder.*—The field at

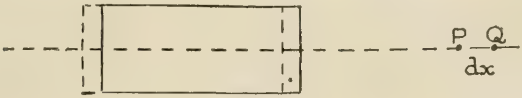


FIG. 166.

any point, P, on the axis of a circular cylinder may be found as follows:—The action of the cylinder at Q (distant dx from P) is the same as it would be at P if the cylinder were moved *away* through a distance dx (Fig. 166). This movement is equivalent to removing a shell of thickness dx from the near end and adding an equal shell to the more distant end. Therefore the potential at Q must be less than that at P by the amount $A dx\,(\omega_1 - \omega_2)$ where ω_1 and ω_2 are the solid angles subtended at P by the near and distant ends respectively. The magnetic field is the rate of decrease of potential and is consequently $A(\omega_1 - \omega_2) = A 2\pi\,(\cos\theta_2 - \cos\theta_1)$.

The same result is obtained by considering the cylinder as a bundle of straight filaments with their ends on the ends of the cylinder. If da is the area of cross-section of any one of these, its potential at P is $A da\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$ where r_1 and r_2 are the distances of

its ends from P. The field of this element, obtained in the way described in **191** (i.), is $\frac{Ada}{r_1^2}$ in the direction of r_1 together with Ada/r_2^2 in the negative direction of r_2. The components of these in the direction of the axis are

$$\frac{Ada\cos\gamma_1}{r_1^2} \text{ and } -\frac{Ada\cos\gamma_2}{r_2^2}$$

where γ_1 and γ_2 are the angles between the axis and r_1 and r_2 respectively. The factor of A in each of these expressions is the solid angle subtended at P by the corresponding end of the filament; consequently we find the effect of all the filaments by replacing each of these elementary angles by the total solid angles which the ends of the cylinder subtend. Thus the axial component of the field is $A(\omega_1 - \omega_2)$ and since the normal components cancel each other in pairs the result is in agreement with that previously obtained.

If P be close to the end of the cylinder $\theta_1 = \pi/2$, or $\cos\theta_1 = 0$, and the field is $2\pi A \cos\theta_2$; which, when the other end is at an infinite distance, becomes $2\pi A$.

If two such long equally magnetised cylinders be placed coaxally and nearly in contact with one another, the field in the gap between them is either $4\pi A$ or zero, according as they are magnetised in the same or in opposite directions. If they are not infinitely long, then an allowance must be made for the solid angles subtended by the distant ends.

The effect of these ends might also be got rid of by bringing them into coincidence, as might be done by bending the cylinders round without displacing the two ends which face each other across the gap. We thus see that the field in the air-gap formed by cutting out a thin slice from a uniformly magnetised anchor ring, as shown in Fig. 167, must likewise have the value $4\pi A$.

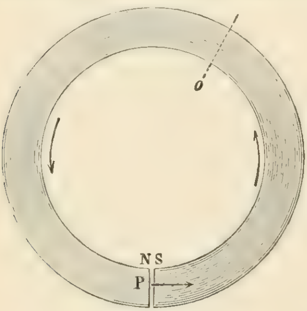


FIG. 167.

This result is also arrived at by observing that a complete anchor-ring has zero potential and field everywhere, since it consists of closed filaments; cutting a thin gap across it is equivalent to removing a magnetic shell; consequently the potential due to the remainder is the same as that due to the shell, but with sign reversed. If the thickness of the gap is e, the average field in the gap is $4\pi\Phi/e$ where $\Phi/e = A$.

(iv.) *Assemblage of a large number of Small Magnets with Axes indiscriminately directed.*—If dM is the moment of a single magnetic particle, the potential due to it is $\frac{dM \cos\alpha}{r^2}$. The angle α may have any possible value, and $\cos\alpha$ will therefore be on the average as often positive as negative, and consequently will cancel in the sum. This case corresponds with the supposed constitution of an unmagnetised mass of iron or steel. The potential due to it is everywhere zero.

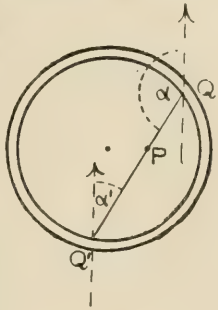


FIG. 168.

(v.) *Thin Spherical Shell uniformly magnetised as to Intensity and Direction.*—(a) *At an internal point.*—Let P (Fig. 168) be the point where the potential is to be found: through P draw any straight line meeting the shell at Q and Q′, and consider the potential due to the corresponding elements of volume dv and dv' subtending the same small solid angle at P. Suppose the direction of magnetisation represented by the arrow-headed lines through Q and Q′. These lines being parallel, the angles α and α' which they make with QQ′ are supplementary, and $\cos\alpha = -\cos\alpha'$. Then the potential due to the element dv at Q is $Adv\,\frac{\cos\alpha}{\mathrm{PQ}^2}$, and that due to the element dv' at Q′ is $-Adv'\,\frac{\cos\alpha}{\mathrm{PQ}'^2}$. But, since they subtend the same solid angle at P, dv and dv' are to each other as $\mathrm{PQ}^2 : \mathrm{PQ}'^2$. Consequently the contributions of these two elements to the total potential at P are equal and opposite, and as the whole shell can be exhaustively divided up into similar pairs of mutually compensating elements, the total potential at any internal point is zero. As the potential is uniform throughout the interior, there is no magnetic field.

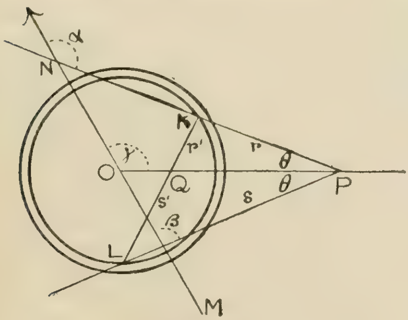


FIG. 169.

(b) *At an external point.*—Let O (Fig. 169) be the centre of the spherical shell, R its radius, and dR its thickness. Let the external point be P, and on OP take Q such that $\mathrm{OP}\,.\,\mathrm{OQ} = R^2$.

Through Q draw a straight line meeting the shell in K and L, and let the uniform direction of magnetisation be parallel to MN. For shortness we will adopt the following notation: PO $= p$, QO $= p'$; PK $= r$, QK $= r'$; PL $= s$, QL $= s'$. Further, we will call the angle between r and MN, α, that between s and MN, β, the angle PON, γ, and the angle OPK and the equal angle OPL, θ. The two angles at P are respectively equal to the angles at the base of the isosceles triangle KOL, and therefore equal to each other. (Compare Fig. 17.)

Consider the potential at P due to an element of the shell at K subtending at Q the small solid angle $d\omega$: the volume of this is $\frac{r'^2\, d\omega\, dR}{\cos\theta}$, and the resulting potential at P is therefore

$$dV = A\frac{r'^2}{r^2}\frac{\cos\alpha}{\cos\theta}\, dR\, d\omega.$$

The geometry of the figure gives $r'^2/r^2 = R^2/p^2$, consequently

$$dV = A\frac{R^2}{p^2}\frac{\cos\alpha}{\cos\theta}\, dR\, d\omega.$$

Similarly the potential at P due to the corresponding element of the shell at L is equal to

$$A\frac{s'^2}{s^2}\frac{\cos\beta}{\cos\theta}\, dR\, d\omega = A\frac{R^2}{p^2}\frac{\cos\beta}{\cos\theta}\, dR\, d\omega.$$

But the angle $\alpha = \gamma + \theta$, and the angle $\beta = \gamma - \theta$; therefore the potential at P due to these two elements of the shell taken together is

$$\frac{2\, A\, R^2\, dR}{p^2}\cos\gamma\, d\omega.$$

Seeing that this expression gives the effect of a pair of elements on opposite sides of Q, we shall get the potential due to the whole shell by integrating $d\omega$ on one side of a plane through Q perpendicular to OP. This gives

$$V = 4\pi A\, R^2\, dR\frac{\cos\gamma}{p^2}.$$

But $4\pi R^2 dR$ is the volume of the shell, and the product of this into A, the intensity of magnetisation, gives the magnetic moment, M, of the shell. Therefore the potential may be written

$$V = M\frac{\cos\gamma}{p^2}$$

which is the expression for the potential at P due to a magnet at O of infinitesimal size but finite moment, M, the axis of which makes an angle, γ, with OP.

Since a uniformly magnetised sphere may be regarded as being built up of concentric spherical shells, to each of which the above expression would apply, we see that if we take M as standing for the magnetic moment of the sphere $(M = A \cdot \frac{4}{3}\pi R^3)$, the same formula will represent the potential due to the sphere.

For a point on the surface of a uniformly magnetised sphere the potential becomes

$$A \cdot \frac{4}{3}\pi R \cos\gamma,$$

since in this case $p = R$. The potential varies from $A \cdot \frac{4}{3}\pi R$, or $\frac{M}{R^2}$, at the positive end of the axis to $-A \cdot \frac{4}{3}\pi R$ or $-\frac{M}{R^2}$, at the negative end of the axis, and is zero in the equatorial plane.

At a point within the mass of a uniformly magnetised sphere, the potential due to the part at a greater distance from the centre than the point is nothing, and the whole value is that due to the sphere whose radius is equal to the distance of the point from the centre. The potential, say $\frac{4}{3}\pi r A \cos\gamma$, is thus directly as the distance of the point from the centre.

The Magnetic Field due to a Uniform Spherical Magnet.—(a.) *At an internal point.*—Take the centre as origin of rectangular coordinates, and take the direction of magnetisation as axis of x. Then $r \cos\gamma = x$, and we may write for the potential at the point

$$V = \frac{4}{3}\pi x A.$$

The equipotential surfaces are therefore a series of equidistant planes perpendicular to the magnetic axis; the polar field within the sphere is therefore uniform, and directed parallel to the axis of magnetisation in the negative sense. This is also expressed by the formulæ

$$-\frac{dV}{dx} = -\frac{4}{3}\pi A, \quad -\frac{dV}{dy} = 0, \quad -\frac{dV}{dz} = 0.$$

(b.) *At an external point.*—Externally the magnet acts like an infinitesimal magnet of moment $\frac{4}{3}\pi R^3 A$ at the centre. The field due to it is therefore given by the formulæ already deduced in **(188)**. The same results may also be deduced by differentiating the expression for the potential.

(vi.) *Ellipsoid of Revolution.*—It is only in rare cases that uniformity of internal field, as in the above example, is a consequence of uniformity of magnetisation.

It can be shown to be so also in the case of an ellipsoid of revolution magnetised uniformly in the direction of its axis of revolution (Fig. 170). Calculation shows that the polar field at internal points due to the magnetisation may be expressed by $F = -PA$, P being a coefficient which depends only upon the ratio of the axes, and which is smaller as the ellipsoid is more elongated. When the axis of revolution a is much greater than the equatorial diameter b the value of P is given approximately by $P = 4\pi \frac{b^2}{a^2}(\log \frac{2a}{b} - 1)$.

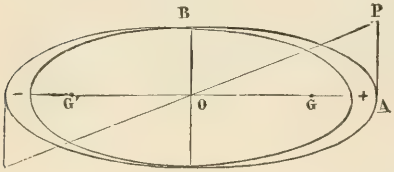


FIG. 170.

For $a = 100b$ and $a = 500b$ (where a and b are the lengths of the axes) calculation gives respectively $P = 0{\cdot}0054$ and $0{\cdot}0003$.

192. Line-Integral of Polar Field.—Since the value of the polar field at a point is obtainable from the space-rate of decrease of the potential, it follows that conversely we may calculate the change in potential between any two points if we know the values and directions of the polar field everywhere along any path connecting them.

Or symbolically, since $$H_x = -\frac{dV}{dx}$$

$$-\int dV = \int H_x dx = \int H \cos\theta \, dx$$

where H is the polar field, θ the angle its direction makes with the path, and H_x its component in the direction of the element of the path dx. Such an integral is known as a line-integral; it may assist the reader if we point out that if H were a mechanical force, the value of the integral would represent the work done by it while the body upon which it acted moved between the same two points.

Since if we return to the starting-point the change in the potential is zero, it follows that the line-integral of polar field taken along any closed path is zero to whatever distribution *of magnetised bodies* it may be due. We shall see later (**265**) that it is possible to produce a magnetic field for which the line-integral taken round a closed path is not zero.

193. Gauss's Theorem—Lines of Field.—The value of the potential of a *very small* magnet, viz., $-\frac{M \cos \alpha}{r^2}$ (**189**), is the same

as if we supposed the magnet to consist of two equal and opposite magnetic charges, obeying Coulomb's *Law of Force*, and of such a magnitude that the magnetic moment equals the strength of the positive charge into the distance between the two. Since this is so, it follows that all the theorems which were established in Chapter V. in respect to electric force will be true also for polar field. Thus, if a closed surface be drawn embracing the magnet, the total outward normal component of field will be zero, since the algebraic sum of the charges inside is zero. This will still be so however many magnets may be inside the closed surface; and also

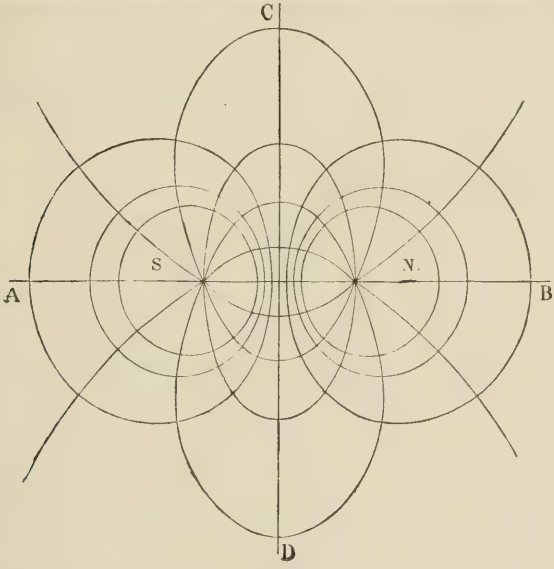


FIG. 171.

if the magnets lie entirely outside the surface. If, however, the surface be drawn so as to enclose an unequal number of positive and negative charges—for example, if in the case of the uniformly magnetised filament (**191**, i.) it surrounds only one of its ends—then the total outward normal field is 4π multiplied into the algebraic sum of the charges enclosed. It also follows from the considerations in (**22**) that the field may be represented everywhere by means of lines, the interpretation of which exactly corresponds to that of lines of electric force. We shall call them lines of *polar field*.

For a thin uniformly magnetised filament (**191**, i.) the lines are as shown in Fig. 171.

They are considered as proceeding in the positive direction of the field (*i.e.* from the north to the south end). The potential at a point distant r_o from the north end and r_1 from the south is given by the formula

$$V = \frac{S}{r_o} - \frac{S}{r_1}.$$

The equipotential surfaces are obtained by giving successively to V the values 1, 2, 3, . . . These surfaces are closed ovoids surrounding each "pole"; they become more and more nearly spherical as we get nearer the poles. All those which correspond to positive potentials surround the north end; those which correspond to negative values surround the south end. They are separated by a plane of symmetry CD at potential zero. The lines produced by the intersection of these surfaces by a plane containing the axis are everywhere perpendicular to the lines of the field traced in this plane.

An equation characterising the lines of field due to any number of poles of given strength situated in a straight line can be found as follows: let m_1, m_2, . . . be the strengths of poles situated respectively at the points Q_1, Q_2 . . .; and let P (Fig. 171A) be a point in the field; then the forces exerted on a unit of magnetism at P by the several poles act along Q_1P, Q_2P . . . and if we denote

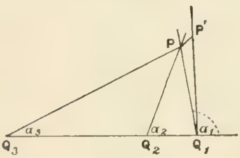


FIG. 171A.

these distances by r_1, r_2 . . . the respective forces are $\frac{m_1}{r_1^2}$, $\frac{m_2}{r_2^2}$ . . . Suppose P′ to be a point very near P on the line of the resultant force through P, and let the angles which r_1, r_2 . . . make with Q_1Q_2, the line of the poles be a_1, a_2, . . . : then the projection of PP′ on a line at right angles to r_1 is $r_1 da_1$ and therefore

$$\frac{m_1}{r_1^2} \cdot r_1 da_1 = \frac{m_1}{r_1} da_1$$

is the moment about P′ of the force through P due to the pole m_1.

Similarly $\frac{m_2}{r_2} da_2$ is the moment about the same point of the force through P due to the pole m_2, and so on for any other poles. Hence

$$\frac{m_1}{r_1} da_1 \text{[1]} + \frac{m_2}{r_2} da_2 + \ . \ . \ . \ = 0.$$

But $r_1 \sin a_1 = r_2 \sin a_2 =$ . . . ; therefore, multiplying the several terms of the last expression by these equal quantities, we have

[1] As the figure is drawn da_3 would be of opposite sign to da_1 and da_2.

$$m_1 \sin \alpha_1 d\alpha_1 + m_2 \sin \alpha_2 d\alpha_2 + \ldots = 0,$$

whence, by integration,

$$m_1 \cos \alpha_1 + m_2 \cos \alpha_2 + \ldots = \text{constant}.$$

As a special case, suppose that there are only two poles and that they are equal and opposite : we then get

$$\cos \alpha_1 - \cos \alpha_2 = \text{constant} = N$$

as the equation characterising the lines of field due to a uniformly magnetised linear magnet. From this equation, it is not difficult to deduce a practical method for drawing the lines of field for the case supposed, which is that of Fig. 171. This figure is to be interpreted as a section, by the plane of the paper, of a figure of revolution about the axis AB or SN. To determine the value of the constant N for a given line of field, we may consider the point where this line cuts the plane CD which bisects SN at right angles. For any such point, $\cos \alpha_1 = -\cos \alpha_2$, and therefore $N = \cos \alpha_1 - \cos \alpha_2 = 2 \cos \alpha_1 = \dfrac{2l}{\sqrt{l^2 + y^2}}$, if $2l$ is the distance of the poles SN and y is the perpendicular distance from SN to the point where the line in question cuts the plane CD.

For two unequal poles of opposite sign, the general expression becomes

$$m_1 \cos \alpha_1 - m_2 \cos \alpha_2 = N.$$

Fig. 26 (**37**) gives the lines of field and equipotential lines for the case where $m_1 = 20$ and $m_2 = -5$.

194. Influence of the Medium.—We must now inquire into the effect that will be produced by changing the medium surrounding a magnet from being air, or other non-magnetic substance, as has tacitly been assumed in the preceding sections, to a magnetisable medium which will itself become magnetised by influence. The result may be most easily obtained by considering an anchor-ring magnet with a thin gap in it (Fig. 167). If the magnetisation of the ring is A_1 we have seen that the field in the gap is $4\pi A_1$ where the magnet is surrounded by air. Replace the air by any medium, such as a solution of ferric chloride ; this will become magnetised by the influence of the ring-magnet : let the intensity of magnetisation of the portion in the gap become A_2. The introduction of the liquid into the gap is equivalent to adding a magnetic shell of this degree of magnetisation, and the polar

field in this shell due to its magnetisation is $-4\pi A_2$. We must add this to the field due to the ring-magnet in order to obtain the total polar field, the value of which therefore becomes $4\pi (A_1 - A_2)$. The polar field in the gap is therefore diminished by inserting any substance which becomes magnetised in the same direction as the ring-magnet. We shall find the relation between A_1 and A_2 in the next chapter.

195. Energy of a Magnet in a Uniform Magnetic Field.—As we have seen (**179**), a magnet of moment M in a field of intensity H is subject to a couple tending to make the direction of its axis coincide with that of the field; also, that the moment of the couple is

$$MH \sin \theta,$$

if θ is the angle between the two directions. If the magnet be turned so as to increase the angle θ by the small amount $d\theta$, work must be done against this couple equal to

$$MH \sin\theta d\theta = -MH\, d(\cos\theta).$$

Consequently if the magnet be turned from a position in which θ has the value θ_o to one in which it has the value θ_1, work is done equal to

$$-MH (\cos \theta_1 - \cos \theta_o),$$

and the magnet gains an equal amount of potential energy. For example, if the initial position be that of stable equilibrium ($\theta_o = 0$), and the final position be the reverse position, corresponding to a rotation of the magnet through two right angles ($\theta_1 = 180°$), the energy of the magnet is increased by $2MH$.

196. Case of a Non-Uniform Field.—The behaviour of a magnet in a uniform magnetic field suggests that the action may be expressed by saying that the product $MH\cos\theta$ tends to become as great as possible, and that the rate of increase of this quantity measures the rate of loss of potential energy by the magnet. When the field is uniform a change of value in this product can result only from a variation of the angle θ, the value of the cosine being greatest when $\theta = 0$. In a non-uniform field, however, that is, in a field where H has different values at different points, the product may also vary with motion of the magnet from one part of the field to another. In such a case it is found that a magnet is not only subject, as in a uniform field, to a couple tending to set the axis along the direction of the field, but that in general it is acted on also by a force of translation urging it in the direction of increasing intensity of field.

For example, if two bar magnets are placed near each other and are each free to move, they will place themselves in contact side by side with opposite ends in the same direction; two plane magnetic shells will place themselves with the south face of one in contact with the north face of the other. In both these cases the position of equilibrium is that in which the relative potential energy of the magnets is zero, and the expression $MH \cos \theta$ has a maximum value for each.

To estimate the magnitude of the force acting in such cases we have to find the rate of decrease of energy corresponding to the motion of translation. For this purpose it is natural to assume that the change of energy due to motion in a non-uniform field is measured by the inverse change in the value of the quantity $MH \cos \theta$, as we have seen that it is in the case of motion in a uniform field. From this point of view we may say that the translatory force acting on a magnet in any direction, x, is given by the rate of loss of energy—or by the rate of increase of the product $MH \cos \theta$—corresponding to motion in that direction. That is, we may write for the force

$$f = M \cos \theta \frac{dH}{dx}.$$

In general the action of a magnetic field on a magnet is represented by a couple, $MH \frac{d \cos \theta}{d\theta} = -MH \sin \theta$, and a force: the couple being proportional to the intensity of the field, while the force is proportional to the space-rate of variation of the field, and accordingly vanishes when the field is uniform.

We have seen (**188**) that the field at a distant point due to a small magnet of moment M is M/r^3 multiplied by a numerical factor depending on the direction of r. It follows that the couple exerted by one small magnet upon another is proportional to the inverse *cube* of the distance between them, but that the force tending to cause mutual displacement in any direction is proportional to $\frac{dr^{-3}}{dx} = -3r^{-4}\frac{dr}{dx}$, that is, to the inverse *fourth* power of the distance. The force thus decreases very rapidly, and may become insensible at a distance where the couple is still appreciable.

The force urging a bit of iron, or other magnetic material, towards the pole of a magnet presented to it, is an example of the tendency of a magnet to move, in a magnetic field, in the direction of increasing intensity of field, for the bit of iron becomes a

magnet for the time being. In such a case, the force urging the iron towards the magnet increases with diminution of distance even more rapidly than the field, for the magnetisation of the iron increases with the strength of the field and therefore both factors of MH become greater simultaneously.

We shall see later (**220**), that *diamagnetic* bodies tend to move from the stronger to the weaker parts of the field, and are therefore apparently repelled by a magnetic pole.

CHAPTER XIX

MAGNETIC INDUCTION

197. Field inside a Magnet—Magnetic Induction.—We have defined the polar field of a magnet—both within and around it—to be the field calculated from the magnetic potential. We have no right to assert that this process will give the actual field in the magnet—as we have already pointed out. We shall now examine the matter further. A magnet, according to modern conceptions, consists in an assemblage of molecules each of which is itself a magnet; and the resultant magnetic effect of the whole is due to the more or less complete alignment of these elements (**174**). We shall consider the space round each of these molecules to be non-magnetic, that is to say, to differ in no essential respect from the medium (air) outside the magnet. The magnetic field in these interspaces could therefore be calculated without ambiguity in terms of our definition of the magnetic field *in air*, provided we knew the magnetic moment, shape, and position of each of the molecular magnets, or provided we could find experimentally the turning-moment on a minute standard magnet placed at the point in question. Although this is impossible experimentally, there is no theoretic difficulty involved in the conception. Now our knowledge of the field round actual magnets makes clear that this internal field must vary very rapidly from point to point. Considering its value at any point as the resultant effect of all the molecules, we may separate this value into two parts: there is the part contributed by the molecules in the immediate neighbourhood of the point, and the part contributed by the more distant molecules. The rapid fluctuations from point to point of the field are due to the former; the effect of the latter, on account of their greater distances, being much more uniform. This more uniform portion is what we have called the *polar field at the point*. That this is so may be ascertained by the following considerations applied to a uniformly magnetised bar. We regarded the potential at a point inside the magnet as being the sum of the potentials due to the uniformly magnetised filaments of which the bar may be built

up, the potentials due to these depending only upon their ends. Now, if we leave out of consideration the particular filament which passes through the point, or, what comes to the same thing, if we suppose it replaced by air, the value of the field in the air-gap so formed may be found by differentiating the potential arising from the remaining filaments. But if the filament so left out of account has a cross-section comparable with only a small multiple of the dimensions of a molecule, its omission will not make any sensible difference in the calculated value of the field; at the same time, it is clear that its omission is equivalent to leaving out of consideration the very molecules which must be most effective in producing the actual field in their neighbourhood. Thus what we have defined as the polar field at a point in the magnet cannot in every case be the actual field there. Moreover, while our experi-

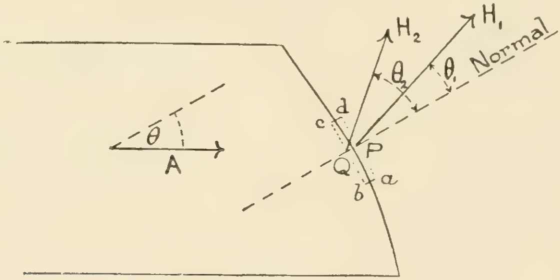


FIG. 172.

mental methods remain as coarse as they are, the actual value is altogether beyond our reach.

198. Magnetic Induction.—But, as we shall now proceed to show, its *average value* over any cross-section, large compared with the magnitude of a molecule though small compared with the magnet itself, can be found both experimentally and theoretically. We shall call this average value the *magnetic induction.*

It has been shown (**193**) that the number of lines of polar field which leave any closed surface is $4\pi \times$ magnetic charge enclosed. If we take, for example, a bundle of magnetised filaments ending in a plane surface of area a (Fig. 172), which may be inclined to the direction of magnetisation, the total magnetic charge which may be supposed on the boundary is $Aa \cos\theta$, where θ is the angle between A and the normal to the boundary; and the total outward normal component of field through any surface Σ which includes only this charge is consequently $4\pi Aa \cos\theta$. If, in particular, this surface be drawn so as just to enclose the ends of the

filaments (see dotted lines, Fig. 172), in which case it will consist of two plane surfaces connected by a rim of infinitesimal width, the total outward normal component may be considered as sensibly that through the two plane faces, and is consequently

$$(H_1 \cos \theta_1 - H_2 \cos \theta_2)a,$$

where $H_2 \cos \theta_2$ and $H_1 \cos \theta_1$ are the normal components of field (measured from left to right) inside and outside respectively. Since this expression must equal $4\pi Aa \cos \theta$, we may state that the normal component of polar field at the boundary of iron and air is greater outside than inside by an amount equal to $4\pi \times$ the normal component of intensity of magnetisation in the iron.

If now we find the line-integral of the polar field round the rectangle *abcd*, we obtain $H_1 \sin \theta_1 \times ad - H_2 \sin \theta_2 \times bc = 0$ **(192)** and since $ad = bc$

$$H_1 \sin \theta_1 = H_2 \sin \theta_2$$

—that is to say, the component of the polar field which is parallel to the boundary has the same value on both sides of it.

These results are obtained on the supposition that the magnet is a continuous magnetic substance and that the field within it is to be calculated from the same potential as for points outside.

If now we consider a magnet to be built up of molecular magnets with non-magnetic interspaces, the value of A can only be an *average* value over a space small compared with the magnet but large compared with its constituent molecules. In order to apply Gauss's theorem to such an assemblage without ambiguity, it is necessary to specify more completely how the closed surface is drawn. If it is described so as to include *only the north ends* of the elementary magnets which lie near the surface, in which case it must cut through these molecules, we obtain a result in exact agreement with that deduced above, that is to say, this mode of reckoning gives us the *polar field* of the magnet. If, however, the surface be described so as to include an equal number of north and south ends of the elementary magnets—that is to say, so as to pass entirely through non-magnetic material—the total outward normal component of field through it must equal zero **(193)**. If the surface have the same shape as in Fig. 172, it will be seen that the average normal component of field inside is, in this case, *equal* to that just outside; *i.e.* it is greater than when only the north poles were inside the surface by the amount $4\pi A \cos \theta$. Now the essential difference between the two cases is that in the former the surface under consideration passes partly through air and partly through magnetic molecules, while in the latter it passes wholly through non-magnetic material. It is only in the

latter case that we can assert definitely that we are dealing with the actual field in the interspaces (or at least its average value); for, in the former case, the same uncertainty exists with regard to the significance of the polar field *inside the molecules cut by the Gauss surface* as with regard to the polar field of the entire magnet. We have seen, however, that the result in the former case was in agreement with that previously obtained for the polar field of the entire magnet; thus the average field (*i.e.* the magnetic induction) in any direction making an angle θ with the magnetisation is greater than the polar field in the same direction by the amount $4\pi A \cos\theta$. In non-magnetic material—where A is zero—there is no necessity to distinguish between the two quantities; hence *the normal component of magnetic induction has the same value on the two sides of a surface of separation between the magnet and air.*

This theorem leaves undetermined the *direction* and magnitude of the resultant magnetic induction. This may be determined as follows:—We must regard the quantity $4\pi A \cos\theta$ as the component in the given direction of that portion of the total field which is due to the molecular magnets in the neighbourhood of the point in question (that is to say, those whose influence was left out of consideration in determining the polar field at the point). Hence the quantity $4\pi A$ represents the total field (supposed parallel to the magnetisation) which arises therefrom. The value of the magnetic induction in the interior of a magnet is therefore the resultant of this field, $4\pi A$ (arising from the magnetisation in the neighbourhood), and the polar field; to which must be added the field due to other magnets, if there are any such. Thus we may write

$$B = H + F + 4\pi A$$

where B = induction, F = the polar field, H the field due to other sources, and the terms on the right are to be added by the parallelogram law. When H, F, and A are parallel to one another, B will also coincide in direction with them; in general, however, this parallelism does not exist.

In a later section (**300**) we shall describe phenomena which depend upon the rate of change of the number of lines of total field threading through an area. When this area encloses a magnet, it is clear that it is the induction and not the polar field which is the quantity concerned; for the polar field only corresponds to a portion of these lines.

The continuity of the normal component of induction at an interface between iron and air enables the induction to be represented there by continuous lines (**50**). If the air be replaced by a magnetisable medium the continuity still exists. For the medium

itself becomes a magnet to which the same theorem would apply if it were surrounded by air. But the induction everywhere is the sum of the inductions due to these two magnets respectively, and the theorem, if true for each, is true for their sum. Thus, in the case of every distribution of magnetised and non-magnetisable media, *the induction may everywhere be represented by re-entrant lines.*

199. Refraction of Lines of Induction.—Though the lines are continuous across an interface, their direction, in general, is different on the two sides. For the tangential component of the polar field is the same on the two sides (**198**). To obtain the tangential component of magnetic induction we must add on each side $4\pi A \sin\theta$, when A is the intensity there. Since $A \sin\theta$ is, in general, different on the two sides (*e.g.* if the medium be air its value is necessarily zero *in air*, while it may have a finite value in the magnet), the tangential components of induction differ from one another. The normal components being alike, the resultants must in general differ in magnitude and direction. In the important case in which the external medium is non-magnetisable (*e.g.* air)—in which case the field and induction outside coincide—we may write, using suffixes 1 and 2 for inside and outside respectively :—

$$B_2 = F_2' \qquad \theta_2 = \phi_2$$
$$B_1 \cos\theta_1 = B_2 \cos\theta_2 = F_2' \cos\theta_2$$
$$F_1' \sin\phi_1 = F_2' \sin\phi_2 = F_2' \sin\theta_2$$
$$B_1 \cos\theta_1 = F_1' \cos\phi_1 + 4\pi A \cos\psi$$
$$B_1 \sin\theta_1 = F_1' \sin\phi_1 + 4\pi A \sin\psi,$$

where $F' =$ resultant of H and F, and θ, ϕ, and ψ are the angles which B, F', and A respectively make with the normal to the interface.

We shall call the total number of lines of induction passing through any area *the total magnetic induction*, or *the magnetic flux* through the area. By analogy with the corresponding theorems in electrostatics (**50**) all space may be divided into tubes whose boundaries are everywhere parallel to the lines of induction; a tube through which the flux equals unity, we shall call a unit tube. All the theorems which apply to tubes of electric induction *in regions free from electric charges*, apply equally to tubes of magnetic induction. Where there are electric charges the tubes of electric induction cease to be continuous, whereas we have seen that lines of magnetic induction (and consequently the tubes they trace out) are continuous *everywhere*. There is therefore no magnetic property corresponding to that which we describe when we say that a body

is electrically charged. The reader must therefore be warned against hastily concluding that, in the few cases in which we have spoken of a magnetic charge from which the values of the field can be calculated by an application of Coulomb's law of force, there is a complete analogy of properties between it and an electric charge. The properties of the magnetic induction and field within and in the neighbourhood of a magnet correspond, in fact, to those of the electric induction and electric force of a rod of shellac, say, of the same shape as the magnet, immersed in an electric field; with the exception that, in the electric case, induction and force are proportional to one another, while in the magnetic case, this proportionality between the corresponding quantities, as we shall see, does not generally exist.

200. Special Cases.—The above theorems may be illustrated by considering the case of a permanently magnetised sphere as in (**191**). The lines of induction and polar field are shown in Fig. 173. The polar field inside, as we have seen, is equal to $-\frac{4}{3}\pi A$. The magnetic induction is therefore (assuming no magnets near, whence $H=0$) $-\frac{4}{3}\pi A + 4\pi A = \frac{8}{3}\pi A$. Thus the magnetic induction in the magnet is twice as strong and is oppositely directed to the polar field. Outside, the polar field and induction coincide. At the interface the lines of induction are continuous, while those of the field are not so. The figure is so drawn that where the induction is uniform it is represented by equally distant parallel straight lines. These lines must be regarded as representing the boundaries in the plane of the paper of square-sectioned unit tubes. Since the cross-sectional area is proportional to the square of the distance between two lines, a field of $\frac{1}{2}$ the strength is represented by lines $\sqrt{2}$ times as far apart. In the outer region where the lines are curved, the lines forming the boundary of a tube diverge at different angles in different planes; in consequence, the section of the tube is no longer square, and no simple statement can be made by aid of which the value of the induction in this region can be inferred from the closeness of the lines. At every point, however, the *direction* of the induction is correctly indicated.

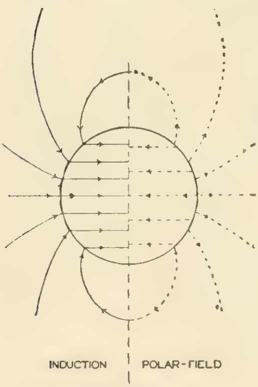


FIG. 173.

In Fig. 174 is shown the case of a nearly closed ring-magnet, supposed to be uniformly magnetised along its length (**191**). Although the magnetisation is everywhere parallel to the length of the magnet, the induction is not so; for the induction is obtained by compounding the polar field with $4\pi A$. The consequence is that the lines of induction diverge as they approach the "poles," and some of them leave the iron. In fact, since induction and polar field coincide in the surrounding air, it follows that a line of induction must leave the iron wherever a polar line crosses over the surface.

In a real magnet of the form shown in Fig. 174 the intensity of magnetisation falls off as the neighbourhood of the terminal faces is approached. This decrease causes a further divergence of the lines of induction.

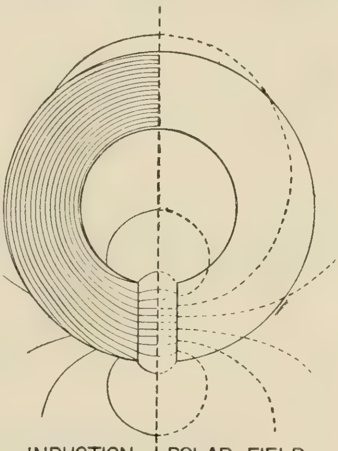


FIG. 174.

201. Susceptibility. Permeability.—We have seen that iron and other magnetic bodies become magnetised when placed in a magnetic field; it is further found experimentally that the intensity of magnetisation produced depends upon the strength of the field and in general increases with it. The ratio of the intensity of magnetisation to the field producing it is called the *magnetic susceptibility* of the substance; we shall denote it by κ. Thus if A is the magnetisation and F' the field,

$$A = \kappa F'.$$

The equation for the magnetic induction may now be written $B = F' + 4\pi A = (1 + 4\pi\kappa) F'$. The factor $(1 + 4\pi\kappa)$, which expresses the relation between the induction and the field, is called the *magnetic permeability* of the substance. Denoting it by μ we write

$$B = \mu F'.$$

The value of κ is positive for magnetic, and negative for diamagnetic, substances; but for all, except iron, cobalt and nickel, its numerical value is small. Accordingly, μ is greater than unity for magnetic substances, and less than unity, but still positive, for diamagnetic substances.

For diamagnetic substances, and for very slightly magnetic

substances, κ and μ may be considered as constant numerical coefficients, but for iron and similar substances they are very complicated functions. For a given substance under otherwise uniform conditions they vary with the magnetising field.

202. Influence of the Form of a Body.—It must be understood that F' is the *magnetic field* at a point of the substance *after the magnetisation is produced;* that is to say, in order to obtain its value we must compound the influencing field with the field (if any) calculated from the potential arising from the magnetisation. In the case of a closed anchor-ring placed in a concentric circular field, so as to become uniformly magnetised, no such potential exists, and F' is then the influencing field alone. If a straight rod be placed with its length parallel to a uniform magnetic field, the field due to the magnetisation produced is not uniform, and therefore neither is the induction.

It is only in rare cases that the polar field of the resulting magnet is itself uniform. We know that it is so in the case of a sphere or of an ellipsoid of revolution with its axis parallel to the field. To calculate the magnetising field F' we must subtract the reaction of the ellipsoidal magnet from the influencing field H. Representing, as in (**191**, vi.), this reaction by PA, we have

$$F' = H - PA,$$

and since

$$A = \kappa F'$$

it follows that

$$\kappa = \frac{A}{H - PA}.$$

The magnetising field expressed as a fraction of the influencing field is given by

$$\frac{F'}{H} = \frac{1}{1 + \kappa P}$$

—that is to say, it is smaller as P is larger. For a sphere $P = \frac{4}{3}\pi$ and for a value of κ equal to 100, the magnetising field would not be $\frac{1}{400}$ that of the influencing field. In order that F' may have a considerable ratio to H it is necessary to have recourse to very elongated ellipsoids of revolution. In an ellipsoid of revolution for which $a = 500b$ ($2a$ being the axis of revolution and $2b$ the equatorial axis), the magnetising field, with $\kappa = 100$, would be 0·971 of that of the field. (Comp. **191**, vi.)

Most frequently very long cylinders are used, with a length 400 or 500 times the diameter, and these are placed in a uniform field and parallel to it. In these conditions the action of the ends may be considered negligible.

203. Observations on Permeability.—A magnetic body

placed in a uniform field, by the very fact of its magnetisation, produces a perturbation of the lines of field; near the body these cease to be parallel and equidistant; at the same time lines of magnetisation (**198**) to the number of $4\pi A$ per unit area traverse the body, the magnetisation of which we assume to be uniform. The lines of induction, as we have seen, are a continuation of those

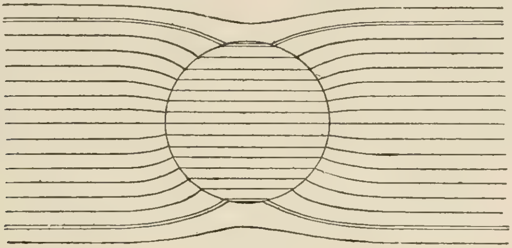


FIG. 175.

of the field, but are packed more closely together, or are further apart than in the undisturbed field, according as the body is magnetic or diamagnetic. Figs. 175 and 176 represent two spheres placed in a uniform field, and such that for the former $\mu = 2{\cdot}8$, and for the latter $\mu = 0{\cdot}48$. We may regard the state of the body submitted to influence as resulting from a modification of the medium which constitutes it, analogous to that which existed in the medium, air

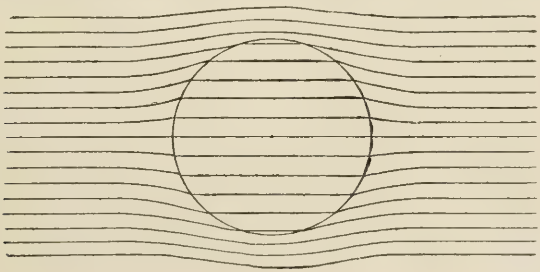


FIG. 176.

or vacuum, whose place it has taken, with this difference, that the flux of induction for unit surface instead of being H has become equal to $\mu F'$.

A comparison with currents will render the idea more precise. Suppose a sphere of metal introduced into a mass of mercury traversed by a uniform current: the lines of flow, which were originally parallel, would tend to pass in greater number through

the sphere if it were a better conductor than mercury, and, on the contrary, in smaller number if it were a worse conductor. Figs. 175 and 176 give a representation of the lines of flow in the two cases. The words *conductivity* for lines of flow, and *permeability* for lines of magnetic induction, thus correspond to analogous ideas.

204. Magnetic Screens.—If a hollow block of iron be placed in a magnetic field, the perturbation of the lines by the iron is such as to make the field within the cavity less intense than the undisturbed field. In the case of a hollow cylinder placed with its axis transverse to a uniform field the lines of induction are as shown in Fig. 177, in which the lines are the boundaries of tubes of rectangular cross-section, whose depth (perpendicular to the

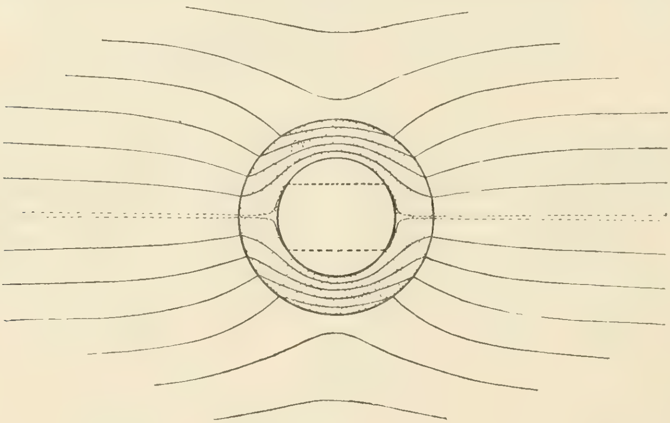


FIG. 177.

plane of the paper) is supposed constant. The reciprocal of their distance apart at any place is therefore a measure of the intensity of induction there. The figure represents the case where the permeability is 100. It will be seen that the two dotted lines of induction which lie close to the axis of symmetry diverge rapidly on approaching the inner surface, and are widely separate inside the cavity; the field inside (which is uniform) is in fact only ·06 of that outside. The iron acts as a kind of screen protecting the inside of the cavity from the influence of the external field.

It should be noted that many of the lines pass round through the iron. Although they are more crowded there than they are in the influencing field which gives rise to them, they are not so numerous as the high value of the permeability might lead one to expect. The result is, of course, due to the large negative polar

field which diminishes the effectiveness of the influencing one. This crowding together of the lines in the iron has practical applications which are referred to in the sequel (**422**).

205. Influence of the Medium.—We can now further examine the influence that a change in the medium round a permanent magnet will have. Refer to (**194**), where we found that in a narrow air gap in a ring-magnet the polar field is changed by the introduction of a magnetisable medium from $4\pi A_1$ to $4\pi (A_1 - A_2)$, where A_2 is the magnetisation in the material occupying the gap. The magnetic induction in the gap is this field added to $4\pi A_2$; that is, it is $4\pi A_1$; thus the magnetic induction is not changed by the alteration in medium. The ratio of the induction to the polar field is the permeability μ_2 of the medium introduced: therefore

$$4\pi A_1 = \mu_2 4\pi (A_1 - A_2),$$

or

$$\mu_2 A_2 = A_1 (\mu_2 - 1).$$

The ratio of the new to the original value of the field is $(A_1 - A_2)/A_1$, and this by the first equation is equal to $1/\mu_2$.

The couple exerted by one magnet on another depends upon the magnitude of the field and not upon the induction. It may therefore be written equal to $MH \sin \theta$ whether the medium is air or not; but it must be remembered that, to produce a field of given strength H, requires a stronger magnet in a medium of higher permeability; for example, the field on the axis of a small magnet in a medium of permeability μ is $h = \dfrac{2M}{\mu r^3}$. That the couple depends on the field may be seen by supposing the air-gap in the ring-magnet to be filled up with steel identical in properties to the ring-magnet and test-magnet themselves; it is evident that no action can then occur. In such a case it is the field that vanishes, while the induction retains its old value $4\pi A_1$.

206. Curves of Magnetisation.—Suppose now that an infinitely long cylinder is placed parallel to the lines of induction in a uniform field, and that then, while gradually increasing the strength of the field, we measure the intensity of magnetisation for every value of F', which, in this case, is the same as H. The results may be represented by a curve, the values of F' forming the abscissæ, and those of A the ordinates. And since $B = F' + 4\pi A$, or $= 4\pi A$ nearly for highly magnetic bodies, the same curve will represent the induction, provided we change the scale of the ordinates, and take the unit of length as indicating 4π times as great a value.

In the case of iron or its varieties, or cobalt, or nickel, the curve has always the same general appearance. Three parts may be distinguished: the first (Fig. 178), corresponding to very small magnetising fields, in which the intensity increases very slowly, but at first proportionally to the field; a second, where the curve rises very rapidly, and has a point of inflection to which the maximum value of κ corresponds; lastly, a third part corresponding to high values of the magnetising field, and where the magnetisation increases very slowly and evidently tends to a maximum.

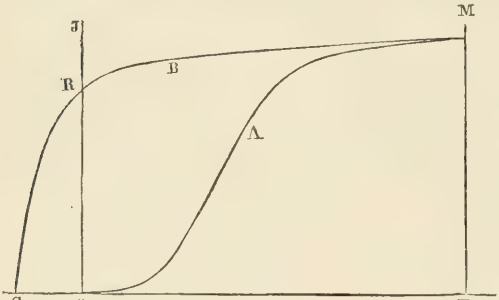


FIG. 178.

207. Remanent Magnetism.—Suppose now that after having raised the magnetising field to the value $F'' = \text{OF}$, we make it gradually decrease, the magnetisation does not by any means again pass through the same values. The course of the phenomenon is represented by the curve B. When the magnetising field has become nothing, the bar has still an intensity of magnetisation represented by OR. This is called the remanent or residual magnetism. Its value is generally a large fraction of the total magnetism, and it is more or less stable according to the nature of the body.

208. Coercive Force.—If after having reduced the magnetising field to zero, it is again increased, but in the opposite direction, the magnetisation is quickly reduced to zero. The inverse field OC which just destroys the residual magnetism may be taken as a measure of the coercive force, and it is in this sense that we shall use this term in what follows.

209. Cycles of Magnetisation.—Imagine that after having exposed the bar to fields increasing from zero to F', the field is decreased from F' to zero, then again raised to F', and so on successively.

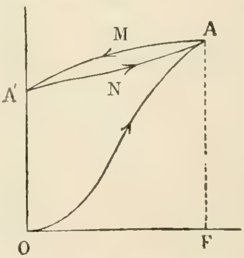


FIG. 179.

The results will be such as are represented by Fig. 179. The substance after having passed through the changes represented by

the curve O to A in the first period, then from A to A′ along AMA′ in the second, will return from A′ to A by A′NA in the third. As the magnetising field continues to pass through the same cycle, the point which represents the state of the body at each instant will continue to describe the same loop AMA′N.

If instead of varying the magnetising field from zero to F' and conversely, it is made to oscillate between two equal values of opposite sign $+F'$ and $-F'$, the curve representing the magnetisation is like Fig. 180. The curve OA corresponds to the initial magnetisation when the magnetising field increases for the first time from 0 to F'; the portion ACA′ to values of the force decreasing from $+F'$ to $-F'$; finally the portion A′BA to values increasing from $-F'$ to $+F'$. The same cycle is gone through every time the same changes of magnetising field are repeated.

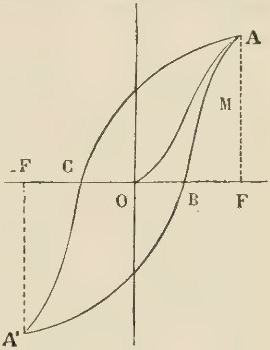


FIG. 180.

210. Retardation of Magnetisation — Hysteresis.—Thus, to the same value of the magnetising field, intensities of magnetisation correspond which depend not only on the existing conditions, but also on the previous ones. In particular, when the successive conditions form a cycle like that in Fig. 180, it may be noted that, for the same value of the magnetising field, the intensity is greater in the descending than in the ascending period. This may be expressed by saying that there is a *retardation* or lag of magnetisation in respect of the magnetising field. Professor Ewing, to whom most of these observations are due, has given the name *hysterēsis* to this tendency of bodies to persist in a previous condition.

This lag may be regarded as an effect of coercive force. The two curves which form the cycle of Fig. 180 diverge more from each other as the coercive force, represented by OC, is greater. The lag is diminished if the body is subjected to vibrations while it is under the influence of the magnetising field. The general effect of the vibrations is to make the curve of increasing magnetisation rise more rapidly, and to make the curve of decreasing magnetisation fall more rapidly. One consequence of the effects of lag is that the coefficients κ and μ, defined by the equations of (**201**), have no precise meaning, except for a body submitted to a definite magnetising field applied in a definite manner.

211. Neutral State.—If starting from any point, M, of the cycle, corresponding to the field F', we decrease the field to zero, and then increase it from zero to F', the curve will form a loop like AA′ in Fig. 179. By suitably choosing the point M, on the positive part of the ascending branch, or on the negative part of the descending branch, we may obtain zero magnetisation when the magnetising field becomes zero. Although there is no magnetisation, the condition of the bar is not identical with that which it had before being magnetised at all; it acquires more easily a magnetisation of the opposite sign to that which it has lost than a magnetisation of the same sign.

If the bar is to be restored to a symmetrical state, which is really neutral, it must be submitted to the action of fields alternately in opposite directions, gradually decreasing to zero; in other words, it must be made to pass through a series of cycles like that of Fig. 180, and corresponding to values of F' gradually diminishing to zero.

For bodies with a very small coercive force, the same result is obtained by repeated blows. The neutral state to which we attain by these methods is a constant one, but it is not necessarily that which the bar had before any magnetisation. A red heat also restores the bar to the neutral state.

212. Work of Magnetisation.—Let F' be the magnetising field, and A the intensity of magnetisation: if the latter be increased by dA, the work for each unit of volume is that which must be expended in order to bring unit volume, with magnetisation dA, from a place where the field is zero to the point in question. But this work is equal to the product of the moment, dA, by the normal component of the field (**196**), which is here the field F' itself; it is thus equal to $F'dA$. On the other hand, the product $F'dA$ is the area of the infinitely small trapezium $mnqp$ (Fig. 181); it follows, therefore, that the work expended in causing each unit of volume to pass from the condition M to the condition N is represented by the curvilinear area MNQP.

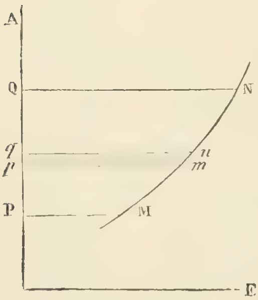


FIG. 181.

If the curve of demagnetisation coincided with the curve of magnetisation, the work corresponding to a cycle would be nothing, and we should therefore conclude that the work expended was altogether represented by the increase of the energy of the magnet.

As the two curves do not coincide, owing to hysteresis, it is easy to see that the area which they enclose represents that portion of the work of magnetisation which has been expended in each unit volume against forces analogous to friction and can only be transformed into heat.

If the magnetising field and the intensity of magnetisation are expressed in C.G.S. units, and the figure is drawn to such a scale that 1 cm. measured horizontally represents unit of field-intensity, and 1 cm. measured vertically represents unit-intensity of magnetisation, the area measured in square centimetres will give the work transformed into heat expressed in ergs per cubic centimetre, and dividing by $4{\cdot}18 \times 10^7$, we shall have the quantity of heat in gramme-degrees. If the density of iron be taken as $7{\cdot}8$, and its specific heat as $0{\cdot}11$, the thermal capacity of one cubic centimetre is $0{\cdot}858$. The rise of temperature for each erg per cubic centimetre will therefore be

$$\frac{1}{0{\cdot}858 \times 4{\cdot}18 \times 10^7} = 2{\cdot}8 \times 10^{-8} \text{ degrees Centigrade.}$$

In the case of Fig. 180 the area of the cycle, as readily seen, is sensibly equal to four times the product of the maximum magnetisation by the coercive force.

Although it is thus possible to find the energy converted into heat in each cycle, nothing is known as to the amount so converted in any separate part of a cycle. The area of the trapezium $mnqp$ represents the sum of the energies stored and dissipated, and there is no means known of separating them. If the heat produced in any part of the cycle could be measured thermometrically this additional knowledge could be obtained; its amount, however, is too small for this to be done.

213. Numerical Values—Soft Iron.—Fig. 182 gives the curves of magnetisation for soft iron, steel, cast-iron, cobalt, and nickel. They all show very definitely the three features mentioned above (**206**), but with differences in the general aspect, and particularly in the intensity.

For soft iron the curve rises with extreme rapidity; the point of inflection which corresponds to the maximum of κ occurs for a value of the magnetising field equal to about 5 units. For ordinary iron κ is then equal to about 150, but it may go up to 250 for very soft iron. For the sample represented, it is about 180.

It will be seen that saturation is nearly reached for $F' = 15$. This corresponds to a value of A of over 1200. It has been

possible to reach 1700 by the use of a very strong magnetising field.

These curves show well the existence of a maximum, although they only refer to values of the field comprised between zero and 150 C.G.S. Experiment shows, moreover, that the intensity of magnetisation undergoes no appreciable variation when the field is altered from 2000 or 3000 units to the highest values which have been attained, about 20,000. The induction presents no maximum, for it consists of two parts, one $4\pi A$, which soon becomes constant, and the other, F', which increases without limit.

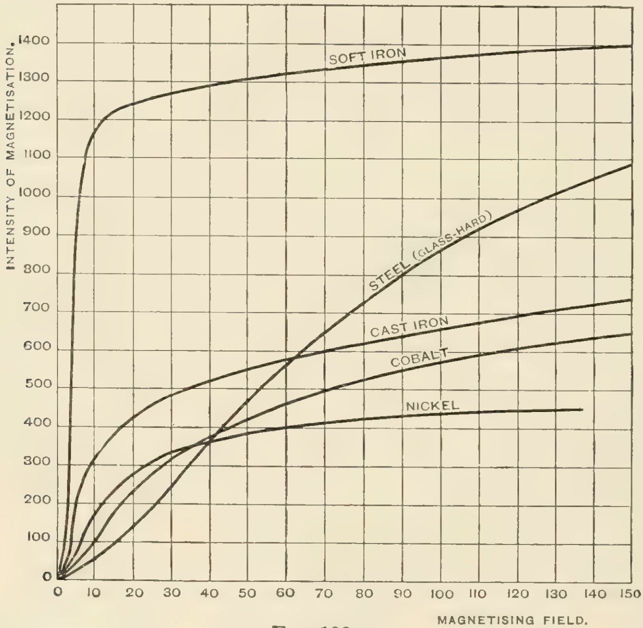


Fig. 182.

The residual magnetism may reach 90 per cent. of the total magnetism, but it is extremely unstable. If the reaction of the ends has a sensible value, or if the bar is submitted to vibration, the magnetism disappears to within a few hundredths. Thus practically we may say that the magnetism of soft iron is purely temporary.

The coercive force does not exceed 2·5, and the work corresponding to a complete cycle of magnetisation is about 15,000 ergs per cubic centimetre. For very weak fields (from ·00004 to

·04 C.G.S. units) Lord Rayleigh found that there was no hysteresis, and that the permeability through this range was about 81.

214. Cast-Iron and Steel.—The intensity of magnetisation of cast-iron is less than that of soft iron, and it less rapidly attains a maximum. For soft cast-iron the maximum is about 900, or about 1250 as an extreme value. The coercive force is about 12, and the work of the complete cycle of magnetisation varies from 40,000 to 60,000 ergs.

The curve for steel rises more gradually, and the numerical values vary greatly with the composition of the steel and its state as to hardening or annealing; but the maximum intensity which it can attain is not much lower than that for soft iron.

The coercive force is much greater. It increases with the proportion of carbon, but it is increased more particularly by the presence of certain metals. Chromium-steel or steel containing 1 per cent. of chromium has a coercive force of 40; steel containing 3 per cent. of tungsten a coercive force of 51. The work in ergs per cubic centimetre for a complete cycle is 40,000 to 60,000 for mild steel, 95,000 for piano-wire which has been annealed, and 116,000 for the same wire which has not been annealed. In like manner it is 65,000 for chromium-steel which has been annealed, and 167,000 for the same steel unannealed. A steel containing tungsten gave 216,000.

The presence of manganese takes from iron its magnetic properties; the alloy with 12 per cent. of manganese known as manganese-steel is not magnetic.

215. Cobalt—Nickel.—The curve for cobalt differs little from that for cast-iron. The maximum intensity of magnetisation is from 1000 to 1300; the coercive force 12, and the work corresponding to a complete cycle 40,000 to 50,000.

The maximum magnetisation for nickel is about 500; the effect of straining or annealing is much more marked with nickel than with iron. The work per cubic centimetre for a complete cycle of magnetisation, which is 25,000 for strained nickel, is only 11,000 for the same metal annealed.

216. Influence of Temperature—Critical Temperature.—Temperature has a considerable influence on magnetisation. The cardinal fact is that there is a temperature for the three magnetic metals beyond which they completely lose their exceptional properties, and become non-magnetic like ordinary bodies. This temperature is 785° for soft iron, 340° for nickel. It has not been determined exactly for cobalt, but is higher than for iron. The property disappears suddenly, and, so to speak, without any transi-

tion. Thus at 770° soft iron is at least 10,000 times more magnetic than at 785°.

At lower temperatures the permeability of iron undergoes very different variations, according to the value of the magnetising field. For very small fields, 0·3 for instance, it increases slowly with the temperature, then more rapidly, and a little below the critical temperature it attains a considerable maximum, disappearing suddenly a little above. For moderate and for strong fields, it is virtually constant up to 500°, and then decreases gradually.

The critical temperature, which is 785° for soft iron, is only 735° for steel. It is lower as the proportion of carbon increases. The phenomenon is due to a profound change of state which at the same time modifies other physical properties. Thus the electrical resistance, which varies up to that point in an abnormal manner with an increasing coefficient, after passing the critical temperature, has a constant coefficient which differs little from that of non-magnetic metals.

This change of state is also characterised by an absorption or a disengagement of heat, according as the temperature is rising or falling. When a piece of iron wire which has been heated to a very high temperature is allowed to cool, several points are observed at which there is an abnormal disengagement of heat corresponding to some change of state. Two such points are specially marked: one between 800° and 700° corresponds to the critical point of magnetisation and to a change of state of the iron; the other is at 660°, and has no relation to magnetism, but appears to be due to a modification of the carbon contained in the iron. At this latter point the quantity of heat disengaged is sometimes so great that a portion of iron wire which has sunk to a dull red heat again becomes incandescent.

A curious illustration of these changes of state is afforded by the alloys of nickel and iron. The alloy of 25 per cent. of nickel when raised to a temperature of 580° is no longer magnetic at lower temperatures; but when cooled to 20° below zero, it becomes magnetic, and remains so for all temperatures below 580°; so that at mean temperatures the same metal may exist in two different conditions, one magnetic, and the other not so, both conditions being equally stable.

217. Other Bodies Magnetic or Diamagnetic.—Certain alloys of copper, aluminium, and manganese have considerable magnetic permeability. One in which the proportions are copper 61·5, aluminium 15, and manganese 23·5 has a permeability as

high as 27. These alloys are known as Heusler's alloys. They exhibit a considerable amount of hysteresis.

So far as can be observed, for bodies which are but slightly magnetic, or which are diamagnetic, the values of the coefficients of magnetisation are constant, and the intensity of magnetisation is always proportional to the magnetising field. Moreover, no phenomena of hysteresis have been detected.

When the susceptibility is very small it becomes correspondingly important to define accurately the substance which is taken as a standard. Hitherto we have taken it indifferently as being either air or vacuum. Owing to the variability of air it is preferable to take the susceptibility of *vacuum* (*i.e.* space devoid of ordinary matter) as zero, and to reduce to this standard all observations made on any material immersed in air. This reduction is effected with sufficient accuracy by adding the susceptibility of air to the observed susceptibility of the substance under examination. The value of the susceptibility for air at 18° C. and one atmosphere pressure is about $\cdot 027 \times 10^{-6}$.

For diamagnetic bodies, the value of κ is always very small. The highest known value is that for bismuth, for which

$$\kappa = -\frac{1}{400,000} = -2\cdot 5 \times 10^{-6}.$$

The value for water is $-0\cdot 75 \times 10^{-6}$.

218. Extension of the Theory to include Hysteresis.—In developing the theory we supposed the magnetic induction to vary with the magnetising field in such a way that, to each value of field, there corresponds a definite value of induction. The phenomena which have just been described show that the single-valued correspondence exists only under exceptional circumstances: in most cases more than one value of induction corresponds to one value of field. In particular, the existence of permanent magnets shows that a finite induction can exist where there is no influencing field, and in consequence the polar field inside the magnet—which is that due to the magnetisation—is either oppositely directed to the induction or, in the case of closed magnets, is zero. The phenomena are therefore more complicated than the simple theory would imply.

This theory would be made more complete by considering the observed induction to consist of two components, one of which is a definite function of the field and is parallel to it, while the other cannot be expressed in terms of the field without ambiguity. Thus we may write

$$B = B_0 + \mu F'$$

where F' is the magnetising field and the two terms on the right are to be added by the parallelogram law. When F' is zero, $B = B_0$ and this represents the remanent induction. For other values of F' the values of B_0 would require to be known, and these depend upon the previous history of the iron. There is, however, no experimental method of determining them, and in practice B is always plotted against F'.

It must be borne in mind that when any hysteresis exists B and F' are not necessarily parallel. This can be well seen if we consider the case of a permanent magnet lying in a field—that of the earth, for example; if we turn the magnet round we may place it with its direction of magnetisation making any angle we please with the direction of the field.

As another example, we will take the case of a spherical shell magnetised uniformly as to both direction and intensity (**191**, v.). The lines of induction in the air complete their circuit by bending round through the substance of the iron itself. That this must be so may be concluded from the fact that the field in the cavity, and, therefore, the induction there, is zero. The cyclic character of the lines of induction can only be preserved by a return of the lines through the substance of the iron. Since the induction in the iron is the resultant of the polar field and of a field $4\pi A$ due to the magnetisation and the latter is parallel to the axis while the induction is not, it follows that the induction and field are not, in general, parallel to one another.

219. Molecular Theory of Magnetisation.—We have in several places made use of the view, suggested by the experiment in (**174**), that a magnet, or indeed an unmagnetised piece of iron, consists of molecules each of which is a complete magnet. In the neutral state these small magnets have no external action, either because they form closed chains, or because they are arranged indiscriminately in all directions (**191**, iv.). The process of magnetisation has the effect of arranging the molecular magnets in some predominating direction, and the maximum is reached when their axes are all parallel, and their similar poles in the same direction.

It remains to establish the theory of the very complicated phenomena of magnetism on this basis. Professor Ewing's experiments throw much light on this subject. He arranges a great number of very small magnetic needles movable on pivots, at small regular distances apart. Experiment shows that these small magnets may assume a great number of different stable configurations that do not exert external force. These configurations

represent so many neutral states which are not identical with each other.

If one part of the system is disturbed it falls into a different configuration; each single magnet takes up a position of stable equilibrium after oscillations of greater or less amplitude. These oscillations represent the loss of energy which the system experiences in passing from the first configuration to the second. The change in general is not reversible.

Now, suppose the system exposed to the action of a uniform field gradually increasing from zero, the magnets are at first but slightly deflected, and if the directive force is removed, they revert to their first positions, and there is no permanent effect. As the field continues to increase, a point is reached at which the equilibrium is suddenly broken, and the system falls into a new configuration in which all the elements have approximately the same direction as the field. From this point the effect of an increase in the strength of the field is only to make the arrangement more complete.

These three phases correspond to the three parts of the curve of magnetisation (**206**).

If now the strength of the field is decreased, the system does not pass through the same states; it tends towards whatever condition of stable equilibrium is nearest to the existing configuration. Thus without any hypothesis of a coercive force analogous to friction, we may explain the effects of hysteresis and of residual magnetism.

Experiment shows that if the system is homogeneous, that is to say, if the small magnets are uniformly distributed, the transition to quasi-parallelism is made suddenly. This is what happens in the case of soft iron. If the system is not homogeneous, and the magnets are unevenly grouped, they do not equally and simultaneously obey the external action; the magnetisation curve rises more gradually, and does not so soon attain a maximum; on the other hand, the residual magnetism is more stable. This is the case of steel.

It is in accordance with this view, and the fact affords additional confirmation of its truth, that a piece of iron spun round in a very strong field is found experimentally to exhibit little hysteresis; the molecular magnets turn round so as always to lie parallel to the field and no unstable positions occur.

CHAPTER XX

EQUILIBRIUM AND MOTION OF MAGNETIC BODIES

220. Movements of very Small Bodies in a Field.—If the field is not uniform, the same conclusions that apply to a sphere so small that the field may be considered uniform throughout the space which it occupies, apply also, at any rate very approximately, to a body of any shape provided it is very small; the disturbance it causes in the field is then so small that it may be neglected; the magnetisation may consequently be regarded as uniform, and as due solely to the action of the field.

Let us consider an isotropic sphere of very small volume, u; let H be the intensity of the field, and κ the magnetic susceptibility; the sphere may be compared to an infinitely small magnet, with its axis parallel to the field, and of moment $M = u\kappa H$. In the case of an *invariable* magnet the energy would be equal and of opposite sign to the product of M into the component of H parallel to the axis. As the axis is here parallel to the field $W = -u\kappa H^2$. This would be the work required to bring this very small magnet from a place where the field is zero to the position which it occupies; but in an actual case the moment is not invariable, and neglecting any effects of hysteresis, it may be assumed to be, at each point, proportional to the intensity of the field at that point. The average moment is thus the mean between 0 and the value corresponding to the final position; consequently, the work done, and therefore the energy required, is only half that which an invariable magnet would have required, and we have

$$W = -\tfrac{1}{2}u\kappa H^2.$$

The body left to itself will remain in equilibrium in a uniform field, but in a variable field it will tend to expend the energy which it possesses, and, like a falling body, to move along the line of most rapid loss of energy. If κ is positive, the energy diminishes when H increases; it will move then not in general along a line of field, but in the direction along which the field

varies most rapidly, and will ultimately come in contact with the magnet; it will be *attracted* by the magnet (**196**). If κ is negative, as in the case of a diamagnetic body, it will move in the opposite direction, or in that in which the field diminishes most rapidly, and will seem to be *repelled* by the magnet. This is the true interpretation of the action exerted by magnets, or by electrified bodies, on bodies originally in the neutral condition and submitted to their influence. This special tendency of polarised elements to move, not in the direction of the lines of field, but towards a position of maximum or minimum intensity, accounts for some peculiarities which are met with in the way in which iron filings arrange themselves under the influence of a magnet. The particles of iron outline the bar very sharply, and, except near the centre, they do not remain above the magnet itself (Fig. 155). The reason is that the strength of the field varies very rapidly near the magnet, and is a maximum at the edges of the bar. Hence, too, when the glass plate is gently tapped, the lines formed by the filings are displaced, they shrink together and move towards the magnet. Each granule of the filings moves parallel to itself, and transversely to the lines of field, towards regions of greater intensity.

The direct consideration of the forces acting on the "poles" of the particle leads, moreover, to the same result. Take a particle whose direction coincides with the line of field: the two forces applied at the ends are equal and contrary, but not directly opposite. They form with each other an obtuse angle equal to that between the tangents to the line of field at the two ends of the particle, and their resultant is normal to the line. This resultant is directed towards the concave side of the line of field if the body is magnetic, and towards the convex side if it is diamagnetic: it is evident that the first direction is that in which the force increases; the second that in which it decreases.

221. Equilibrium of a Magnetic Body in a Uniform Field.—An isotropic sphere placed in a uniform field is in neutral equilibrium in all positions. This is not the case with a sphere cut out of a crystallised body. The coefficient of magnetisation, like other physical constants, has different values in different directions, and the body has three principal coefficients, κ, κ', κ'', in three directions at right angles to each other. The sphere has, in this case, three positions of equilibrium, those in which one of the axes is parallel to the field. Of these three positions, one is stable; if the body is magnetic, the equilibrium is stable when the axis parallel to the field is the axis of maximum magnetisation; and if the body is diamagnetic, when the axis of minimum magne-

tisation is parallel to the field. If the sphere is movable about one axis, the couple which keeps it in a position of equilibrium is proportional to the difference of the coefficients relating to the other two axes.

222. Equilibrium of an Elongated Body.—Let us now consider a body of an elongated shape, a cylindrical needle, for example, suspended freely in a uniform field. If each element of volume behaved as if it were alone, it would acquire a magnetisation parallel to the field, and the needle would be in neutral equilibrium in any position (Fig. 183). Experiment shows that this is the case for all feebly magnetic bodies, but that if the body is strongly magnetic the needle takes a direction parallel to the field.

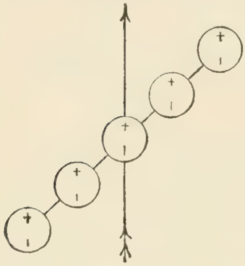


FIG. 183.

It behaves then like a needle having crystalline structure, the axis of figure corresponding with the axis of greatest magnetisation, and the reason is the same. The action which the several elements exert on each other in consequence of induced magnetism is greater when the needle is transverse to the field than when it is parallel, and the *apparent* coefficient is consequently smaller in the first case than in the second. The needle has thus a position of stable equilibrium, and the couple which maintains it there is proportional to the difference of the two apparent coefficients (**221**).

223. Suppose now the needle is in a non-uniform field. If it is isotropic, and feebly magnetic or diamagnetic, it will merely obey the tendency of each element to move towards a maximum or minimum field as the case may be. For instance, in a field symmetrical in reference to a centre like that between the opposite and equal poles a and b of an electro-magnet (Fig. 184), a magnetic needle, ef, will set along the lines of field, or *axially*, in the direction ab; while a diamagnetic needle will set across the lines of field, or *transversely*, in the direction cd; hence the term diamagnetic given to bodies which possess this property.

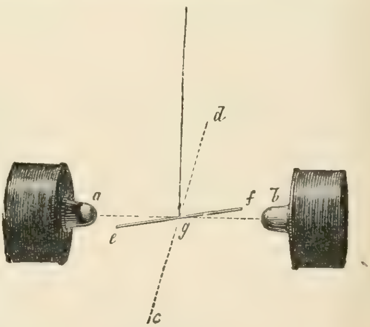


FIG. 184.

If the body is crystallised or strongly magnetic, it has to follow a twofold tendency, that which urges each element towards the points of maximum field, and that which tends to place the axis of greatest magnetisation parallel to the lines of the field. In the symmetrical field in question these two actions concur, and the position of equilibrium coincides with the axis of magnetisation; but in more complex fields they may give rise to effects which are apparently very strange.

The resulting phenomena are often referred to as *magneto crystallic phenomena.*

224. Motions of Liquids and of Gases in a Non-Uniform Field.—The same effects are produced upon liquids and upon gases. Figs. 185 and 186 represent the effects produced by putting a drop of liquid in a watch-glass between two powerful equal and opposite poles. In the first case, the liquid is magnetic; it is

FIG. 185.

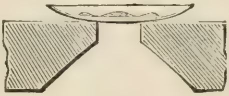


FIG. 186.

heaped up towards the points where the field is a maximum; in the second it is diamagnetic, and moves towards points where the field is a minimum. A drop of liquid in a glass tube which is placed transversely between the two poles moves towards the centre if it is magnetic, and away from it if it is diamagnetic. In like manner the flame of a candle, which is diamagnetic, is repelled in the transverse plane (Fig. 187).

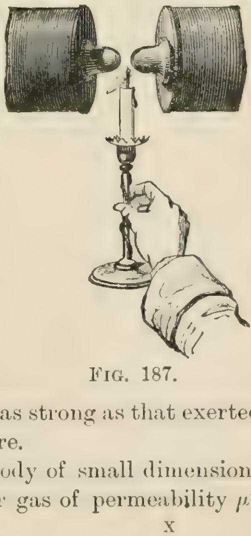


FIG. 187.

Among gases, oxygen is strongly magnetic; nitrous oxide, carbonic acid, ethylene, and cyanogen are diamagnetic; nitrogen and hydrogen seem to be neutral.

In order to demonstrate this phenomenon, a glass globe is suspended to the beam of a delicate balance over the pole of a powerful magnet. After the globe has been exhausted, it is filled with the gas to be tested, and is then counterpoised. In the case of oxygen, there is a very perceptible attraction, which is four or five times as strong as that exerted on air at the same pressure and temperature.

225. Relative Magnetisation.—A body of small dimensions and of permeability μ, placed in a liquid or gas of permeability μ',

behaves like a body whose permeability is $\mu_1 = \frac{\mu}{\mu'}$, or, what comes to nearly the same thing if the susceptibilities are small, it acts as though the susceptibility were $\kappa_1 = \kappa - \kappa'$.

This result leads to consequences analogous to those which are deduced from the principle of Archimedes for floating bodies. Three cases may present themselves: First, $\kappa > \kappa'$, the body behaves like a magnetic body; second, $\kappa = \kappa'$, the body appears neutral; and third, $\kappa < \kappa'$, the body behaves like a diamagnetic body. It is possible that diamagnetism might be explained by a cause of this kind.

CHAPTER XXI

PERMANENT MAGNETS

226. Permanent Magnets.—Permanent magnets are usually made in the form of straight prismatic bars, but they are also sometimes bent in the form of a horse-shoe. It is important that the metal of which they are made should have considerable coercive force. Highly tempered steel, especially such as contains tungsten, is found to answer best.

It is impossible to obtain uniform magnetisation in a prismatic or cylindrical bar. The effective magnetising field is less near the ends of the bar than it is at the middle; consequently the intensity of magnetisation diminishes towards each end.

The action of such a magnet at external points cannot be represented by means of "poles" located on its ends, as we showed to be possible in the case of a magnet that is magnetised longitudinally and uniformly. If the idea of magnetic charges be introduced at all, it is necessary, when the intensity changes from point to point, to introduce a distribution of such charges in the body of the magnet, as well as upon the end surfaces. Thus, in (**191**, i.), in the case of a non-uniformly magnetised filament, the potential is given by

$$V=\frac{S_0}{r_0}-\frac{S_1}{r_1}+\int_0^1\frac{dS}{r}$$

where $S=A\alpha$; that is to say, each element of the filament would require a magnetic charge equal to the change in the value of $A\alpha$ from end to end of it, in order that the potential might be calculated as for electric charges. In the more general case where the directions of magnetisation are not parallel to each other throughout any cross-section, a more complicated expression is necessary in order to give the requisite charge. In brief, if the transition from a finite magnetisation A to zero magnetisation take place abruptly, the equivalent magnetic charge is merely superficial; if the transition take place gradually through a measurable distance, a gradual distribution is required. The result of this in its application to an actual bar magnet, is that the poles or mass-

centres of the north and south magnetic charges respectively are within the bar at some distance from the ends.

The non-uniformity is increased if the bars are thick, since the metal is not homogeneous throughout; the hardening more particularly affects the superficial parts, and the magnetism seems to reside to a greater or less depth in this layer. If a bar is magnetised alternately in different directions, magnetic layers are formed which seem to superpose themselves. This fact can be shown by removing the surface layer either mechanically as by grinding, or chemically by dissolving the surface away with weak acid.

227. Distribution of Magnetism.—We have already spoken of the impossibility of ascertaining the internal constitution of a magnet from its external effects. The investigation of these effects, speaking generally, does not even determine the distribution of the surface layer by which the internal magnetism might be supposed to be replaced. This layer, in fact, is not in equilibrium according to the law of inverse squares, and consequently Coulomb's theorem cannot be applied; therefore, even if we knew the value of the normal component at each point, we could not deduce from it that of the density. Thus most discussions as to the distribution of magnetism in magnets are of little value.

Coulomb caused a very small magnetic needle to oscillate in front of various parts of the bar to be examined (Fig. 188). The needle, *ns*, was suspended by a cocoon fibre, and in order to make the oscillations slower it was fixed at right angles to a bit of stout copper wire. The needle was first made to oscillate under the action of the earth alone, and then under the action of the earth and of the bar placed at a fixed distance in the plane of the meridian of the needle.

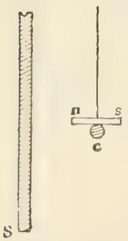


FIG. 188.

Having regard to the smallness of the needle, we may treat it as though it were suspended in a uniform field and apply to it the formula for the pendulum. If F is the magnetic field due to the earth, and f that due to the bar, and if n and N respectively are the numbers of oscillations which the needle makes in the same time, when it oscillates under the action of the earth alone, and under that of the earth and magnet together, we have

$$\frac{N^2}{n^2} = \frac{F+f}{F}$$

and therefore

$$\frac{N^2 - n^2}{n^2} = \frac{f}{F}$$

If N' is the number of oscillations in the same time when the needle is opposite another point of the bar, we have

$$\frac{N'^2 - n^2}{N^2 - n^2} = \frac{f'}{f}.$$

The ratio $\frac{f'}{f}$ thus found may be regarded as being that of the normal components of the magnetic field at the two corresponding points of the bar. In order to make the number obtained for the end of the bar comparable with the others, Coulomb doubled it, a correction which is certainly somewhat arbitrary.

In another series of observations Coulomb used the method of torsion. A long magnet, a knitting-needle, for instance, was suspended horizontally by a metal wire, so that the wire was without torsion when the needle was in the magnetic meridian; he then measured the torsion which had to be applied to keep the end of the needle at a small fixed distance opposite various points of a long magnetised bar placed vertically in the meridian of the needle. This torsion would give an approximate measure of the normal component, if it could be assumed that the magnetism of the needle remains the same and does not vary when it is placed opposite different parts of the bar.

A third method consists in measuring the force needed to pull off a small sphere, or a small cylinder of soft iron placed in contact with different parts of the bar. To be able to assume, as is ordinarily done, that the effect measured is proportional to the square of the normal component, we must suppose that the magnetic susceptibility of the test-sphere is independent of its magnetic intensity, and that the presence of this sphere does not alter the magnetic state of the bar at the part which is being investigated.

We shall afterwards (**312**) learn a method which gives much more correctly the value of the normal component at each point of the bar.

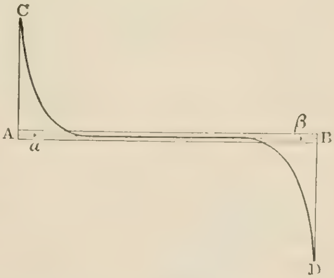


FIG. 189.

228. If at each point of the bar we draw ordinates proportional to the numbers found, we obtain a curve which may be called the curve of the normal components (Fig. 189); but which does not, as Coulomb thought, represent the surface densities. In particular, the points of the bar α and β which correspond to the centres of figure of the areas under these

curves, are not the true poles of the magnet. A very simple example will show this: for a uniformly magnetised cylinder the method would still give a curve analogous to Fig. 189, which would fix the positions of the poles at a and β, whereas the surface layer is restricted to the terminal faces, and the true poles are therefore on the extreme ends.

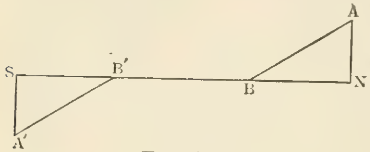


FIG. 190.

The curve resulting from experiment is approximately a triangle. In long magnets, that is to say, those in which the length is more than 50 times the diameter, the base, BN, of the representative triangle is about 25 times the diameter of the bar (Fig. 190). The centres of figure are at a fixed distance from the ends equal to 7 or 8 times the diameter. For bars which differ only as to length, the effective length may be expressed by $l-x$, where l is the length of the bar and x a constant.

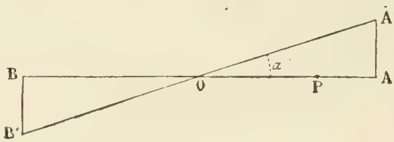


FIG. 191.

For magnets whose length is less than about 50 times the diameter, the distribution is represented approximately by the sloping line A′B′ (Fig. 191), which cuts the bar in the middle. The points a and β are then at a distance from the ends equal to one-sixth the length.

229. Magnetic Moment of a Bar.—The moment, MH (179), of the couple exerted on a bar by a field of uniform intensity, H, that of the earth, for example, is directly accessible to experiment.

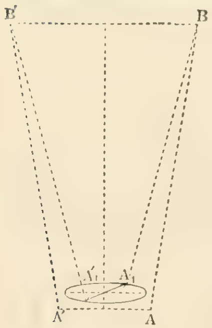


FIG. 192.

One method is to suspend the magnet horizontally by placing it in a stirrup attached to a fine metal wire, which is without twist when the axis of the magnet is parallel to the field: this is the case if the direction does not change on replacing the magnet by a copper bar of the same weight. The angle through which the wire must be twisted to bring the magnet into a position at right angles to the field measures the couple, MH. If θ is the angle of twist, and C the coefficient of torsion of the wire, we have

$$MH = C\theta.$$

Instead of a metal wire, a bifilar suspension may be used (Fig. 192); in this case, if θ is the angle of torsion necessary to keep the bar in the transverse position, we have

$$MH = C' \sin \theta,$$

C' being a coefficient depending on the suspension; if AA$' = 2a$, BB$' = 2b$, AB $= l$, and if P is the weight of the suspended system, we easily find as an approximate formula

$$C' = P\frac{ab}{l}.$$

230. We may also deduce the couple, MH, from the duration, T, of the small oscillations of a bar suspended horizontally by a silk fibre without torsion. As the moment of the directive couple is at each instant proportional to the sine of the angle of divergence from the position of equilibrium (**179**), and therefore, for small angles, to the angle itself, the magnet makes isochronous vibrations of which the period is

$$T = 2\pi\sqrt{\frac{I}{MH}},$$

where I is the moment of inertia of the oscillating body in respect of the axis of motion, we get thus

$$MH = 4\frac{\pi^2 I}{T^2}.$$

In practice, in order to obtain an accurate measurement of T, the period of one complete vibration, the time occupied by a considerable number is observed, and this is divided by the number. The result gives the product MH, but to get the magnetic moment M we require to know the value of H: how this can be found will be explained in the next chapter (**243**). Without this knowledge, however, the method of vibration affords an accurate way of comparing the moments M and M' of two magnets. If I and I' are their respective moments of inertia and T and T' their periods of vibration when suspended successively in the same position, so that H is the same for both, we get for the ratio of the magnetic moments

$$\frac{M}{M'} = \frac{T'^2 I}{T^2 I'}.$$

231. Intensity of the Magnetisation of Steel.—If we divide the moment of a bar by its volume, we have the mean intensity of magnetisation (**190**). In ordinary bars this varies from 200 to 400 C.G.S. units; with very long thin bars it may attain 800, that is to say, almost half the maximum intensity of soft iron. Taking 7·8 as the density of steel, the *specific moment* or the moment per gramme of steel is from 25 to 50 for ordinary magnets, and it may reach a maximum of 100 C.G.S. units.

These numbers are, however, only averages; for the intensity, as has already been pointed out, is far from being uniform throughout the mass of the bar.

232. Influence of Hardening and of Annealing.—Hardening and annealing have considerable influence on the coercive force, which is greater the higher the temperature to which the steel has been raised, and the more suddenly it has been cooled. Very hard steel is magnetised with difficulty, but retains almost all the magnetism developed. Annealing lessens the effects of hardening. The steel for bar magnets is commonly tempered *blue*, that is, at the temperature at which steel acquires a blue tint in consequence of the oxidation of the surface.

Hardening and annealing have also a very marked influence on the electric resistance, and on the thermo-electric power of steel. The specific resistance at 0° C. varies from 15,000 C.G.S. for soft steel to 47,000 C.G.S. for glass-hard steel. The resistance thus furnishes a scale by which the condition of the steel may be defined. It can be ascertained in this way that the effects of annealing occur at relatively low temperatures, such as 100° C. When a bar is kept at a given temperature, the effect of annealing increases with time, but tends towards a limit.

233. Influence of Temperature.—The moment of a magnet diminishes in general as the temperature rises. Part of the diminution is permanent, and the magnet does not regain its original moment when it returns to the original temperature. Experiment shows, however, that if a magnet is repeatedly heated to a given temperature for a sufficient length of time, and is each time remagnetised to saturation, its permanent magnetisation is ultimately unaffected by the action of any temperature below that to which it has been raised.

Steel magnets intended for instruments of observation, by being treated as above for a total time of thirty or forty hours, at the temperature of 100°, assume a condition in which the moment at a given temperature may be considered fixed. Within the limits of atmospheric variations, the effects produced are then proportional to changes of temperature. If M_0 and M are the moments of one and the same bar at the temperatures 0° and $t°$, we may put

$$M = M_0 (1 - \alpha t),$$

α being a coefficient the value of which varies with different kinds of steel, but which is always less than 0·001.

234. Methods of Magnetisation.—In order to obtain a somewhat powerful magnetisation in steel, it is not sufficient

merely to place it in a magnetic field; the particles must be shaken, as it were, so as to overcome the coercive force. Instead of causing a uniform field to act upon it, it is better to act in succession on the various parts. At the present day electric currents are almost exclusively used for magnetising large pieces of steel. Among the older methods the one in most frequent use is that of *double touch* (Fig. 193). The opposite poles NS of two strong bar magnets are applied in the middle of the bar to be magnetised, *ns*; the magnets are held at an angle of about 30°, and are moved simultaneously to the opposite ends of the bar.

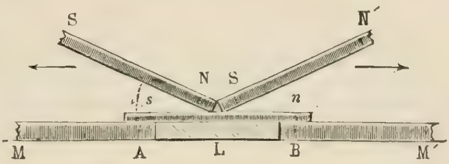


FIG. 193.

The operation is repeated several times on each face. A north pole, *n*, is produced at the end last touched by the south pole, S, and a south pole, *s*, at the end left by the north pole N. The action is increased by allowing the ends of the bar to be magnetised to rest on the poles of two fixed magnets, these poles being of the same kind as the poles of the movable magnets acting on the same parts of the bar.

Instead of drawing the two magnets apart, they may be separated to a small fixed distance, regulated by a bit of wood put between them, and then moved together from the middle towards one end, then to the other, and so on backwards and forwards repeatedly, leaving off in the middle after giving each half the same number of passes.

235. Magnetic Batteries.—The magnetisation of steel being chiefly confined to the surface layers, in order to get large magnets strongly magnetised it is usual to construct them of thin plates, which are magnetised separately, and then put together. The poles of the same kind all on the same side are usually placed in slits in pieces of soft iron, which become magnetised by influence, and give at the terminal faces poles of the same kind as those which they connect (Fig. 194).

FIG. 194.

The magnetic moment of a battery is far from being the sum of the moments of the plates of which it is made up, and if it is taken to pieces after some time, each of the magnets will be found to be weaker, but more especially the central ones.

This is obviously due to the action of the plates on each other, which tends to develop in each plate a magnetisation opposite to that of its neighbours.

236. Preservation of Magnets — Keepers. — In order to preserve magnets, great care must be taken not to let them be jarred, and not to expose them to their own demagnetising action when not in use. It is known that this action is nothing in a ring-shaped magnet, in which all the magnetic filaments are closed upon themselves, and are therefore without action both internally and externally (**191, i.**). In practice it is attempted to realise this condition as nearly as may be. Prismatic bars, as equal as possible, are placed two together parallel to each other in boxes, with opposite poles near each other, and joined by pieces of soft iron, which, so to speak, complete the magnetic circuit (Fig. 195). Such pieces of soft iron are called *Keepers*.

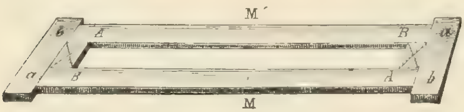


FIG. 195.

In horse-shoe magnets the two poles are joined by a single piece of soft iron, which forms a keeper (Fig. 196). The shapes of the keeper, and of the soft iron pole-pieces which terminate the battery, should be such as to make the magnetic intensity as nearly uniform as possible all round the circuit.

Fig. 196 represents a horse-shoe magnet formed by Jamin's method of flexible plates of steel magnetised separately, then fitted into pole-pieces, A and B, of soft iron. A piece of soft iron, C, completes the circuit, and acts as a keeper.

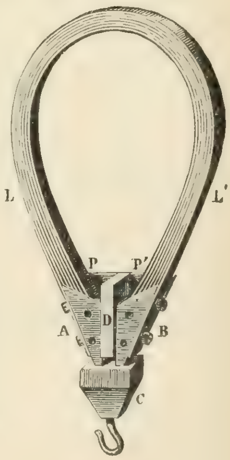


FIG. 196.

It is often said that a magnet should support a permanent weight in order to retain its magnetism. By gradually increasing this load a magnet may ultimately be made to support a weight far greater than it would do at first. When contact is broken, owing to the weight being too great, the carrying power falls most frequently below its original value.

237. Lifting Power.—The lifting power of a magnet can easily be calculated when the magnetisation is uniform. If we suppose that the two opposed surfaces have densities $\pm\sigma$ equal

to A in absolute value, the attraction for each unit of surface is $2\pi A^2$ (**45**). For the total surface of contact S, the force in dynes is $2\pi A^2 S$, and if P is the number of grammes whose weight would just pull off the keeper, we have—

$$gP = 2\pi A^2 S = \frac{B^2 S}{8\pi}.$$

If we take $A = 800$ for steel, we get as the greatest value for P, 4 kilogrammes per square centimetre; this is, in fact, what is given by the best magnets. With soft iron raised to saturation, 10 or 11 kilogrammes per square centimetre is easily obtained.

238. Magnetisation by the Action of the Earth.—It is very rarely that we find a piece of steel or iron, unless perhaps it is perfectly pure soft iron, which does not present magnetic polarity. Magnetisation is produced in it if it is struck or undergoes any other mechanical disturbance while under the influence of the earth. A bar of soft iron which is placed parallel to the earth's field acquires a temporary magnetisation, which is reversed when the magnet is turned end for end. But if the bar is struck with a hammer, the magnetisation is raised to a maximum, and the iron at the same time acquires perceptible coercive force.

A fresh blow on the bar, when placed transversely to the magnetic field, takes from it its magnetisation. A length of soft iron wire twisted while being held in the direction of the dipping-needle, also acquires permanent magnetisation.

The value of κ (**201**) is about 40 for the intensity 0·47, which is that of the earth's field at London. A long, thin, soft iron wire, for example, 1 millimetre in diameter and 50 centimetres long, acquires a magnetic intensity of about $0{\cdot}47 \times 40 = 18{\cdot}8$; and, as its volume is nearly 0·4 cubic centimetre, its moment is $18{\cdot}8 \times {\cdot}4 = 7{\cdot}5$.

CHAPTER XXII

TERRESTRIAL MAGNETISM

239. Terrestrial Field.—The magnetic field due to the earth, though practically uniform throughout any moderate space that is not affected by the presence of magnetic bodies, varies greatly both in intensity and direction from one place to another on the earth's surface; and even in one and the same place it changes in course of time.

We know that its action on a magnet is that of a couple (**171**). In order that a magnet may indicate the direction of the field at a given point, it must be acted upon by the earth's field independently of any other; the direction which the magnetic axis then takes is that of the field due to the earth.

In this part of the world this direction is roughly north and south, but it makes a large angle with the horizon, the north pole pointing downwards.

We may define this direction by means of two angles, the *declination*, and the *inclination* or *dip*.

The *magnetic meridian* is the vertical plane which contains the direction of the earth's field.

The *declination* (or *variation*) is the angle which the magnetic meridian makes with the astronomical meridian.

The *inclination* or *dip* is the angle which the direction of the earth's field makes with its projection on the horizontal plane.

The declination, D, the inclination, I, and the intensity, T, of the field are three elements which characterise the earth's magnetism at a given time and place.

240. It is not practicable to construct an instrument in which a magnet is suspended freely at its centre of gravity. In practice, two instruments are employed, one in which the magnet is movable only about a vertical axis: this is called a *declination compass;* the other, in which the needle moves only about a horizontal axis passing through its centre of gravity: this is the inclination compass or *dip needle.*

Suppose the magnet in any position (Fig. 197), and let α be the angle which the vertical plane, OA, containing the axis makes with the magnetic meridian, OM; the field, T, may be resolved into two components independent of the angle α; the one, vertical,

$$Z = T \sin I;$$

the other, horizontal,

$$H = T \cos I;$$

this latter may be again resolved into two others, also horizontal, namely,

$$X = H \cos \alpha, \text{ and } Y = H \sin \alpha,$$

respectively parallel and perpendicular to the plane of the magnet. We have thus—

$$X = T \cos I \cos \alpha,$$

$$Y = T \cos I \sin \alpha,$$

$$Z = T \sin I.$$

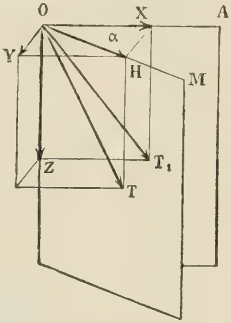


FIG. 197.

If the magnet is free to turn in the plane OA, it will be in equilibrium when its axis is along OT, making with OA an angle AOT, whose tangent is $Z/X = \tan I / \cos \alpha$.

241. Declination Compass.—If the needle is movable only about a vertical axis, it is affected only by the horizontal component; its position of equilibrium is that in which its axis is in the magnetic meridian. When it is deflected through an angle α, the couple which acts on the needle and tends to bring it to its position of equilibrium is $MH \sin \alpha$; M being the magnetic moment of the needle. This couple is proportional to the sine of the deflection; consequently, the law of the motion of the needle when disturbed and left to itself is that of the pendulum (229).

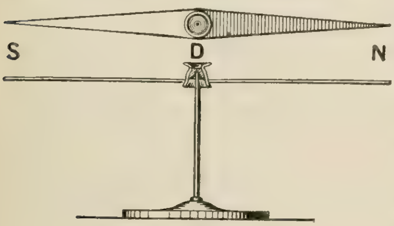


FIG. 198.

In order to realise the fundamental condition of the apparatus, it is not needful for the vertical axis of rotation of the magnet to be rigid. The magnet maybe suspended, when properly balanced, by a cocoon thread, or it may be supported by an agate cap, D, on a pivot (Fig. 198).

If the needle is suspended at its centre of gravity, and we load

the south end, s, by a mass, m, placed at a distance, d, from the point of suspension, such that

$$mgd = MZ = MT \sin I,$$

the needle will rest horizontally in any azimuth, since the vertical component is independent of a.

The measurement of declination consists in determining the astronomical meridian, and measuring the angle which the magnetic axis of the needle makes with this plane. In reality the angle observed is that between the meridian and the *axis of figure* of the needle, and this does not necessarily coincide with the magnetic axis. The correction for this is readily made by inverting the face of the needle (Fig. 199); the magnetic axis, ab, takes the same direction in each case, and the axis of figure assumes symmetrical positions, AB, A′B′, with respect to this direction, first on one side and then on the other. The mean of the two angles observed, AON, A′ON, gives the required angle, MON.

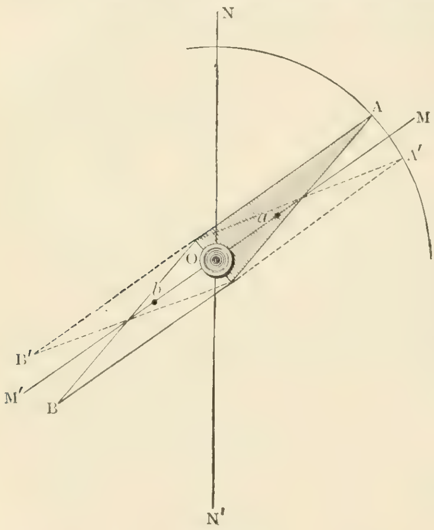


FIG. 199.

Conversely, if the declination is known, it is easy to find the astronomical meridian. When the needle is at rest, the vertical plane, which makes an angle equal to the declination, east or west of the north pole, according as the declination is west or east, is the astronomical meridian.

Fig. 200 represents the Kew pattern instrument mounted as a declination compass. The magnet, suspended by a cocoon fibre, is viewed through a telescope, T. The magnet is hollow, and a scale photographed on a glass disc inserted at the end of it furthest from the telescope is in the focus of a lens inserted in the other end. The case carrying the telescope and suspension tube is turned round until the image of the middle division of the glass scale is on the cross-wires of the telescope: the optic axis of the telescope then coincides in direction with the geometric axis of the magnet.

The magnet is then inverted as in the simple compass, the telescope is readjusted and the mean of its two directions—read on the graduated circle C—is taken as the magnetic meridian. In order to obtain the geographical meridian a mirror *m* is employed which is capable of being turned about either a horizontal or a vertical axis, and which must be adjusted so that the horizontal axis is at right angles to the vertical plane containing the optic axis of the telescope. This is effected by first illuminating the cross wires and then turning the mirror round until their reflected image coincides with themselves. A known celestial object (*e.g.* the sun) is now brought into the field of view by turning the mirror about its horizontal axis and at the same time turning the case along with the telescope, and the exact instant of the transit of the object across the wires is observed. The position of the telescope is now read on the graduated circle. The position of the object at the time of transit, as derived from the *Nautical Almanack*, gives the angle between the last position of the telescope and the geographic meridian. The declination is obtained by adding this angle algebraically to the angle through which the telescope has been turned from the magnetic meridian—the signs of these angles being taken alike when the positions of the object and the magnetic meridian lie on the same side of the geographical meridian.

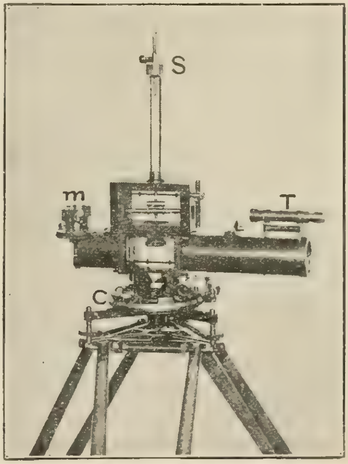


FIG. 200.

242. Dip Circle.—For the determination of magnetic dip, a needle that can turn about a horizontal axis, therefore moving in a vertical plane, is used: a needle thus arranged moves only under the action of the component of the earth's field which lies in the plane of its motion.

Let a be the angle which this plane makes with the magnetic meridian: the effective components are Z and $H \cos a$, which have a resultant

$$T_1 = \sqrt{Z^2 + H^2 \cos^2 a}.$$

This makes with the horizontal an angle i given by the equation

$$\cot i = \frac{H \cos a}{Z} = \cot I \cos a.$$

This direction i is that which the needle will take; it may be called the *apparent dip* in the azimuth α. For $\alpha = 0$ we have $i = I$, and the apparent dip is thus the true dip. If $\alpha = \frac{\pi}{2}$ we get $\cot i = 0$, that is, the needle is vertical.

For angles α and $\alpha \pm \frac{\pi}{2}$, which differ from each other by $\frac{\pi}{2}$, the apparent dip takes values i and i' respectively, which satisfy the equations

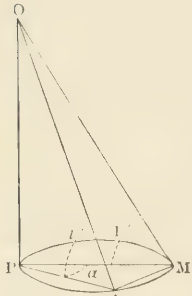


FIG. 201.

$$\cot i = \cot I \cos \alpha$$
$$\cot i' = \pm \cot I \sin \alpha\,;$$

squaring both sides and adding, we get

$$\cot^2 i + \cot^2 i' = \cot^2 I,$$

a formula which is frequently used in determining the dip.

These equations show that in a complete rotation about a vertical axis, OP (Fig. 201), the axis of the needle will trace on the horizontal plane a circumference whose diameter is equal to OP cot I.

Fig. 202 represents a *dip circle*. The needle is a strip of steel in the shape of an elongated acute-angled lozenge, traversed in the centre by a cylindrical steel axis, which rests on agate edges, carefully adjusted to be in the same horizontal plane. The reading is made by an arm, M, which is moved along the limb of the circle, and at each observation is set exactly parallel to the needle. With this object the arm is furnished with two small concave mirrors with their centres of curvature in the plane of the needle. The mirrors thus give in this same plane real and inverted images of the points of the needle, and the arm is moved until these reversed images are brought into contact with the points themselves.

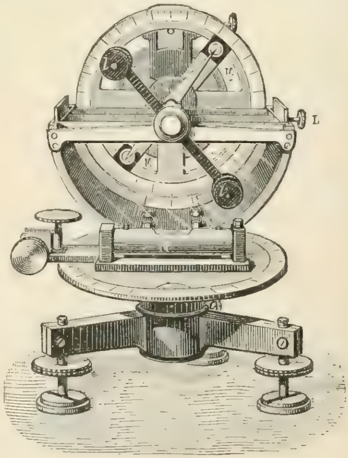


FIG. 202.

In making an observation, the vertical circle is brought into the magnetic meridian, and the angle I is directly measured; in order to determine the plane of the meridian, the azimuth is found

for which the needle is vertical, and the circle is then turned through 90°.

The two angles i and i', which correspond to two positions of the circle at right angles to each other, are also often determined. As we have seen above, the sum of the squares of the cotangents of these two angles gives the square of the cotangent of the dip.

Several errors must be eliminated. That due to possible eccentricity of the axis of rotation is eliminated by reading the two ends of the needle; error of the zero of graduation of the circle, by turning it through 180°; that due to obliquity of the magnetic axis, by reversing the face of the needle. There may also be an error arising from the fact that the centre of gravity is not exactly in the axis of rotation. This is eliminated by re-magnetising the needle so that the poles are reversed, and repeating the whole series of observations. The measure of the dip in a given azimuth is thus the mean of sixteen readings.

243. Measurement of Intensity — Gauss's Method of Deflection. — The horizontal component H is measured, and the total force T is deduced from it by the formula—

$$T = \frac{H}{\cos I},$$

I being the dip.

The method consists in determining for the same bar the product $MH = A$ and the quotient $\frac{M}{H} = B$; whence—

$$M = \sqrt{AB} \text{ and } H = \sqrt{\frac{A}{B}}.$$

Determination of MH.—The product MH is obtained either by the method of torsion or by that of oscillation. In the latter case we have—

$$MH = \frac{4\pi^2 K}{t^2},$$

K being the moment of inertia of the magnet (and of whatever is attached to it) relatively to the axis of rotation, and t the period of a vibration of infinitely small amplitude; this is deduced from the observed period, t_1, by a well-known formula—

$$t = t_1\left(1 - \frac{k^2}{16}\right),$$

k being the semi-arc oscillation in circular measure.

Measurement of $\frac{M}{H}$.—The bar is made to act on a very small magnetic needle, the deflections of which are observed by reflexion

(84). The bar is placed in one of Gauss's principal positions **(186)**, with its axis perpendicular to the magnetic meridian (Fig. 203).

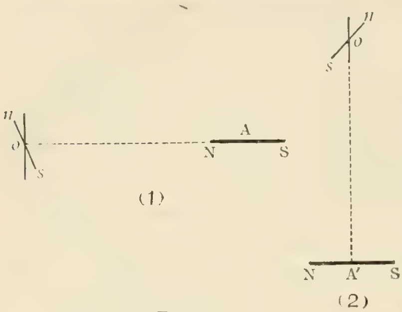


FIG. 203.

In the first position the line joining the centres of the magnet and needle is perpendicular to the meridian; in the second position, the line joining the centres is in the meridian.

If the bar were infinitely small in comparison with the distance OA $= R$, the field at O would be **(187)**—

$$F_1 = 2\frac{M}{R^3}, \text{ in the first position;}$$

$$F_2 = \frac{M}{R^3}, \text{ in the second position.}$$

If the needle itself is very small, the field may be considered uniform throughout the space which it occupies, and the moment of the couple which tends to increase its deflection is, for the first position, $M'F_1 \cos\alpha = 2M'\frac{M}{R}\cos\alpha$, where α is the deflection of the needle. The condition of equilibrium is—

$$2\frac{MM'}{R^3}\cos\alpha = M'H\sin\alpha,$$

whence

$$\frac{M}{H} = \frac{1}{2}R^3\tan\alpha.$$

If the magnets cannot be treated as though they were infinitely small, a rather more complicated formula is required—

$$F = 2\frac{M}{R^3}\left(1 + \frac{a}{R^2} + \frac{b}{R^4} + \ldots\right)$$

where a, b, . . . are constants for the particular pair of magnets. This leads to the formula

$$\frac{1}{2}R^3\tan\alpha = \frac{M}{H}\left(1 + \frac{a}{R^2} + \frac{b}{R^4} + \ldots\right).$$

The fact that the angle α is scarcely altered by transferring the magnet to an equal distance on the other side of the needle, proves that the series on the right of this expression can contain only even-numbered powers of R. Odd-numbered powers might be introduced by irregularities of magnetisation, but unless these are great their effect is got rid of by taking the mean of observations with

the magnet at equal distances east and west of the needle, and in each case with first one pole and then the other turned towards the needle. If R is even a moderate multiple of the length of the magnet, the above series converges rapidly, so that it is usually sufficient to include the term in R^{-2}, writing

$$\frac{1}{2} R^3 t = \frac{M}{H}\left(1 + \frac{a}{R^2}\right),$$

where t is the arithmetic mean of the four observed values of tan α. If similar observations be taken at a second distance, we obtain in like manner

$$\frac{1}{2} R'^3 t' = \frac{M}{H}\left(1 + \frac{a}{R'^2}\right).$$

Eliminating a from these, we finally get the value—

$$\frac{M}{H} = \frac{1}{2}\frac{R^5 t - R'^5 t'}{R^2 - R'^2}.$$

It is best to take the two distances nearly in the ratio of 3 to 4.

In measurements with the Kew pattern magnetometer (Fig. 204) the line joining the centres of magnet and needle, along which the axis of the magnet is set, is at right angles to the axis of the deflected needle, instead of being at right angles to the meridian, and the sine of the angle of deflection is used instead of the tangent.

The case carrying the torsion-head and telescope T can be turned round a vertical axis. The position is adjusted so that a mirror attached to the suspended needle reflects an image of the zero of a small scale so as to be on the vertical wire in the eye-piece of the telescope, and the angular position of the instrument is read on a fixed graduated circle. The deflecting magnet M is then placed in a carrier on the long bar B, which is initially east and west; the case is turned round until the image of the zero of the scale S is again on the vertical wire, thus showing that the telescope, and with it the whole instrument, is in the same position with respect to the suspended needle as at first, and the new angular position is read. It follows that the axis of the deflecting magnet is at right angles to the suspended needle, and the couple due to the earth is HM' sin θ (where θ is the angle through which both the telescope and magnet have turned), and that due to the deflecting magnet is FM'. Since these balance one another

$$\frac{F}{H} = \sin\theta.$$

Measurements are made as in the previous method with the magnet turned end for end, and successively on both sides of the

suspended needle, and in each case at two distances therefrom. Combining these so as to eliminate the unknown constant, we obtain finally

$$\frac{M}{H} = \frac{1}{2} \frac{R_1^5 \sin \theta_1 - R_2^5 \sin \theta_2}{R_1^2 - R_2^2}.$$

One advantage of this method—due to Lamont—over the

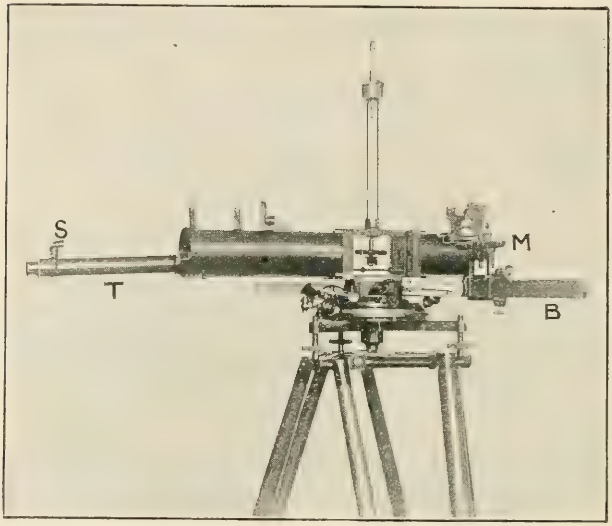


FIG. 204.

former one is that the suspending fibre remains untwisted, and consequently no correction for torsion is required.

244. Magnetic Charts.—The terrestrial magnetic elements vary from one point of the globe to another.

The simplest plan of showing the results is to record the observed numbers on a geographical map, and to draw curves so as to connect all those places where the declination, the dip, and the intensity respectively have the same value. In this way we get *isogonic*, *isoclinal*, and *isodynamic* lines.

When traced on a globe, isogonic lines resemble roughly meridional lines, and isoclinal lines roughly resemble parallels, both referred to the same axis, which is slightly inclined to the geographical axis, and cuts the surface at two points called the *magnetic poles.* At these points the dip is 90°, that is, the dipping

needle stands vertical. The term *magnetic equator* is given to the isoclinal line where the dip is zero.

An interesting isogonic is that of no declination, or, in other words, that in which the magnetic and geographical meridians coincide. It cuts the old world from the North Cape to the Persian Gulf, and passes through Western Australia. In the other hemisphere another such line crosses the eastern part of Brazil, Florida, South Carolina, and Lake Superior. The surface of the globe is thus divided into two parts: in one, which may be called the Atlantic region, the declination is westerly; in the other, the Pacific region, it is easterly.

The *isogonic* lines traverse the British Islands approximately from S.S.W. to N.N.E. The declination for the year 1891 was least in the extreme south-east, being about 16° at the South Foreland, and greatest on the north-west coast of Ireland, where it was about 23°. It had very nearly the same value, about 19°, at Penzance, Lancaster, and Berwick-on-Tweed.

The general direction of the *isoclinal* lines for the same year was from about 18° S. of W. to 18° N. of E. The extreme values were rather less than 67° on a line through Eastbourne and Dungeness, and nearly 73° in the Shetlands.

The lines of *equal horizontal field* ran from about 16° S. of W. to 16° N. of E. The extreme values were about 0·186 at Beachy Head and 0·148 in the Shetlands.

For London (Kew Observatory, lat. 51° 28′ 6″ N., long. 0h. 1m. 15·1s. W.), the estimated mean values of the magnetic elements for various years are given in the following table: —

Year.	Westerly Declination.	Dip.	Horizontal Field.	Vertical Field.	Total Field.
1866	20° 50′·7	68° 6′·1	0·1769	0·4401	0·4743
1871	20° 10′·5	67° 56′·6	·1781	·4395	·4743
1876	19° 26′·1	67° 46′·7	·1791	·4385	·4737
1881	18° 44′·6	67° 41′·2	·1799	·4385	·4739
1886	18° 11′·5	67° 37′·2	·1808	·4391	·4748
1891	17° 41′·9	67° 32′·5	·1819	·4396	·4762
1896	17° 10′·8	67° 22′·3	·1831	·4392	·4759
1901	16° 48′·9	67° 9′·5	·1845	·4380	·4753
1906	16° 28′·5	67° 2′·2	·1852	·4371	·4747
1908	16° 16′·9	67° 0′·9	·18515	·4365	·4741

245. Law of the Distribution of Terrestrial Magnetism. —As a first approximation, and considering only the broad features, the distribution of magnetic force on the globe admits of a very simple statement, it being the same as would result from a very short magnet, *ns*, situated at the centre of the earth, and

with its axis, PP′, inclined at an angle of about $11\frac{1}{2}°$ from the axis of rotation, and with its *north* pole in the southern hemisphere (Fig. 205).

The investigation of the properties of an infinitely small magnet (**188**) enables us to trace out directly the consequences of this hypothesis, which, as we may infer from (**191**, v.), is equivalent to supposing the earth to be a sphere uniformly magnetised parallel to the axis PP′.

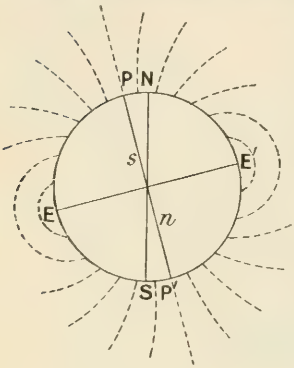


FIG. 205.

The field is symmetrical in respect of the axis PP′ of the magnet; that is, the magnitude and direction of the field are the same at all points of any circle perpendicular to PP′; the circle which passes through the centre is the magnetic equator EE′; the others are magnetic parallels. At all points of the magnetic equator the field is horizontal and the dip zero; the inclination, for a parallel of latitude λ reckoned from the magnetic equator, is given by the formula—

$$\tan I = 2 \tan \lambda$$

and the field by the formula—

$$T^2 = T_e^2 (1 + 3 \sin^2 \lambda),$$

T_e being the field at the equator. The field is doubled on passing from the equator to the points P and P′, where the axis of the magnet cuts the surface of the earth. At these points, which are sometimes called the *terrestrial magnetic poles*, the field is vertical.

Any great circle passing through the axis PP′ is a *magnetic meridian.* The declination at a point M (Fig. 206) is the angle which the great circle, through this point and the axis, makes with the geographical meridian. It varies from point to point along the same meridian, except for the great circle which passes through the two axes PP′ and NS. This is both the geographical and the magnetic meridian, and at every point of it the declination is zero, and the needle points

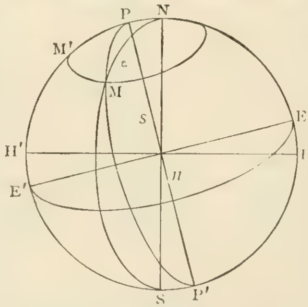


FIG. 206.

exactly north and south. This circle forms the periphery of Fig. 206. On one side of this plane the north pole deflects towards the west, and the declination is westerly; on the other side it is deflected to the east, and the declination is easterly.

The magnetic moment M of the globe is given by the formula:

$$M = T_e R^3.$$

Taking $T_e = 0{\cdot}33$ and $R = 6{\cdot}37 \times 10^8$ centimetres, we have

$$M = 8{\cdot}5 \times 10^{25} \text{ C.G.S.}$$

Supposing the earth to be a uniformly magnetised sphere, its intensity of magnetisation must be

$$A = \frac{M}{\frac{4}{3}\pi R^3} = \frac{T_e R^3}{\frac{4}{3}\pi R^3} = \frac{3}{4\pi} T_e = 0{\cdot}079.$$

That is to say, about $\frac{1}{3500}$ the intensity of magnetisation of a magnet of average strength (**231**).

246. Gauss's Theory.—The preceding theory only gives a rough approximation. Gauss has treated the problem more generally, supposing the magnetisation which produces the terrestrial field to be distributed in any manner.

Whatever be the distribution it gives at each point a determinate potential. Let us suppose equipotential surfaces drawn in the field corresponding to equidistant values of the potential. These surfaces cut the surface of the earth along lines which we may call *magnetic parallels.* These lines have the property of being everywhere perpendicular to the magnetic meridians; for, being drawn on the surface of the earth, which is supposed spherical, they are perpendicular to the vertical, and since they lie in the equipotential surfaces, they are perpendicular to the direction of the field. These lines are *equipotential lines* in reference to the horizontal component; at each point this component is normal to the line, and its mean value varies inversely as the distance between two adjacent lines.

The magnetic equator corresponds to the surface $V = 0$, which, unless the distribution is far from being symmetrical, will pass near the centre: the equator separates those points of the surface for which the potential is positive from those for which it is negative; the dip along this circle is not necessarily zero, nor the field constant. Thus the magnetic parallels are not necessarily either lines of constant dip nor lines of constant field.

The points of the surface where the field is vertical, and which are usually called the poles, are those in which the surface of the globe is a tangent to the equipotential surface which meets it;

these are the points of the surface for which the absolute value of the potential is a maximum. Thus, in order to know the distribution of magnetism, it is sufficient to draw the equipotential lines for the surface of the globe. Gauss showed that in the most general case, and within the degree of accuracy of the observations, these lines may be algebraically expressed by formulæ containing twenty-four coefficients ; so that when these coefficients have been calculated once for all, by means of an equal number of observations corrected for local disturbances, it is only needful to introduce the geographical co-ordinates of any given point into the formulæ to obtain the value of the magnetic elements at this point.

The calculations of Gauss made for the year 1838 fix the following positions for those of the two so-called poles :—

North pole, 73° 35′ lat., 97° 59′ long. west from Greenwich.

South pole, 72° 35′ lat., 150° 10′ long. east from Greenwich.

According to the observations made by the *Discovery* expedition the position of the south magnetic pole in 1903 was 72° 51′ lat. and 156° 25′ long. east. The results obtained by the *Southern Cross* indicate a position only 30 or 40 miles away.

247. Variations of Terrestrial Magnetism.—The elements of the earth's magnetism are not constant at any given place, but undergo variations in the course of time. Among these variations some appear to be irregular, while others, on the contrary, have a well-defined periodic character.

Secular Variations.—Since the date of the earliest exact observations, slow changes have been found to be going on. For many years these proceeded approximately as though they were due to a uniform rotation of the magnetic axis about the terrestrial axis taking place in the direction of the hands of a watch for an observer placed at the north pole, the angle between the axes being about 15° and the period of a complete rotation being about 900 years (Fig. 207). Thus at Paris, which is represented in the figure by P, the declination, which was easterly when first recorded, was zero in 1666 ; since this time it has been westerly, and went on increasing until 1824, when it amounted to 24° ; at present it is decreasing, and on the above supposition will become nothing again about the year 2114, when

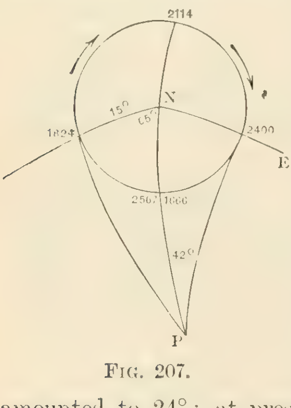


FIG. 207.

the magnetic pole will be on the other side of the north pole in respect to us. As to the dip, it has diminished steadily from 1666, and, on the same hypothesis, will continue to do so until about 2114. The observations of the last few years, however, make it doubtful whether this is a true statement of the law of change.

The values of the declination near London at different times are shown in Fig. 208, while in Fig. 209 are shown the lines of equal

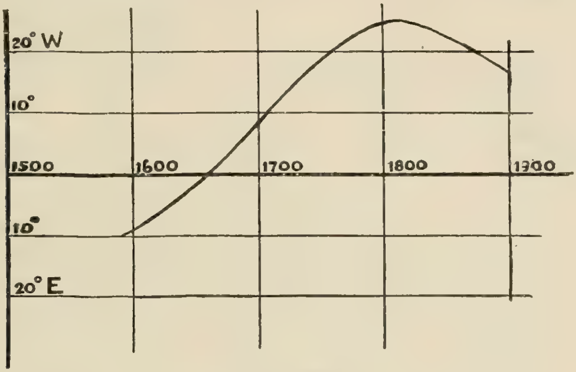


FIG. 208.

declination and equal dip for England and Wales at two epochs, viz. 1837 and 1891.

The mean annual rates of change of the elements through periods each of twenty years are as follows **(244)**:—

Period.	Declination.	Dip.	Horizontal Field.	Vertical Field.	Total Field.
1866–1886	−7′·7	−1′·4	+·00019	−·00005	+·00003
1871–1891	−7′·7	−1′·2	+·00019	+·00001	+·00009
1876–1896	−7′·0	−1′·2	+·00020	+·00003	+·00011
1881–1901	−6′·0	−1′·6	+·00023	−·00002	+·00007
1886–1906	−5′·2	−1′·75	+·00022	−·00010	−·00001

Daily Variations.—Other variations have a short period, and appear to be connected with the apparent motion of the sun, the moon, &c., and follow laws which are not yet known.

These variations affect particularly the declination.

Observations of the sun have shown that regular alternations of maxima and minima of sun-spots recur in a period of eleven years, and it is found that the range of the regular diurnal variations of declination is from 40 to 50 per cent. greater in a year of maximum frequency of sun-spots than in a year of minimum

frequency. For example, the mean daily variation observed at

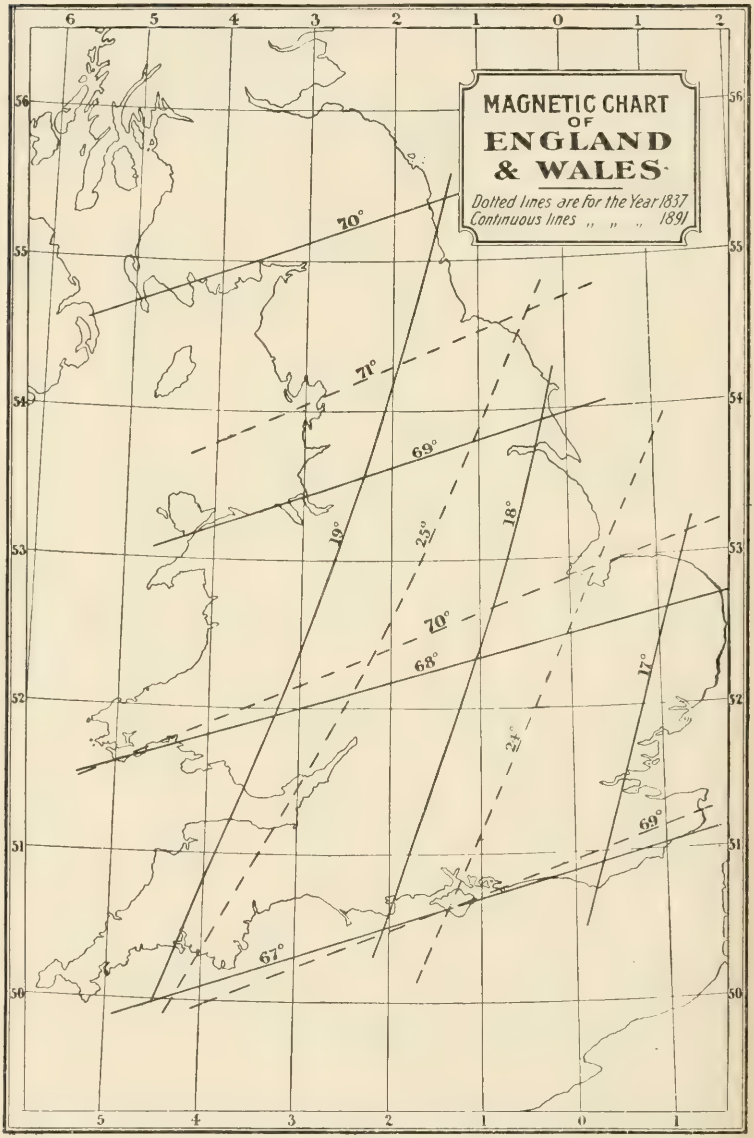


FIG. 209.

Kew in 1893 was 10′·74, while in 1900 it was 6′·83. The relation

between the two phenomena is so regular as to leave no doubt that it rests on some causal connection. Taking a complete 11-year period, the mean of the hourly observations at Kew, leaving out only the highly disturbed days, gives the range of the regular diurnal variations of declination for the different months of the year as follows:—

Jan.	Feb.	March.	April.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.
$4'\cdot9$	$6'\cdot1$	$9'\cdot1$	$10'\cdot95$	$10'\cdot7$	$10'\cdot9$	$10'\cdot6$	$11'\cdot0$	$9'\cdot5$	$7'\cdot7$	$5'\cdot4$	$4'\cdot5$

The hours at which the greatest and least deflections are reached vary from place to place and at the same place in the course of the year. At Kew, from November to February, the principal daily minimum occurs between 10 and 12 P.M., but, taking the mean of the whole year, this is inconspicuous in comparison with that which occurs about 8 A.M.

At Kew the declination is greatest at about 1 or 2 P.M. The maximum and minimum of declination show the greatest difference at the equinoxes, thus for the eleven years 1890 to 1900 the mean absolute daily range at Kew for all days was $15'\cdot93$ in March, $13'\cdot65$ in June, $14'\cdot57$ in September, and $9'\cdot80$ in December.

Irregular Variations.—Irregular variations, often, when they are considerable, called *magnetic storms*, are simultaneously produced over a great part of the surface of the earth. They have often been supposed to be connected with the aurora borealis, but it is not clear that there is a causal relation between the two phenomena.

Both daily and irregular variations are observed by special apparatus, known as *registering magnetometers;* the small movements of suspended magnets, magnified by the method of reflexion, are registered continuously by photography. The variations commonly observed are those of declination and of the horizontal and vertical components. The first is observed by means of a small bar suspended horizontally in the meridian by cocoon threads; the second is measured by means of a horizontal magnet suspended at right angles to the meridian by a twisted metal wire, or quartz fibre, or by a bifilar suspension; the third, by a magnet that is movable like the beam of a balance about a knife-edge, and being exactly counterpoised for a given value of the vertical component, dips more or less as this value changes.

ELECTRO-MAGNETISM

CHAPTER XXIII

ELECTRO-MAGNETISM

248. Electro-Magnetism.—In studying the effects of electric currents, we have hitherto been concerned only with *internal actions*, that is to say, effects produced by the current in the conductors traversed by it. It now remains to investigate the actions produced outside the conductor, which, for this reason, are called the *external actions*. The part of Science which relates to these phenomena is called *electro-magnetism* ; it originated in an experiment made by Oersted in July 1820, and it owes its principal developments to the researches of Ampère and of Faraday.

249. Oersted's Experiment—Ampère's Rule.—When a conductor through which a current is passing is brought near and parallel to a magnetic needle, the latter is deflected from its ordinary position (Fig. 210); this is the fact which Oersted observed. According to the relative positions of the needle and the circuit, the direction of the deflection is different. It may, however, be determined in each case by what is known as *Ampère's Rule*. Suppose an observer swimming in the direction of the current, so that it enters by his feet and emerges by his head: if the observer has his face turned towards the needle, *the north pole is always deflected to his left*. We shall speak of this as the left of the current.

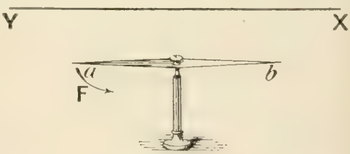


FIG. 210.

250. Ampère's Astatic Needle.—If it were not for the action of the earth, the needle would always set at right angles to the current. In order to show this, Ampère used a needle arranged like a dip-needle (**242**), so that it moved only about a fixed axis passing through the centre of gravity, and he placed this axis parallel to the direction of the earth's field (Fig. 211). As

the couple due to the earth acts in a plane containing the axis, it has no effect on the position of the needle. The needle then always sets at right angles to the current, whatever be the strength of the latter. Hence it follows that the action of the current is exerted in a direction at right angles to the conducting wire, and that, at least in part, it is of the nature of a couple.

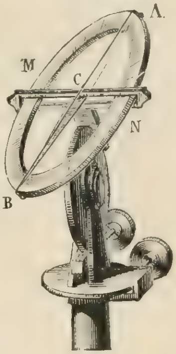


FIG. 211.

251. Galvanometer.—If we stretch the wire above a horizontal needle (Fig. 210) when it is at rest in the magnetic meridian, the needle is under the action of two systems of forces acting in planes at right angles to each other, and the needle takes up an intermediate position.

Experiment shows that, other things being equal, the extent of the deflection is independent of the degree of magnetisation of the needle, which proves that the two actions are in a constant ratio, and therefore that the couple due to the current, like that due to the earth, is *proportional to the magnetic moment of the needle* upon which it acts.

The deflection increases, moreover, with the strength of the current as measured by its chemical action, and may, therefore, serve as a measure of it. This is the principle of the electromagnetic measurement of the current, and of the apparatus which Ampère called a galvanometer (see Chapter XXIX.).

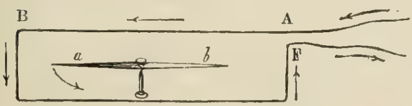


FIG. 212.

The action of the current can be increased by coiling the conductor on a frame; the needle is put at the centre of the frame (Fig. 212), and the plane of the frame is placed in the magnetic meridian. It is easily seen from Ampère's rule that all parts of the current tend to deflect the needle in the same direction. When the conductor is coiled many times round, the arrangement is known as *Schweigger's Multiplier*.

252. Magnetic Field due to a Current.—The essential fact which results from the experiments of Oersted and of Ampère is that an electric current creates a magnetic field, and acts on magnets or on other currents in the same sort of way as magnets themselves. Ampère showed that the field of a current is a true magnetic field, that is to say, a field of the same kind as that which is created by magnets. The identity is not merely an identity of

form, like that which we have found between the electrical field and the magnetic field (**191**), but a real and absolute identity.

The field of a current has all the characteristic properties of a magnetic field; it exerts a directive couple on a magnet, which at each point is proportional to the magnetic moment of the magnet placed at this point. The form of the magnetic field may be rendered evident in the ordinary way with iron filings, and is the same for all values of the current.

253. Field of an Infinitely Long Rectilinear Current.—Let us consider the case of a rectilinear portion of a circuit so long that it may be treated as an infinite right line. If this passes at right angles through a sheet of cardboard on which iron filings are sifted, it is seen that the lines of field are concentric circumferences with their centre in the axis of the wire (Fig. 213).

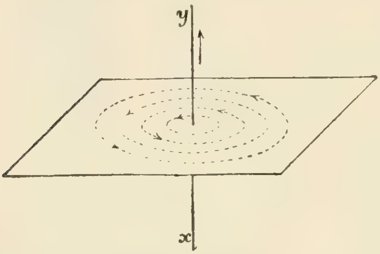


FIG. 213.

Ampère's rule shows that they go from the right to the left of an observer placed in the current. As might be anticipated from reasons of symmetry, it is found by experiment, that the intensity of the field is the same at all points of the same circumference. It follows that the equipotential surfaces are planes passing through the axis of the current, and making equal angles with each other. It can thence be directly concluded that the field varies inversely as the distance (**53**).

254. Biot and Savart's Experiment.—The law just mentioned was established experimentally by Biot and Savart. The experiment consists in causing a very small needle to oscillate first under the influence of the earth alone, and then under the simultaneous influence of the earth and of a very long straight current. The current being vertical, the needle is suspended horizontally in the plane which passes through the current at right angles to the magnetic meridian (Fig. 214). If

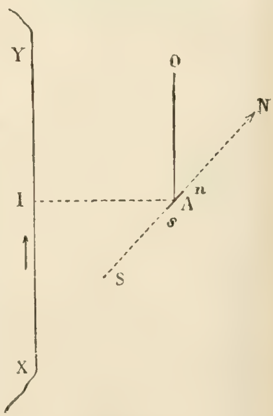


FIG. 214.

the needle is very small in comparison with its distance from the current, it may be regarded as being acted upon by a uniform

field, which, with the direction of the current shown by the arrows, adds itself to that of the earth. The needle retains its position of equilibrium in the magnetic meridian; if it is displaced therefrom, it vibrates according to the pendulum law.

If G is a constant which depends on the moment of inertia and on the magnetic moment of the needle, H the field due to the earth, h and h' the fields due to the current at the distances a and a', and if n, N, and N' are the numbers of oscillations made respectively in the same time under the action of the fields H, $H+h$, and $H+h'$, we have, from the formula for harmonic vibrations—

$$\begin{aligned} n^2 &= GH \\ N^2 &= G(H+h) \\ N'^2 &= G(H+h'). \end{aligned}$$

As experiment gives

$$\frac{N^2-n^2}{N'^2-n^2}=\frac{h}{h'}=\frac{a'}{a},$$

it follows that the *field of an infinitely long current varies inversely as the distance*; it is, moreover, proportional to the strength of the current; we may therefore write

$$h=2k\frac{C}{a},$$

k being a coefficient which depends on the unit in terms of which the strength of the current is expressed.

The unit, which will be afterwards described as the *electro-magnetic unit*, is that for which $k=1$. The unit in question is equal to 10 amperes. An infinitely long current of 10 amperes would, therefore, act on a magnet of unit magnetic moment at a distance of a centimetre with a couple of moment two dyne-centimetres.

If C is the strength of the current expressed in terms of the electro-magnetic unit, the law of Biot and Savart is expressed by

$$h=2\frac{C}{a}.$$

It is to be observed that ah is the moment of the magnetic field about the line of the current as axis. This moment is constant for the same strength of current whatever be the distance.

255. Equivalence of Current and Magnetic Shell.—It may be stated as the result of experiment that a current flowing in a wire is equivalent to a uniform magnetic shell (**191**, ii.) of which the wire is the outer boundary. This may be expressed otherwise by saying that the magnetic field due to a current is (with a certain restriction to be referred to later, **265**) derivable from a potential, and that the value of this potential is proportional to

the solid angle which the current subtends at the point under consideration. The value of this field may be determined experimentally in the same way as that due to a magnet. Thus, if we have a current flowing in a wire carried round a circle, the circle may be placed so that its plane is in the magnetic meridian and its centre movable east or west of a compass needle. It is then found that the field F at any point on the axis is directed along the axis itself (as indeed might be expected from considerations of symmetry), and further, that it is independent of the magnetic moment of the needle, is proportional to the ratio a^2/r^3 (if a is the radius and r the distance from a point on the circumference to the point considered) and is also proportional to the amount of copper set free in a given time in a voltameter placed in the current circuit. Any number of circuits carrying currents produce magnetic effects independent of each other. If the single turn of wire be replaced by a coil of n turns all at sensibly the same distance from the needle, through each of which the same current passes, the couple on the magnet is sensibly n times as great; that is, a current of strength C carried n times round a circle produces the same magnetic effect as a current of strength nC carried once round. The product nC is sometimes called the *effective current* round the coil: in the sequel, when we speak of the current round any area, the effective current as thus defined is to be understood, unless the context implies otherwise. All these results may be expressed in the single statement—

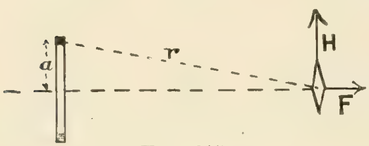


FIG. 215.

$$\text{Couple on magnet due to current} = k\frac{na^2}{r^3}CM' \cos\theta$$

where k is a constant depending on the unit of measurement for the current, C is the current measured by its electrolytic effect, M' is the magnetic moment of the needle, and θ is the deflection.

It was shown in (**191**, ii.) that the field at a point on the axis of a uniform circular magnetic shell of strength ϕ is $2\pi\phi\frac{a^2}{r^3} = 2\frac{\phi A}{r^3} = 2\frac{M}{r^3}$, the symbols a and r having the same meanings as before, and therefore $\pi a^2 = A$, being the area of the shell, and ϕA its magnetic moment. The field due to the current and that due to the shell are accordingly equal, if, with the unit employed above,

$$\frac{k}{2\pi}\cdot nAC = M.$$

256. Influence of a Change in Surrounding Medium.—When the medium surrounding the current is magnetisable, it becomes magnetised under the action of the current field. If the magnetised medium fills all the surrounding space, its magnetisation can be represented by closed lines parallel to those of the current field. Since these lines are closed, they have no polar field; consequently we should expect the field due to the current to be *the same whatever the medium may be.*

This is not the case, as we have seen, when the medium surrounding a magnet is changed (**194, 205**); the abrupt termination of the magnetisation of the medium at the surface of the magnet produces a modification in the value of the field.

A modification of the same kind is produced in the case of a current if the magnetisable medium only partially fills the surrounding space. For example, let it consist of a plane iron shell contained within the contour of the current. The magnetisation of this shell under the influence of the field changes the value of the field at external points; that is to say, the field outside it will not be the same whether the iron shell is present or not. Or again, the iron may fill all space except that of a similar plane shell *of air*. In this case, the field in the iron depends upon the permeability of the iron in the same way and for the same reason (**205**) as if the current and air shell were replaced by a permanent magnetic shell.

It follows that the magnetic shell which is equivalent to a particular current when the medium is air is not equivalent to it when the air is *wholly* replaced by a magnetisable medium. The strength of the equivalent shell is in all cases equal to μC where μ is the permeability of the surrounding medium. That this is so follows from the fact that the field of a shell of *given* strength varies inversely as μ; its strength must therefore be taken directly as μ in order that the field may remain unchanged.

The magnetic induction in any region is obtained from the field by multiplying it by the permeability. Hence surrounding a current by iron instead of air increases the magnetic induction at all points.

257. Electromagnetic Unit of Current.—By properly choosing the unit of current the coefficient $k/2\pi$ in the last equation may be made equal to unity. The unit thus arrived at is called the absolute electromagnetic unit. If a wire be bent into a circular arc, each element of length, if traversed by a current, must be supposed to exert the same force at the centre of the circle. Hence the definition of the electromagnetic unit of current may be stated

in the following form: a current is of unit strength if unit length of it carried along the circumference of a circle of unit radius produces unit magnetic field at the centre; or, what comes to the same thing, if when carried once round a circle of unit radius it produces a magnetic field at the centre numerically equal to 2π. A current of one-tenth of this strength is the unit commonly employed in practical applications and is called an *ampere*. In the following pages either the electromagnetic unit or the ampere will always be used for expressing the strength of current.

258. Action of a Current on a Current.—Since in the cases mentioned above a current acts as the equivalent of a magnet, it might be expected that two currents would behave to one another as two magnets do, and experiment shows this to be the case. This result completes the proof that a current is a magnet in every essential particular; that is to say, that there is not only a similarity in behaviour under certain circumstances between the two, but that each may be replaced by the other as far as magnetic effects are concerned.

259. Weber's Experiments.—The results of very careful experiments made to test the law of the action of two currents on each other were published by Wilhelm Weber in 1846. He proved that two coils of wire, through each of which the same current is flowing, exert upon each other a turning moment proportional to the square of the strength of the current and depending otherwise on the dimensions of the coils and their relative positions. More generally, if C is the strength of current through one coil and C' that through other, the turning moments are proportional to the product CC'. Weber further found that, if two coils were set with their axes in the same horizontal plane, one of them fixed with its axis at right angles to the magnetic meridian, and the other suspended so as to be able to turn about a vertical diameter, and with its axis in the magnetic meridian when it is in equilibrium with no current through it (Fig. 215A), the turning moment exerted by the fixed coil on the suspended one varies with distance according to the same law as applies in the case of two magnets (**186**, **243**), and alters in the same way as with two magnets when the line joining the centres of the coils is changed from being in the magnetic meridian to being at right angles thereto. Confirmation was thus given of the conclusion arrived at theoretically by Ampère and supported by him by less satisfactory experiments, that *the magnetic*

Fig. 215A.

field due to a closed current is the same as that of a magnetic shell whose outline coincides with that of the current and whose magnetic strength is numerically equal to the strength of the current. It follows that, in calculating the magnetic action at any point due to a closed circuit, we may, whenever the calculation is facilitated thereby, suppose the circuit to be replaced by the equivalent magnetic shell and apply to it the conclusions arrived at in Chapter XVIII.

260. Extension to Large Circuits.—Even if it be considered that Weber's results do not amount to a complete proof of the above conclusion, its truth follows from the fact that, while it is being constantly assumed in electromagnetic measurements and calculations, no error in the results has ever been traced to this assumption. Again, if it be admitted as experimentally proved for small circuits, it can be shown to be true in all cases.

Consider a surface of any shape bounded by a contour s (Fig. 216): let us suppose this surface cut by two systems of intersecting parallel lines, which divide it into small elements, and suppose the boundaries of all these elements traversed by currents of the same strength C, all flowing round in the same direction. Each line which separates two adjacent elements is traversed by two equal currents in opposite directions, which neutralise each other. Consequently only those portions of the current which correspond to the elements of the contour come into account, and the force due to these is evidently the same as that of a single continuous current of the same strength flowing round the contour s. On the other hand, each of the elementary currents may be replaced by the equivalent magnetic shell, and all such shells, taken together, form a shell having the same contour as the closed circuit considered, and a strength equal to the strength of the current.

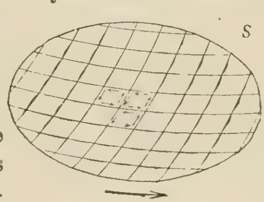


FIG. 216.

261. Laplace's Rule.—Experiments with steady currents can be made only with complete circuits; yet many attempts have been made to calculate the resultant effect as the sum of effects due to each elementary length of the circuit. Laplace showed that it was sufficient to assume that the field at any point due to an element of current is given by the formula

$$dF = \frac{Cds \sin\alpha}{r^2}$$

ds being the length of the element, r the distance of the point from the middle of the element, and α the angle which the element

makes with the straight line r; and that the direction of the field is perpendicular to the plane rds. This rule in every case gives, when the summation is effected, the same result as that which is obtained by replacing the current by its equivalent shell, and it affords a very convenient mathematical artifice for finding the field due to the current. Since, however, it is impossible to produce a steady current through a detached piece of a conductor no physical validity should be attributed to it. Nevertheless, owing to the utility of the formula, it is instructive to see how its formal truth may be most directly arrived at. Taking the case of the field at the centre of a circular current the proportionality of effect to the number of turns is evidence that each similarly situated element acts equally, and that, therefore, other things being equal, the field is proportional to the length of the acting current. Again, taking coils of various radii, each is found to produce the same effect when the number of turns is directly as the radius, that is to say, the field due to each turn (*cæteris paribus*) is inversely as the radius; and since the length of each turn is proportional to the radius, the field due to an element must be inversely as the radius squared. Further, it is shown experimentally that a simple circle and a zigzag, sinuous, or spiral wire bent into circles of the same average radius are equivalent to one another. It follows that the effective component of the element of current is that which is perpendicular to the line joining the element with the point at which the field is determined. Thus the whole rule is proved.

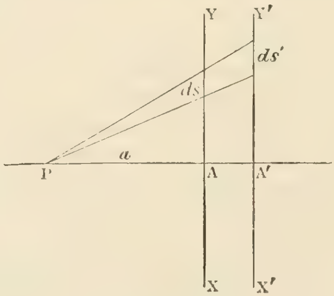


FIG. 217.

261.* Field due to a Long Straight Current.—As an application of this rule we will calculate the field due to a straight current extending equally in both directions. In this case the elementary fields at P (Fig. 217) are perpendicular to the plane of the paper; the resultant field is therefore their arithmetic sum,

and may be written, if θ is the value of α at the most distant point of the current,

$$F = \int_{\pi-\theta}^{\theta} \frac{ds \sin \alpha}{r^2} C.$$

Representing the shortest distance PA by a, we have

$$\frac{ds}{-r d\alpha} = \frac{r}{a} \text{ or } \frac{ds}{r^2} = -\frac{d\alpha}{a},$$

whence

$$F = -\frac{C}{a}\int_{\pi-\theta}^{\theta} \sin \alpha d\alpha = \frac{2C}{a} \cos \theta,$$

which, when the wire is very long, becomes $\frac{2C}{a}$, which agrees with the result found experimentally by Biot and Savart (**254**).

We may note some consequences of this result:—

i. Suppose a long straight current through the point O at right angles to the plane of the figure (Fig. 217A), and flowing outwards; the magnetic field at P is $\frac{2C}{a}$ perpendicular to OP in the direction of the arrow. The component along b making an angle ϕ with OP is $2C \sin \phi/a$. If a unit magnetic pole be moved from P through the very short length db along b, the work done upon it by the current is $2Cdb \sin \phi/a$. Draw a line from O to the further extremity of db, and let $d\beta$ be the angle which it makes with OP: then $db = \frac{a d\beta}{\sin \phi}$; hence the work is

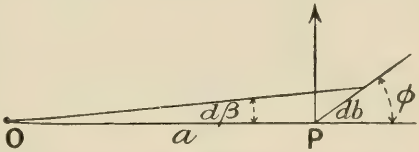


FIG. 217A.

$$\frac{2C \sin \phi}{a} \cdot \frac{a d\beta}{\sin \phi} = 2C d\beta.$$

If the pole is moved along any path so that the radius vector from the nearest point of the current turns through a finite angle β, the work done is $2C\beta$, which becomes $4\pi C$ for any closed curve encircling the current.

ii. If the conductor of a current is a long, thin-walled tube, of uniform thickness and material, so that the current is uniformly distributed within it, there is no magnetic field inside the bore of the tube due to the current in the walls. To prove this, consider any point inside the tube, and take two planes through this point parallel to the axis and intersecting each other at a small angle. These planes will intercept two longitudinal strips of the tube, one on each side of the selected point, which will carry portions of the

whole current proportional to their respective distances from the point; and since equal currents in these strips would produce at the point oppositely directed fields of strengths inversely proportional to their distances, the resultant field due to both together will be nothing; and, as the whole tube can be similarly subdivided into pairs of mutually compensating elementary strips, the total magnetic effect of the current is zero at any internal point.

iii. It follows from ii. that on the axis of a straight cylindrical rod or wire, carrying a current uniformly distributed through the cross-section, the magnetic field due to the current is nothing, but that at any internal point not on the axis there is a field proportional to the distance from the axis. For the portions of the wire lying farther from the axis than the selected point may be regarded as forming a hollow tube which, as we have seen, has no internal field. The effective part of the current is that conveyed by the internal cylindrical portion of the wire whose radius, say a, is the distance of the selected point from the axis. If the total current is C and the radius of the wire is r, the strength of the effective portion of the current is $C\frac{a^2}{r^2}$, and the field due to this at the distance a from the axis is

$$2C\frac{a^2}{r^2}\cdot\frac{1}{a}=2C\frac{a}{r^2}.$$

262. Field due to a Circular Current at a Point on the Axis.—As another application of Laplace's formula we will calculate the field due to a circular current at a point P on the axis (Fig. 218). Let a be the radius, and r the distance of a point of the circumference from the point P. The angle α between the direction of the current and the line r is $\frac{\pi}{2}$; the formula therefore reduces for a unit current to

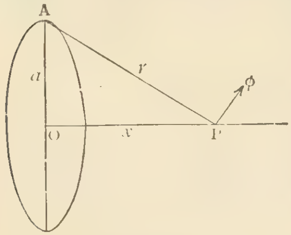


FIG. 218.

$$dF=\frac{ds}{r^2}.$$

The field due to an element may be resolved into two others, one parallel and the other perpendicular to the axis. The components perpendicular to the axis neutralise each other in pairs, and the resultant is equal to the sum of the components along the axis.

The component along the axis is obtained by multiplying dF by the cosine of the angle PAO, or by a/r, which gives $a\,ds/r^3$. The resultant is then equal to the product of the constant factor a/r^3

by the sum of the elements ds, that is to say, by the circumference $2\pi a$. We have thus

$$F = \frac{2\pi a^2}{r^3} = \frac{2A}{r^3},$$

A being the area of the circle. The same result is readily got by comparing a circular current with a circular magnetic shell of the same radius. The potential of a shell of strength Φ, at a point where it subtends the solid angle ω, in $\Phi\omega$ (**191,** ii.); and, that the current may be equivalent to the shell, we must have $nC = \Phi$ (**255**), n being the number of times the current goes round. At a point on the axis of a circle where its radius subtends the plane angle θ, the circle subtends the solid angle $2\pi(1 - \cos\theta)$; hence the potential at this point becomes $V = 2\pi nC(1 - \cos\theta)$, and the field is $F = -\frac{dV}{dx} = -2\pi nC \sin\theta \frac{d\theta}{dx} = 2\pi nC \frac{a^2}{r^3}$ since $\sin\theta = \frac{a}{r}$ and $-rd\theta/dx = \frac{a}{r}$. At the centre we have $r = a$ and therefore $F = 2\pi nC/a$. Thus at the centre of a circle of 10 centim. radius the field due to a unit current carried once round is 0·628 absolute units.

263. Field of a Circular Current.—Except for difficulties of the calculation, we might use Laplace's formula to calculate in like

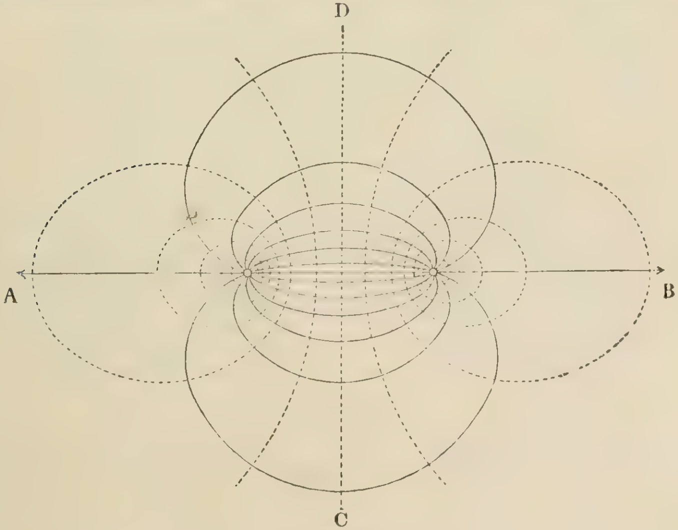


FIG. 219.

manner the strength of the field at each point. Fig. 219 represents the field of a circular current; the plane of the circle is supposed

horizontal and perpendicular to the plane of the figure which it intersects at the two small circles on the line AB. The dotted lines represent the directions of the field in a plane passing

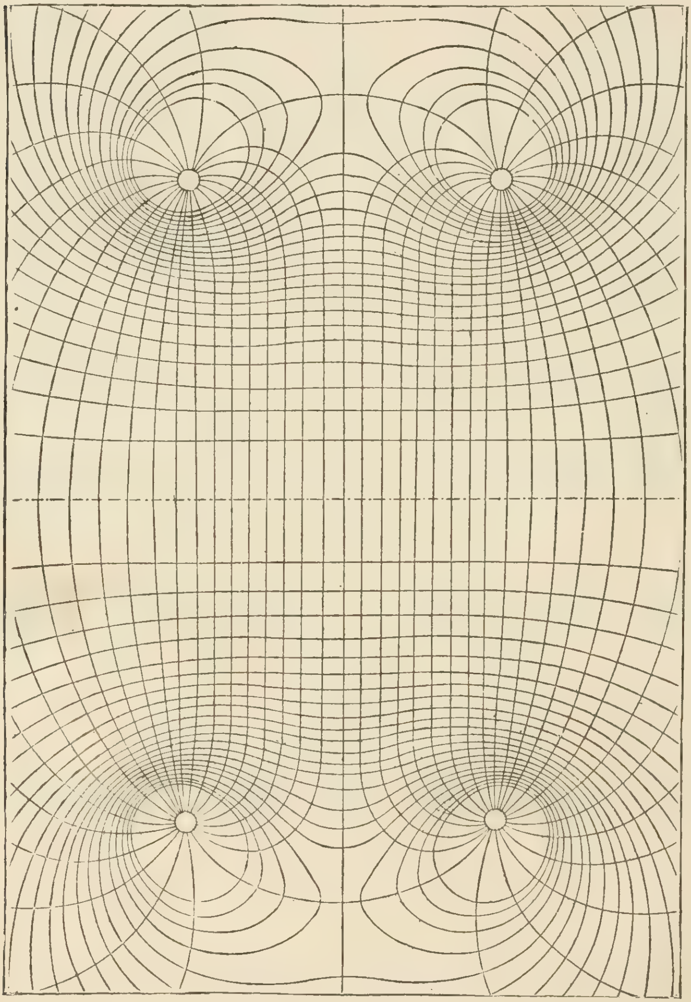


FIG. 220.

through the axis CD of the current. These lines traverse the circle from the right to the left for an observer who looks towards the middle of the circle and is imagined in the conductor so that the

current flows from foot to head. We shall take this as the positive direction of the axis of the current, and we shall call that face of the current which is on the left of the observer the *positive*, and that which is on the right the *negative*; hence the axis traverses the circuit from the negative to the positive face. If the circuit were replaced by the equivalent magnetic shell, the north magnetic face of the shell would correspond with what we have called the positive face of the current. The full lines which cut the dotted lines at right angles are the intersections of the equipotential surfaces by the plane of the figure.

264. Field due to Two Circular Currents.—Fig. 220 represents the lines of field and the equipotential surfaces for the case of two equal and parallel circular currents with their planes at a distance equal to the radius. Throughout a considerable region the lines of field are very nearly parallel to the common axis of the two currents, and the equipotential surfaces are equidistant. The field may therefore be considered to be uniform (**357**).

265. Restriction to Theorem of Equivalence of Current and Shell.—In an important particular the field of a current is different from that of a magnetic shell. We saw (**191**, ii.) that in passing from any point close to a magnetic shell to an adjacent one on the opposite side of the shell the potential changes through the value $4\pi\phi$, but that in passing back to the starting point through the substance of the shell the original value of the potential is recovered if the magnetic field in the substance of the shell be suitably defined. In the case of a current, however, though the former part of the statement still applies, there is no counterpart to the magnet in the plane of the current by aid of which the original value is restored. Thus, if a closed path be traversed so as to thread through the circuit—in which case the solid angle subtended changes through 4π—the potential changes through the amount $4\pi C$: if the path surrounds the circuit n times, the potential increases by $4\pi nC$; if the closed path lies altogether outside the contour of the current the potential resumes its original value. We may therefore write, in general,

$$V=(\omega \pm 4\pi n)C$$

the positive sign corresponding to the case in which the path has threaded the circuit from the positive to the negative face.

As the potential is the same, within a constant, for a closed circuit, as for a magnetic shell, the field at any point is the same for both (except in the substance of the shell) as it depends only upon the space-rate of decrease of potential at the point. On the

other hand, the line-integral of magnetic field (**192**) is only zero for a closed path if no current be embraced by the path ; if the path is linked with a current the line-integral is equal to $4\pi C$ where C is the total current with which it is linked.

266. Applications to Calculation of Magnetic Field.—In certain cases where assistance is obtained from considerations of symmetry and otherwise, the above theorem enables us to deduce the value of the field. Take the case shown in Fig. 221 in which a coil is wound in a close spiral on the surface of an anchor ring. If we regard this system as sensibly consisting of a series of circular currents whose centres are all on the same circular axis (which is equivalent to neglecting the component of current parallel to the circular axis of the ring), the lines of field consist in circles having their centres on the straight axis. Line-integrating round any such circle, the result must vanish unless the circle lies inside the ring itself, and since the field must have the same value at each point of any one circle the zero result can only be possible if the value of the external field is zero. The field inside the ring can be obtained by evaluating the integral. Let the point considered be at distance a from the straight axis. Then

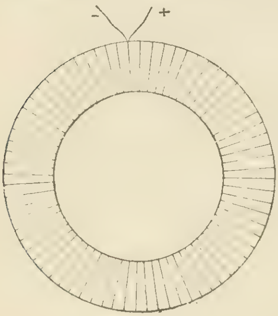


Fig. 221.

$$H \cdot 2\pi a = 4\pi \times \text{current enclosed}$$
$$= 4\pi n C$$

where n is the total number of windings on the ring. Thus the field $H\left(=\dfrac{2nC}{a}\right)$ varies inversely as the distance from the straight axis. If the radius of the convolutions is small in comparison with the radius of the circular axis of the ring, the internal field is nearly uniform and the same as if a had its mean value.

266.* Cylindrical Coils.—In many pieces of electrical apparatus, cylindrical coils of wire of one or more layers are employed, and it is of interest to examine the magnetic properties of such a coil when a current is sent through the wire. We shall investigate the strength of the field at points on the axis, assuming that the radius of the wire is small in comparison with that of the cylinder and that the convolutions are close together. Each convolution of the spiral may be replaced by its projections on the axis, and on a plane perpendicular

to the axis. For each turn of the spiral, the projection parallel to the axis is a straight line equal to the pitch of the spiral, and the projection perpendicular to the axis is a circle of the same radius as the cylinder. If the section of the cylinder is small, the combined effect of the projections parallel to the axis is neutralised by carrying the wire back along the axis. The compensation is still more perfect if the cylinder is wound with an even number of layers, in which the inclination of the turns is alternately in opposite directions.

Consider first a coil of a single layer with n turns of wire per unit length: we have to determine the field at a point P on the axis. Let a be the radius of the coil, r the distance from P to any part of the coil, and θ the angle between r and the axis. A very short length of the coil dx, taken parallel to the axis, will contain ndx turns of wire, and the current $Cndx$ in this length of the coil may be treated as being all at the same distance from the point P. Hence the field at P due to the element of the coil is (262)

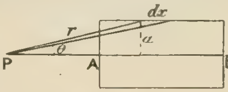


Fig. 221A.

$$dF = 2\pi nC\frac{a^2}{r^3}\,dx.$$

By reference to the figure (Fig. 221A) it will be seen that

$$\frac{dx}{-r\,d\theta} = \frac{r}{a}, \text{ or } dx = -\frac{r^2 d\theta}{a}.$$

Hence we may write

$$dF = -2\pi nC\frac{a}{r}\,d\theta = -2\pi nC\sin\theta\,d\theta = 2\pi nC\,d\,(\cos\theta).$$

Hence for the field at P due to the whole length of the coil we have

$$F = 2\pi nC\,(\cos\,\theta_B - \cos\,\theta_A),$$

where θ_B stands for the value of θ corresponding to the far end B of the coil, and θ_A for its value at the near end.

If the point P moves up towards the coil, the negative term, $\cos\,\theta_A$, in the above gets less and less and vanishes when P comes to A, at the same time $\cos\,\theta_B$ also diminishes, but less rapidly. If l is the length of the coil, the field at either end is

$$F = 2\pi nC\frac{l}{\sqrt{a^2+l^2}},$$

which becomes $2\pi nC$ if a^2 may be neglected in comparison with l^2.

If P moves still further along the axis so as to come to the centre of the coil, the two angles θ become supplementary to each other and the field may be written

$$F = 4\pi nC \cos\theta = 4\pi nC \frac{\frac{1}{2}l}{\sqrt{a^2 + \frac{1}{4}l^2}},$$

which becomes $4\pi nC$ if we may neglect a^2 in comparison with $\frac{1}{4}l^2$. The last expression gives approximately the strength of the field inside a coil, whose length is great as compared with its diameter, at any point except near the ends.

The direction of the field is always through the coil from the negative to the positive end (**263**), or inwards at the end where the conventional direction of the current round the coil is that of the hands of a watch with its face outwards, and out at the other end.

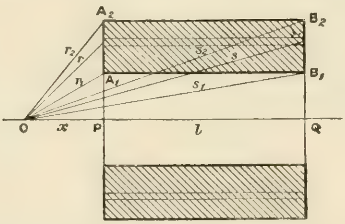


FIG. 221B.

Coil of Several Layers.—Let the figure (Fig. 221B) represent a section, by a plane through the axis, of a coil formed by many layers of wire uniformly wound one upon another. We will determine the field at the point O distant x from the near end of the coil, the length of the coil being l. Let P and Q be points on the axis where it is cut by planes through the two ends of the coil; take a for the radius, PA_2 or QB_2, of the outermost layer, and b for the radius, PA_1 or QB_1, of the innermost layer. Call the distances of A_1 and A_2 from O, r_1 and r_2 respectively, and the distances of B_1 and B_2 from the same point s_1 and s_2. Now consider the field at O due to a cylindrical element of the coil of radius y and thickness dy. If there are m layers of wire per unit thickness of the coil measured at right angles to the axis, the number of layers in the element in question is mdy and the resulting field at O is (see above)

$$dF = 2\pi nC.mdy(\cos\theta_2 - \cos\theta_1)$$

where n as before is the number of turns of wire per unit length

of the coil, θ_1 is the angle with the axis made by a line drawn from o to a point at the near end of the elementary cylinder considered, and θ_2 the corresponding angle made by a line drawn to the far end: these lines themselves we will call r and s respectively. Then

$$\cos\theta_1 = x/r \quad \text{and} \quad \cos\theta_2 = (x+l)/s,$$

whence

$$dF = 2\pi nmC\left[(x+l)\frac{dy}{s} - x\frac{dy}{r}\right].$$

But $r^2 = x^2 + y^2$ and $s^2 = (x+l)^2 + y^2$, and therefore, since x and l are constant, $rdr = ydy$ and $sds = ydy$, whence

$$\frac{dy}{r} = \frac{dr}{y} = \frac{d(y+r)}{y+r} \quad \text{and} \quad \frac{dy}{s} = \frac{ds}{y} = \frac{d(y+s)}{y+s}.$$

Hence we may write

$$dF = 2\pi nmC\left[(x+l)\frac{d(y+s)}{y+s} - x\frac{d(y+r)}{y+r}\right],$$

and, to get the field due to the whole coil, we have to integrate this expression between the limits $y = b$ and $y = a$, which gives

$$F = 2\pi nmC\left[(x+l)\log\frac{a+s_2}{b+s_1} - x\log\frac{a+r_2}{b+r_1}\right].$$

If o is at the middle point of the axis, $x = -\frac{1}{2}l$, and $r_1 = s_1$ and $r_2 = s_2$ and the field attains its greatest value

$$F_{max.} = 2\pi mnlC.\log\frac{a+r_2}{b+r_1}.$$

If the coil is very long in comparison with its radius, r_1 and r_2 both become nearly equal to $\frac{1}{2}l$; the last factor may then be written

$$\log\frac{1+\dfrac{2a}{l}}{1+\dfrac{2b}{l}} = \log\left(1 + 2\frac{a-b}{l}\right) = 2\frac{a-b}{l},$$

the two last equations being approximate and becoming more nearly

accurate as l is greater in comparison with a and b. Using the last value, the field at the middle of a long coil becomes

$$4\pi mn(a-b)lC\frac{1}{l}=4\pi mn(a-b)C,$$

where $mnl(a-b)$ is the whole number of turns of wire in the coil and therefore $mn(a-b)$ is the number per unit of length; the expression thus takes the same general form as in the case of a coil with only a single layer of wire, and, at all points within the coil and not near either end, the field has sensibly the same value.

266. Experimental Illustration.**—A cylindrical coil has all the properties of a magnetised bar.

This may be shown by means of the arrangement represented in Fig. 222. When the electromagnetic cylinder is suspended horizontally, as in the figure, and a current is passed through it, it sets in the direction of the magnetic meridian, and when the north pole faces us, the apparent direction of the current is opposite to that of the hands of a watch.

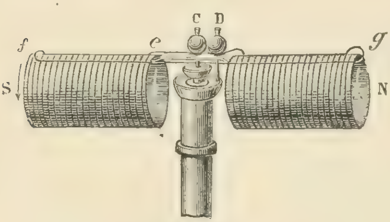


Fig. 222.

All the experiments on attraction of opposite poles and on repulsion of like poles may be repeated by acting on it either with a magnet or with another electromagnetic cylinder.

Nevertheless, an electromagnetic cylinder must not be confounded with a hollow magnet: in the latter all the lines of field, both inside and outside, start from the positive or north end, and terminate at the negative or south end. In the electromagnetic cylinder the lines of field form closed curves, those in the interior being a continuation of the external lines of field. The magnetic induction is, however, the same as for a *solid* magnet of the same form; for at outside points it is equal to the field, which is the same for both, while at inside points it is the continuation of the external induction.

CHAPTER XXIV

ENERGY OF A CURRENT

267. Energy of a Current in a Field.—A wire carrying a current cannot in general be displaced in a magnetic field without the performance of positive or negative work. We regard the system, therefore, as having a store of energy the variation of which represents the work done. The change of energy can be calculated by replacing the current by its equivalent shell, to which the theorems of (**195, 196**) at once apply. Thus, writing μCA instead of M, we obtain, for a very small movement,

Increase of energy

$$= -dW = -HM\,d(\cos\theta) = -H\mu CA\,d(\cos\theta),$$
$$= -CBA\,d(\cos\theta)$$

where B is the magnetic induction, and A the effective area of the circuit.

The quantity $BA\cos\theta$ is the flux of induction N through the area of the shell. Consequently

$$\text{Increase of energy} = -CdN.$$

If the value of the flux of induction entering the negative face changes *in any way* from the value N_1 to N_2, the corresponding increase of energy is $-C(N_2 - N_1)$.

If we suppose the whole region to be mapped out by unit tubes of induction, the total flux through any area can change only by a passage of tubes across the contour of the area. The above theorem may therefore be otherwise expressed as follows: *The increase in energy of a current and a field is equal to the product of the current into the number of unit tubes of induction which pass outwards across its contour.*

If the flux of induction is that due to a second current of intensity C', its value is proportional to C', and otherwise depends only upon the geometry of the system; we may write its value MC';—M is called the *coefficient of mutual induction*, or the *mutual*

inductance of the two circuits. Let M_2 and M_1 be the values which correspond to N_2 and N_1 respectively; then

$$N_1 = M_1 C' \qquad N_2 = M_2 C',$$

and consequently

$$-dW = -CC'(M_2 - M_1).$$

In order to find the couple which either of the circuits experiences in any position, we must calculate the rate of change with angle of the total flux N, as a small rotation takes place; this rate multiplied by the current, or $+\dfrac{CdN}{d\theta}$ is the couple required.

Again, to find the translational force it experiences, we must calculate the value of $+\dfrac{CdN}{dx}$ where dx is a small linear displacement; the result is the force in the direction of x. It follows that when the flux N is a maximum the circuit experiences neither force nor couple, and that in all other positions it tends to move so as to increase the value of the product CN.

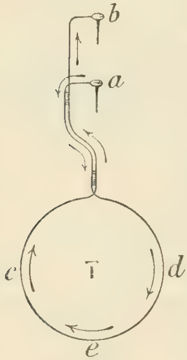


Fig. 223.

Let us consider, for instance, a circuit moving about a vertical axis in the earth's field (Fig. 223); the position of equilibrium will be that in which the plane of the circuit is perpendicular to the magnetic meridian, the current, as seen from the south side, circulating in the same direction as the hands of a watch. If A is the area of the circuit, and H the horizontal component of the earth's field, the flux through the area in the position of stable equilibrium is AH, since μ is unity for air. If the circuit is turned through a right angle, the flux is nothing; if it is turned through 180°, the flux is $-AH$. When the current is left to itself and it turns from the latter position to the former, the total variation of the flux is equal to $2AH$, and the work done by electromagnetic force for a current of constant strength C is

$$W = 2CAH.$$

In like manner if the frame is movable about a horizontal axis, passing through the centre of gravity (Fig. 224), and at right angles to the meridian, its plane sets at right angles to the dip-needle.

Lastly, a circuit is evidently astatic if it consists of two equal areas of opposite signs, since there is no variation of the flux whatever the displacement.

The application of the same principles shows that, under the action of its own flux, a circuit will tend to present the greatest possible surface; if the contour is flexible, it will take the form

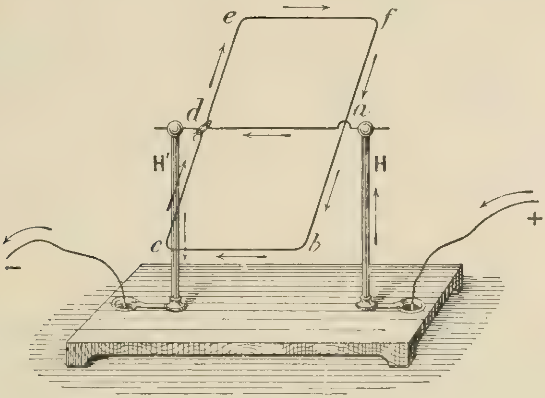


FIG. 224.

of a circle, as this comprises the greatest area within a given perimeter.

This is the true explanation of a well-known experiment of Ampère, which he regarded as proving that two consecutive elements of the same current repel one another.

Two quantities of mercury (Fig. 225), separated by an insulating partition, are connected by an iron wire, bent so as to form two parallel horizontal branches, *np* and *rq*, connected by a cross-piece, *pq*. When connection is made with a battery by the cups E and F, the wire glides on the surface of the mercury away from the points by which the current enters, thus increasing the area of the circuit.

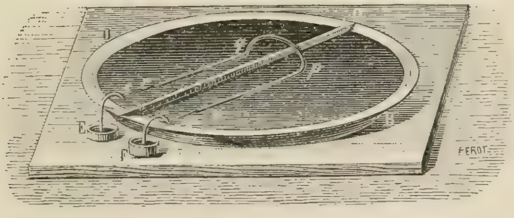


FIG. 225.

268. Force between Parallel Currents.—Let A and B be two infinitely long parallel currents, of strengths C and C', placed

at a distance a (Fig. 226) in a medium of permeability μ. At the point C, the magnetic field due to the current A is perpendicular to the plane of the two wires, and is equal to $H = \frac{2C}{a}$.

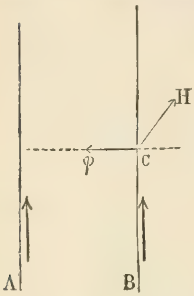


FIG. 226.

The number of lines of induction cut by unit length of B while it moves through a distance dx parallel to itself is

$$\mu H \,.\, dx = \frac{2\mu C}{a} dx.$$

Hence the force upon unit length of B is equal to

$$\frac{2\mu CC'}{a}.$$

It acts in the plane of the two currents, perpendicular to their direction, and tends to bring them together if they are in the same direction, and to separate them if they are in opposite directions. It varies in magnitude inversely as the distance apart.

269. Force between Coaxal Coils.—The force on unit length of each of two equal coaxal circular currents, whose distance apart is very small compared with their radii, is approximately given by the same formula. Hence the total force that each experiences due to the other is

$$\frac{2\mu CC'}{a} \times \text{its circumference},$$

that is

$$\frac{4\pi r\mu CC'}{a}.$$

Again, if one of the coils is very small compared with the other the field of the large one is sensibly uniform over the area of the smaller, and equal to $\frac{2\pi r^2 C}{(r^2+x^2)^{\frac{3}{2}}}$ (**262**), r being the radius of the larger and x their axial distance apart. The force upon the small coil is therefore equal to the rate of variation of this quantity with x,—namely

$$\frac{d}{dx}\left(\frac{2\pi r^2 C}{(r^2+x^2)^{\frac{3}{2}}}\right) = -\frac{6\pi r^2 Cx}{(r^2+x^2)^{\frac{5}{2}}},\text{—multiplied by } \pi r'^2 \mu C',$$

that is

$$-\frac{6\pi^2 r^2 r'^2 x\mu CC'}{(r^2+x^2)^{\frac{5}{2}}}$$

where r' is the radius of the smaller. If the currents are of the same sign this indicates a negative force in the direction in which

x increases; *i.e.* it is an attraction. If each coil consists of several practically coincident turns of wire—say n and n' respectively—the above expression must be multiplied by the product nn'; for the induction through the small coil is proportional to n, and the current in *each* turn of wire thereby experiences a force equal to n times that which we have calculated.

270. Electromagnetic Rotations.—It follows from the fundamental principles of mechanics, that a continuous rotatory motion in the same direction cannot arise among a system of rigid bodies which are acted on solely by the forces they exert upon each other, if these forces are functions only of the relative distances of the bodies between which they act, that is to say, if the forces have always the same values when the bodies are at the same distances. For, regarding one portion of such a system as being at rest, suppose another portion to move relatively to it from a given position along any path, and come back to the same position again. Then any change of the distance between two points of the system that takes place during one part of the motion must be undone during the remainder; for, when the moving part of the system has returned to its first position, the relative distance between any given pair of points must be the same as at first. Hence, if positive work is done during part of the motion by the forces acting between any two points, equal negative work must be done during the rest of the motion, and this will apply to every pair of points that can be taken in the system. From this we see that the total work done by the internal forces, during a complete revolution of any part of the system about an axis, must be zero. Consequently, even if rotation were started by energy supplied from without, it would die out more or less rapidly, since in every real case of motion energy is gradually dissipated by friction, and, under the conditions supposed, no energy is generated within the system to replace that which is thus lost.

This would be the case of a system of magnets or of closed circuits of invariable form, which are equivalent to magnetic shells; but it is not the case if the circuits can be deformed, as when they comprise liquid conductors or sliding contacts. Faraday was the first to show that, in this case, a continuous rotatory motion can be produced. In order that these motions may be kept up, the forces must tend to increase the velocity at every revolution; in this case, the velocity is accelerated until the work done against friction in a given time is equal to that done by the forces tending to increase the velocity, and it is then constant. The energy spent in keeping up the motion, like that

spent in doing any other work by a current, is derived from the chemical energy of the battery.

271. Rotation of a Current by a Magnet.—A conductor carrying a current is movable about an axis which coincides with the axis of the magnet. If the ends of the conductor are on opposite sides of the same pole, as A and C (Fig. 227), the conductor turns with a continuous motion towards the left of an observer going head-first with the current, and looking in the direction of the lines of induction of the magnet.

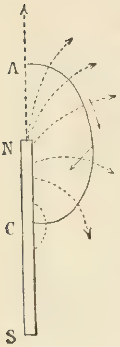


FIG. 227.

When the ends are in this position, each line of induction is only cut once by the conductor during one revolution. It would always be cut twice, once positively and once negatively, if both ends of the conductor were on the same side of one of the poles outside the magnet, or one outside one pole and the other outside the other, or if both were between the poles: the total flux cut would then be nothing, and there would be no motion.

The flux cut by the arc is a maximum if the end C is at the middle of the magnet; for a complete revolution it is then the total flux from one of the poles. If A be the intensity of magnetisation and a the cross sectional area of the magnet at its mid point, the lines of induction threading it there are equal to $(4\pi A + F)a$ where F is the polar field of the magnet at the same point. The lines all leave the magnet near its north end and return through the air to its south end. The conductor therefore cuts all of them once in a complete revolution, and the electromagnetic work corresponding to one revolution is $(4\pi A + F)aC$. The moment of the couple about the axis of rotation is constant and equal to $\dfrac{(4\pi A + F)aC}{2\pi}$; or neglecting F, as we may do for long magnets, it is equal to $2AaC$; it is thus independent of the size and shape of the conductor AC. Since Aa is the strength (S) of the magnet (**191**, i.) we may write this $2SC$.

FIG. 228.

This experiment may be very simply made by means of the apparatus represented in Fig. 228; it consists of a glass tube closed by two corks; through the lower one passes the end of a magnet about which mercury is poured so that it is just below the level of the pole; a copper wire is put through the upper cork and a platinum wire hangs from it with its lower end touching the mercury. As soon as a current

is passed, the platinum wire rotates continuously round the pole of the magnet, and the direction of the rotation changes with the direction of the current.

272. Rotation of a Magnet by a Current.—A cylindrical magnet is loaded by a platinum cylinder so that it floats upright in mercury, one of its poles projecting. The experiment may be made in two ways. In Fig. 229 the current enters near the edge, follows the surface of the mercury, and emerges by a rod fixed in the centre; the magnet moves round this rod as soon as a current passes. In Fig. 230 the part of the magnet which projects above the mercury conveys the current, the fixed rod dipping into a drop of mercury contained in a small hollow in the top of the magnet. The magnet rotates about its own axis. In both cases the projecting pole follows the lines of field of the current, and for a complete turn the work of the electro-magnetic forces is equal to $4\pi SC$.

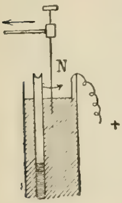


FIG. 229.

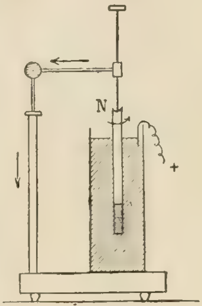


FIG. 230.

With a north pole, and with the direction of the current as shown by the arrows, the rotation is opposite to that of the hands of a watch. This and the previous experiment are, of course, examples of the equality of action and reaction.

It is by virtue of the fluidity of the mercury that the magnet is able to rotate; only one of the poles traverses the closed circuit. If both the circuit and the magnet were rigid, the whole magnet would have to pass through the circuit, and as the amounts of work done on the two poles respectively during a complete revolution would be equal and opposite, the movement could not be maintained. With a flexible magnet, the two poles of which could move independently, the positive pole would be seen to coil itself in one direction about the current, and the negative pole in the opposite direction, the motion being such as to render the lines threading the circuit a maximum.

273. Rotation of a Current by a Current.—This experiment is made by means of the apparatus represented in Fig. 231. The ends g and h of the movable branches fh and eg dip in a solution of copper sulphate in connection with the negative pole of a battery. The vessel, v, is surrounded by a coil of copper wire through which the current passes. The experiment is quite analogous to that of Faraday (**271**). The current, entering at x, passes

by a wire under the base of the apparatus, ascends by the column P, and descends in the two branches *fh* and *eg*. If the currents flow as shown by the arrows in the figure, the part *fh* will be urged forward by the mutual forces between it and the fixed horizontal current, and similarly *eg* will be urged backwards, so that the whole frame will rotate in the opposite direction to the hands of a watch with its face upwards. It may be said generally that a descending vertical current is urged *up-stream* by a horizontal current, and an ascending vertical current is urged *down-stream.*

274. Rotation of Liquids and Gases.—When a current traverses a liquid, the liquid filaments, which are the seat of the current, behave like movable currents under the action of electromagnetic forces.

Davy's Experiment.—The experiment may be made in several

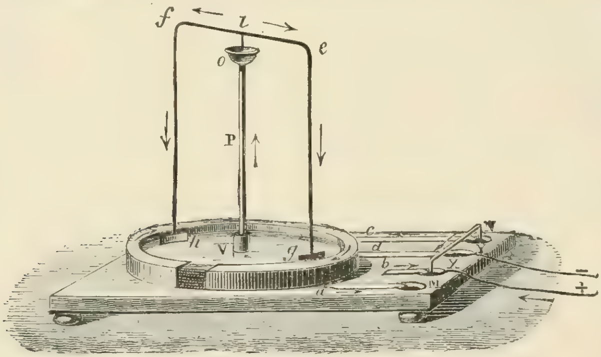


FIG. 231.

ways: the following is due to Davy. Two platinum wires insulated, except at the ends, pass through the bottom of a vessel full of mercury, and end just below the surface. When a current is passed, the mercury rises above each wire as though each wire were a source of liquid. If a north pole is placed just above one of the little heaps thus formed, the mercury is seen to rotate very rapidly in the direction of the hands of a watch in the case of the negative pole, where the current is towards the centre, and in the opposite direction when the magnet is over the positive pole, where the current flows from the centre.

Jamin's Experiment.—The two electrodes of a voltameter are placed in the same vertical line, coincident with the line joining the poles of a horse-shoe magnet placed horizontally. The two electrodes being between the magnetic poles, if they were joined by a rigid conductor rotation would be impossible (**271**). But as

each element of the conducting liquid is independently acted on, the liquid divides into two superposed layers rotating in opposite directions. The gas bubbles in the liquid make the rotation visible.

De la Rive's Experiment.—A rapid succession of electric discharges (from a Holtz machine or an induction coil) is caused to take place between two ring-shaped electrodes, F and A, enclosed in an exhausted vessel, V (Fig. 232), and an iron rod projects upwards through the electrodes, being enclosed by the glass tube G, which is sealed air-tight into the bottom of the vessel V. The iron rod can be magnetised by means of the coil E. If the upper electrode F is positive, and the upper end of the rod is a north pole, the luminous discharge B will be seen, by an observer looking down upon the apparatus, to rotate round the magnet in the direction of the hands of a watch with its face upwards.

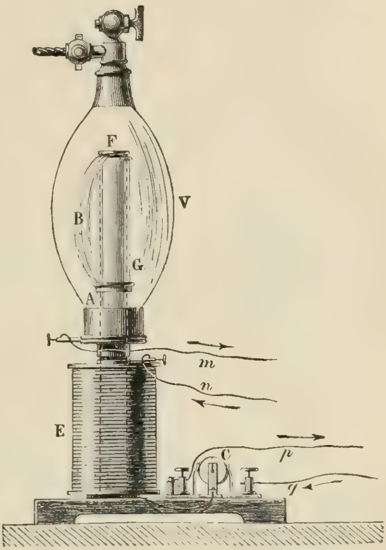


FIG. 232.

Action on the Electric Arc.—Analogous effects are produced on the electric arc formed between two carbon poles: if the pole of a magnet is brought near, the arc seems as if blown aside perpendicularly to the line which joins the pole to the arc.

275. Barlow's Wheel—Faraday's Disc.—*Barlow's Wheel* (Fig. 233) consists of a toothed metal wheel movable about a horizontal axis, so arranged that at the bottom one or more teeth dip in a trough of mercury, FD, on opposite sides of which are the two branches, AB, of a horse - shoe magnet.

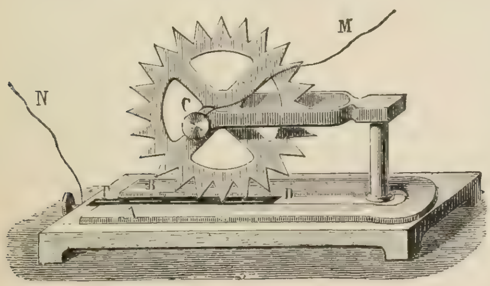


FIG. 233.

If the axis on the one hand, and the mercury on the other, are

connected with the terminals of a battery, the wheel begins to rotate. If the current goes from the centre to the mercury, and the north pole is nearest the observer, the lower part of the wheel moves towards the right.

Suppose the apparatus placed in a uniform field in air, and let H be the component of the field parallel to the axis. Let a be the radius of the wheel, and θ the angle of two consecutive teeth, and suppose the surface of the mercury is arranged so that one tooth touches the liquid the moment the preceding tooth quits it. The current goes from the axis to the tooth in contact with the mercury; the flux which it cuts, while one tooth comes into the position of the next one, is $\frac{1}{2}a^2\theta H$; and the corresponding electromagnetic work is $\frac{1}{2}a^2\theta HC$. For a complete turn the work is $\frac{1}{2}2\pi a^2HC$ or AHC, A being the total surface of the wheel.

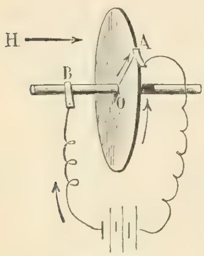


FIG. 234.

Faraday's Disc.—This apparatus (Fig. 234) works like Barlow's wheel; the current enters and leaves by two springs, one of which presses against the axis, and the other against the periphery of the wheel; the work for each turn is equal to ACH, and is independent of the velocity.

276. Rotation of a Current by the Action of the Earth.—This experiment may be made with the apparatus represented in Fig. 231, except that no current is passed through the coil. With the arrangement as shown in the figure, the circuit rotates, under the action of the earth, in the opposite direction to the hands of a watch. Resolve the earth's field into a vertical component Z, parallel to the axis of rotation, and a horizontal one H, perpendicular to this axis. The apparatus is astatic (that is, it is subject to no force or couple) so far as the component H is concerned, but the horizontal part of the current is acted on by the vertical magnetic field Z, in the same way as the current in Barlow's wheel or Faraday's disc is acted on by the component of magnetic field parallel to the axis. If $2a$ is the distance between the vertical branches where they dip into the liquid, the work done during one complete revolution of the apparatus is πa^2ZC, where C is the strength of the current in both branches taken together.

CHAPTER XXV

MAGNETISATION BY CURRENTS

277. Arago's Experiment.—As the field of a current is of the same nature as that of a magnet, it should produce the same magnetising effects. The fact of the magnetisation of iron by a current was discovered by Arago in 1820. He observed that a copper wire traversed by a current and dipped in iron filings attracts them, and is uniformly covered with them. Each particle of iron becomes a magnet, and sets at right angles to the wire along the line of field, the north pole being to the left of the current. The action exerted on the magnet so formed has a resultant directed towards the axis of the wire (**196**).

In like manner a bar is magnetised when it is set crosswise

FIG. 235.

FIG. 236.

with the current. The action is increased, as in the multiplier, by coiling the wire round and round in a plane perpendicular to the axis of the bar. The direction of the magnetisation is that of the lines of field in the interior of the coil. It is to be observed that this direction is independent of the direction of the winding, and only depends on the direction in which the current circulates about the axis. These two directions are related to each other in the same way as the lengthways movement of a right-hand screw is related to its motion of rotation. Several formulæ have been used to connect the position of the poles with the direction of the current. The simplest is that of Ampère: *the north pole is to the left of the current.* It is also often said that the north pole is at that end of the helix where the apparent direction of the current is opposite to that of the hands of a watch.

If the direction of the winding is abruptly changed by bending the wire back on itself, the direction of the rotation is changed, and therefore that of the magnetisation. The experiment is easily made by coiling a copper wire on a glass tube, and placing a knitting-needle inside the tube (Figs. 235 and 236). On passing a current through the wire, the needle is found to be strongly magnetised, and there are as many reversals of polarity, or *consequent poles*, as there are reversals of the current. These poles are alternately of opposite signs. Those of the ends are of the same kind, or of opposite kinds, according as the number of reversals is odd or even.

278. Electro-Magnets.—The magnetisation of soft iron by the current is of extreme importance in the applications of electricity. Not only can the magnets thus obtained be made much

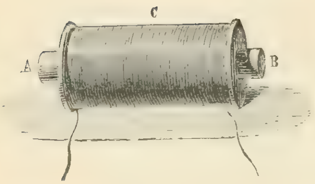


FIG. 237.

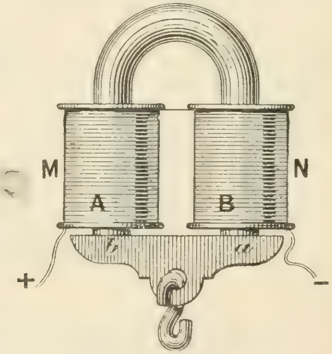


FIG. 238.

more powerful than steel magnets, but the property they possess, more especially if the iron is quite soft and the bars short, of being essentially temporary, of only existing during the passage of the current, of being created and destroyed, so to speak, with it, renders them suitable for a host of applications.

The temporary magnets thus obtained by the action of the current on soft iron are called *electro-magnets*. The conducting wire, insulated by silk or cotton, is wound in one or more layers round the core of soft iron.

The soft iron core may be straight, as in Fig. 237, or in the shape of a horse-shoe, as in Fig. 238. In the latter case the windings are not generally continued round the bend. In order that the poles at the two ends may be of opposite kinds, the wire should go round each branch in the same direction as it would have done if the core had been bent after the winding had been finished.

The windings ought to appear in opposite directions on the two legs to an observer who is looking at the two ends, the current going like the hands of a watch round the south pole, and in the opposite direction round the north pole (Fig. 239).

Owing mainly to the difficulty of bending massive iron bars without imparting to them a coercive force, large horse-shoe magnets are usually formed with parallel coils, the cores of which are connected by a cross-piece or *yoke*, T, of soft iron (Fig. 240).

279. General Observations on Electro-Magnets.—The direction of the field at a given point in the magnetic field of a coil, and its *relative* intensity as compared with that at any other given point in the field, are independent of the strength of the current, and depend only on the geometry of the coil—that is, upon its size and shape, and on the number of turns of wire; but the *absolute* intensity of the field is proportional to the strength of the current. If, however, a piece of soft iron is placed inside the

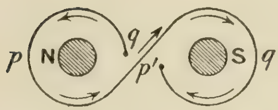


FIG. 239.

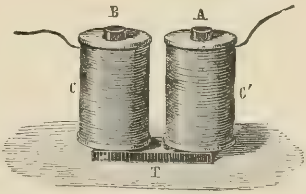


FIG. 240.

coil, its magnetisation varies with the strength of the current in a very complicated way, which it is not possible to include under any general law. In the following remarks, therefore, we shall confine ourselves to a few general considerations and the discussion of some special cases.

We may suppose the cross-section of the channel divided into any number of equal parts, and the whole mass of metal split into the same number of rings; whatever this number may be, the magnetic field is always the same, and so also is the production of heat. But this system of rings traversed by equal and parallel currents is equivalent to a coil of the same number of turns, which would give in the section of the channel the same mean density of current. From this the remarkable consequence follows, that whatever be the size of the wire coiled on the bobbin, for the same volume of wire, and therefore the same weight of copper, the magnetising field is the same, as well as the production of heat, provided the mean density of the current remains constant.

Let d be the diameter of the bare wire, $D = d(1 + \delta)$ that of the

covered wire, l the length of wire, ρ its specific resistance in ohms, and a the density of the current in amperes per square centimetre: we have

$$R=\frac{4\rho l}{\pi d^2} \quad \text{and } C=\tfrac{1}{4}\pi d^2 a,$$

and therefore for the energy spent per second in generating heat

$$W=RC^2=\tfrac{1}{4}\pi d^2 l\rho a^2.$$

As the space filled by the wire is

$$V=\tfrac{1}{4}\pi D^2 l=\tfrac{1}{4}\pi d^2(1+\delta)^2 l,$$

the heat produced per unit of this space is

$$w=\frac{W}{V}=\frac{\rho a^2}{(1+\delta)^2}.$$

Taking $\rho=1600\times10^{-9}$ and $\delta=0{\cdot}035$, a number found experimentally for the thickness of the double layer of cotton which serves as insulator, we get $w=1{\cdot}5\times10^{-6}a^2$.

It is to be observed that the production of heat in the coil is the same with or without the soft iron core. Although energy is expended in producing magnetisation, there is no expenditure of energy in maintaining it, any more than there is in keeping a spring in a fixed position after it has once been compressed. As all the useful expenditure reduces to the heating of the wire of the coil, the conditions of maximum work are satisfied when the resistance of the wire of the electro-magnet is equal to that of the rest of the circuit (**135**).

280. Case of a Long Coil.—The simplest case is that of a long cylindrical coil, uniformly wound, surrounding a thin cylindrical bar parallel to the axis (Fig. 241). The field inside the coil is uniform, and its value for a current of strength C, and a number of turns n_1 for unit length is (**266**)—

$$F=4\pi n_1 C.$$

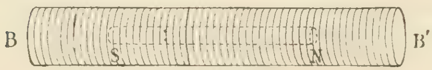


FIG. 241.

The magnetisation of the cylinder, if it is so long that the action of its polar field may be neglected, is also uniform, and the intensity of magnetisation is

$$A=\kappa F=4\pi n_1\kappa C.$$

The value of the induction (**197**) is

$$B=\mu F=4\pi n_1(1+4\pi\kappa)C,$$

and the flux or number of lines of induction for the entire section S of the bar is

$$N = BS.$$

For a given current, the magnetising field thus depends only on the number n_1 of turns per unit length, and not at all on their shape or size. In order to economise wire and reduce the expenditure of energy in generating heat, it is desirable to wind the wire directly on the core. The effect of unit length of the wire is then made as great as possible.

281. Ring Magnet.—If the core has the shape of a *tore* or anchor ring covered with equidistant windings in planes passing through the axis (Fig. 221), the system may be regarded as formed of concentric closed solenoids having no external action. The elementary solenoids are not, however, all of equal strength, seeing that those near the outside of the ring are longer than those near the inside circumference, although the current passes round all of them the same number of times.

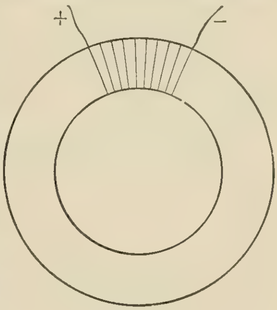


FIG. 242.

But if the section S is small compared with the radius of the ring, it may be assumed, without appreciable error, that the strength of each of the filaments is the same as that of the central filament; if n_1 is the number of turns of wire for unit length of the mean circumference, we have as the approximate value of the flux of induction through a cross-section of the ring, the strength of the current being C (comp. **266**)—

$$N = BS = \mu FS = \mu 4\pi n_1 CS.$$

Multiplying both sides by the length l of the mean circumference, and calling n the total number of turns of wire, we have

$$Nl = \mu 4\pi nCS;$$

or, again,

$$4\pi nC = \frac{1}{\mu}\frac{l}{S}N \quad . \quad . \quad . \quad (1).$$

The quantity N may, as we shall see (**315**), be determined by experiment.

282. If the core is covered for only a part of its length (Fig. 242), some lines of induction will escape at the ends of the coil, but generally in so small a number that we can assume the flux constant and still apply the formula (1).

In this case the magnetisation of the ring depends only on the product nC of the strength of the current into the number of turns of wire, and not in the manner in which these turns are arranged. This product is called the number of *ampere-turns*, when the current is expressed in amperes, in which case the second member of equation (1) should be multiplied by 10. Dividing by the length l of the circumference, we get the number of ampere-turns per centimetre of the ring. The same degree of magnetisation corresponds to the same number of ampere-turns per centimetre of length, whatever be the section and length of the ring. The two terms of the product may be varied at will; ten turns of a wire which conveys one ampere produces the same effect as a single turn which conveys ten amperes.

These consequences are not confined to the special case of an anchor ring; they apply to any closed magnetic circuit from which no lines of induction escape at the surface, and which may therefore be considered as identical with a tube of induction. They are not applicable to the case of a circuit which is not closed—a straight bar, for instance, which allows lines of induction to escape all over its surface.

283. Magnetic Resistance or Reluctance.—Formula (1) of 281,

$$4\pi nC = \frac{1}{\mu}\frac{l}{S}N,$$

may be compared with that of Ohm,

$$E = \rho\frac{l}{S}C.$$

Ohm's law, when expressed in the form of the last equation, may be taken as a statement that the difference of electric potentials, E, between the ends of a conductor is equal to the electric flux, or strength of current, multiplied by a factor, ρ, depending on the material of the conductor and by a geometrical factor l/S. The corresponding magnetic equation states that the difference of magnetic potentials $4\pi nC$ is equal to the magnetic flux N, multiplied by a factor characteristic of the material, $1/\mu$, and by the same geometrical factor as in the other case.

This parallelism of formulæ has led to the somewhat general use of similar terminology in reference to the two classes of phenomena. The *magnetic circuit* is compared with the electric circuit: as E in the case of the latter is called electro-motive force, so $4\pi nC$ in the former is called *magneto-motive force*, and, by

analogy with electric resistance, $\frac{l}{\mu S}$ is called *magnetic resistance*. It may be remarked in passing, that the last analogy implies the comparison of μ with the reciprocal of specific resistance: this would be expressed by calling it specific magnetic conductivity, an idea already embodied in the accepted name *permeability*.

With regard to these comparisons, it is to be observed that there is no known magnetic phenomenon which is physically analogous to electric conduction, whether in the form of disruptive discharge through a dielectric, or in that of conduction through metals or electrolytes. Physically, the analogy between magnetic permeability and specific inductive capacity is much closer than that between permeability and metallic conductivity, and the true electric analogue of a magnetised piece of iron is rather a polarised dielectric than a conductor carrying a current. Just as every terminated magnet has two equal opposite poles, and as new poles of the same strength are generated by dividing the magnet and interposing a stratum of air, so any portion of a polarised dielectric, bounded by conducting surfaces, has equal opposite charges on its two ends, and new charges of the same magnitude make their appearance whenever the dielectric is divided by an interposed conductor. If we were acquainted with liquid or gaseous bodies capable of assuming a high magnetic intensity, and thus comparable with liquid and gaseous dielectrics, the above analogy would be much more obvious and striking.

A further failure in the analogy between the electric circuit and the magnetic circuit arises from the fact that whereas the electric resistance of a conductor is independent of the strength of the current, the so-called magnetic resistance depends essentially upon the magnetic induction. It is, therefore, undesirable to use the same word in reference to both properties. In order to retain the advantage of the real mathematical parallelism between the laws of the electric and magnetic circuits, without implying a closer physical analogy than really exists, the term *reluctance* has been employed instead of resistance in the magnetic case, and we shall adopt it in what follows.

284. Calculations relating to the Magnetic Circuit.—The analogies pointed out above may be employed for the discussion of the conditions existing in a magnetic circuit, and they lead to results which are in close accord with experiment. In making such applications we have to treat the lines of magnetic induction as always forming closed curves, whether they traverse magnetic substances, or air, or other non-magnetic bodies. In circuits of

complicated shapes there are difficulties in the way of precise numerical calculation of exactly the same kind as those which would present themselves in the application of Ohm's law to similar cases; and if a circuit includes relatively large portions of non-magnetic material, further difficulties arise in determining the course of the lines of induction. Allowing for such considerations, we may calculate with magneto-motive forces and reluctances in exactly the same way as with electro-motive forces and resistances.

Kirchhoff's rules (**118**) apply to the case of divided circuits. Thus the fluxes N, N', N'' . . . of the branches which terminate at the same point satisfy the equation

$$N + N' + N'' + . \quad . \quad . \quad = 0 \quad . \quad . \quad . \quad (2).$$

In a closed contour the magneto-motive force is equal to the sum of the products obtained by multiplying the flux of each branch by the corresponding reluctance. Thus if n is the algebraic sum of the turns, C the strength of current, while l l' l'' . . . represent the lengths of the various segments, μ μ' μ'' . . . the permeabilities, S S' S'' . . . the sections, and N, N', N'' . . . the fluxes which traverse them—

$$4\pi nC = \frac{1}{\mu}\ \frac{l}{S}\,N + \frac{1}{\mu'}\ \frac{l'}{S'}\,N' + \frac{1}{\mu''}\ \frac{l''}{S''}\,N'' + . \quad . \quad . \quad (3).$$

285. Construction of Electro-Magnets.—The theory of the magnetic circuit accounts for all the facts presented by electro-magnets, and enables us to establish exact rules for their construction. It leads to the same conclusions as the consideration of the demagnetising actions of the ends. It shows why the force exerted by an electro-magnet upon its armature diminishes so rapidly when the distance between them increases. As the permeability of air is several hundred times less than that of iron, the interposition of a layer of air is equivalent to a great increase in the length of the core; the number of ampere-turns remaining the same, the induction is greatly decreased. Hence the necessity of having the surfaces in contact well worked and smooth; even then a joint always introduces an appreciable reluctance, and, other things being equal, an electro-magnet with a yoke (Fig. 240) is not so good as a bent electro-magnet (Fig. 238).

In planning an electro-magnet, it is generally assumed that a particular value of the induction B is to be obtained with the wire used. Too high a value must not be taken, since, beyond a certain limit, the increased expenditure is out of proportion to

the gain of magnetisation. In the case of an electro-magnet which is intended to support a weight and to act in closed circuit, the weight to be lifted is given. The polar surface is then determined by the equation (**237**)—

$$Pg = \frac{B^2 S}{8\pi},$$

and the number of *ampere-turns* per centimetre by the formula

$$\frac{4\pi n C_a}{10 l} = \frac{B}{\mu},$$

where C_a is the current in amperes.

The value of B being given, that of μ is also defined by the curve of magnetisation (**213**) for the iron in question.

The length, l, of the circuit of soft iron is arbitrary; it is economical to take it as small as possible, and give the electro-magnet a compact form like that in Fig. 243. There must be space enough to contain the wire necessary. The volume of the wire is determined by the condition of not exceeding a certain temperature. The thickness of the wire depends solely on the current to be used.

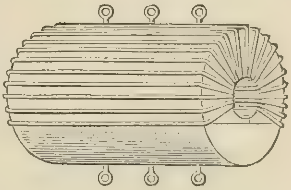


Fig. 243.

The rule to be followed is that the coil should have such a surface that for a given excess of temperature over the surrounding medium, the rate of loss of heat by radiation may be equal to the rate of production of heat by the current. Experiment shows that, with ordinary coils, the radiation per square centimetre per degree may be taken as about 0·001 watt. If we know (**279**) the heat produced per cubic centimetre, it is easy to calculate the necessary surface, and therefore the number of layers of wire for any value of the density of the current.

For a straight electro-magnet, the simplest course is to consider the core as an ellipsoid, and calculate the value of the coefficient P (**202**). Since κ is given, we know the ratio of the magnetising to the influencing field, and, multiplying this by $0{\cdot}4\pi$ times the number of ampere-turns, and by μ, we get the induction.

286. Fig. 244 represents a powerful electromagnet designed by Dubois. The core is an anchor-ring formed of a soft iron cylinder of 5 cm. radius bent so that its axis forms a nearly complete circle of radius 25 cm. The ring is cut through at s in a plane tangential to the innermost circle of the ring. This allows one limb (shown

to the right in the figure) to be moved slightly in or out, by means of a screw turned by the handle G, so as to vary the space Z between the pole faces, while a large surface of contact is maintained between the two parts of the ring. The poles are pierced at L_1 and L_2, for experiments on optical effects; K_1 and K_2 are accurately fitting iron cylinders by which the borings, L, can be plugged when not required. The magnetising coils are wound in twelve sections, each of 20°, marked 1, 2, . . . in the figure. Each section of the coil has 200 turns of wire, and, when all are in place, they cover 240° of the circular core. A brass stay, M_1DM_2, whose length can be adjusted by a screw, is used to prevent bending of the ring by the mutual attraction of the poles, which, when a strong magnetising

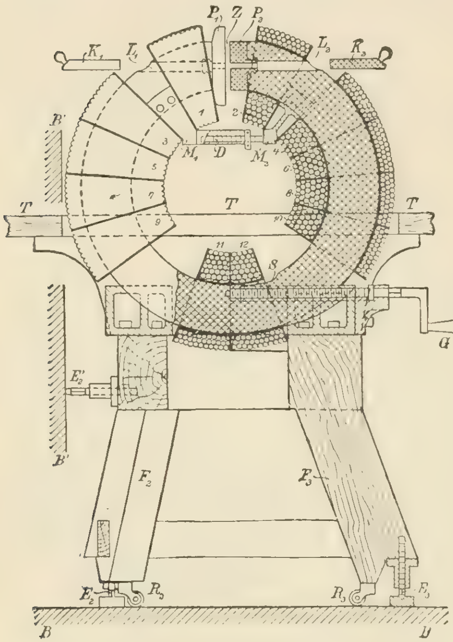


Fig. 244.

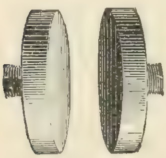


Fig. 245.

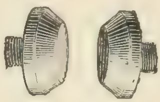


Fig. 246.

Fig. 247.

current is used, may exceed a weight of 1250 kilogr. End-pieces of various shapes (Figs. 245, 246, 247) can be adapted to the poles.

287. Polarised Electro-Magnets.—This term is given to magnets the cores of which are already magnetised by a permanent magnet (Fig. 248). This arrangement is used whenever very

sensitive electro-magnets are wanted, which rapidly obey feeble currents. The magnetisation curve (**206**) rises very slowly for weak currents; but if the soft iron is so arranged that it has a magnetisation nearly corresponding with the point of inflection, the least current produces a considerable variation in the magnetisation.

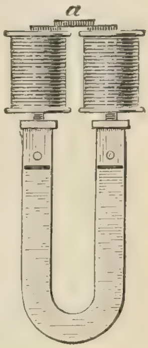


FIG. 248.

288. Transverse Magnetisation. — Suppose that an iron or steel wire is wound spirally on a glass tube, and that a current is passed through a copper wire in the axis of the tube; if C is the strength of the current, a the external radius of the tube, the magnetic field exerted by the infinitely long current on any element of the iron wire in the direction of its axis is equal to $\frac{2C}{a}$ (**266**). The iron wire becomes a solenoidal filament, and has no field at an external point; but if it is uncoiled, its ends have poles of opposite kinds.

The effect is the same on an iron wire traversed by a current. The iron is transversely magnetised, and may be regarded as formed of circular solenoidal filaments perpendicular to the axis. As all these solenoids are closed, there is no external field. But if the wire were sawn longitudinally along its axis, the surfaces produced would *each* present the properties of two polar strips of opposite sign, the axis being the dividing line between them.

289. Magnetisation of Steel.—Currents are ordinarily used instead of permanent magnets for magnetising steel bars. A ring is formed of several turns of stout wire, through which a strong current is passed. The ring is placed round the middle of the bar, and then when the circuit is closed, it is passed from one end of the bar to the other; after having been carried the same number of times over each half of the bar, the ring is brought to the middle again and the current is stopped.

With horse-shoe magnets a double ring is used coiled in the form 8; the limbs of the horse-shoe are placed one in each loop, and the ring is moved backwards and forwards several times along the limbs. The magnetisation thus obtained is both strong and regular.

290. Rotatory Magnetic Power.—Any substance, whether solid, liquid, or gaseous, when placed in a magnetic field acquires the property of rotating the plane of polarisation of a ray of light which traverses it. The effect is greatest when the direction of the ray is the same as that of the lines of field; it vanishes when

these two directions are at right angles to each other. The action is less with doubly refracting than with singly refracting substances. Faraday's heavy glass (borosilicate of lead) among solids, and bisulphide of carbon among liquids, show this effect in an especial degree.

The experiment is easily made with a Ruhmkorff's electromagnet (Fig. 249); the iron cores are hollow, and rays of light from a lamp, L, travel along the axis P′P; at P′ is the polariser, while P is the analyser. Suppose the light is homogeneous; the analyser being set so as to extinguish the light, the light reappears when the current is passed, and in order to extinguish it again

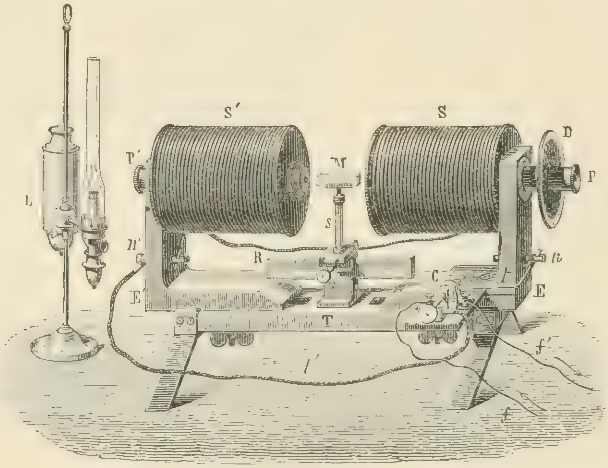


FIG. 249.

the analyser must be turned through an angle which can be read off from the limb D, and is a measure of the rotation. By reversing the current, an equal rotation is obtained in the opposite direction.

For a given substance the direction of the rotation is independent of the direction in which the rays travel; it is the same whether the propagation is in the direction of the lines of field or in the opposite direction. It follows from this that if the ray turns back again, and traverses the substance a second time in the opposite direction, the rotation is doubled. Thus, by increasing the length of the path of the ray by successive reflections (Fig. 250), the rotation can be increased in the same proportion.

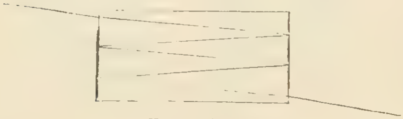


FIG. 250.

For most diamagnetic substances that have been examined, and for iron, nickel, cobalt, and the salts of nickel, cobalt, manganese, and copper the direction of the rotation is that of the current which produces the field. This direction we shall take as *positive.* For most magnetic substances, however, the rotation is negative.

291. Verdet's Law.—*The rotation of the plane of polarisation between two points is proportional to the difference of magnetic potential between these two points.*

If V and V' are the magnetic potentials at the two points in question taken on the path of the ray, the angle θ through which the plane of polarisation is turned is expressed by the equation $\theta = \omega(V - V')$, ω being the rotation which, for the body in question, corresponds with unit difference of potential. This quantity has been called *Verdet's constant;* it defines the rotatory power of the body. For the different rays of the spectrum it varies *nearly* as the inverse square of the wave length.

For the ray D and the temperature 0°, the value of the constant is 0′·043 for carbon bisulphide, and 0′·013 for water. It is less as the temperature is higher; for carbon bisulphide we have

$$\omega_t = \omega_0(1 - 0{\cdot}00104t - 0{\cdot}000014t^2).$$

For iron magnetised to saturation, Kundt found the rotation to be at the rate of one complete revolution for a thickness of 0·02 mm., an effect which is 290 million times as great as in bisulphide of carbon.

292. Optical Galvanometer.—The rotation of the plane of polarisation furnishes a convenient means of measuring the strength of a current in absolute value.

Let a length e of the substance be placed in a uniform field, for instance along the axis of a long coil with n_1 turns for unit length. The difference of the potential per centimetre is $F = 4\pi n_1 C$ (**266**), and if θ is the observed rotation we have $\theta = \omega 4\pi n_1 Ce$, an expression which only requires two measurements, those of e and θ.

Or, more simply still, let the substance (carbon bisulphide, for instance) be placed in a long tube closed at both ends by glass plates, and round the middle let there be a coil formed of n turns of any given shape and size (Fig. 251). If the tube is so long that the force due to the coil at the ends may be neglected, the difference of potential between the ends is $4\pi nC$, and the rotation is therefore $\theta = \omega 4\pi nC$.

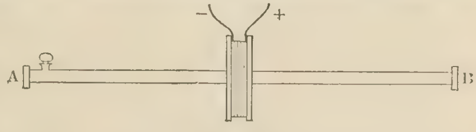


Fig. 251.

The method requires no measurement of the coil, nor accurate placing of it; all that is needed is the number of turns of wire.

The above formula assumes that the tube is infinitely long in both directions. It is easy to find the required correction: the coil remaining fixed, the tube is moved so that the end A comes into the place of the end B; the correction will be twice the rotation measured in this new position.

293. Hall's Phenomenon.—Suppose a very thin metal plate (Fig. 252) provided with four electrodes, A, B, a, b, the two first of which can be connected with a battery, and the two others with a galvanometer. When the current is made between A and B, the electrodes a and b can be moved about, and points found by trial such that no current passes in the galvanometer. These two points are then at the same potential.

When this adjustment has been made, if the plate is placed in a strong magnetic field at right angles to the lines of field, the deflection of the galvanometer shows that a portion of the current AB passes through the galvanometer. It is also found that an instantaneous discharge, other things being equal, divides in the same way and in the same proportion as a continuous current.

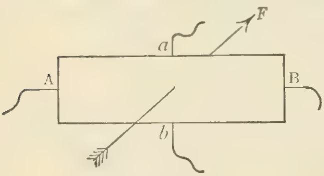


FIG. 252.

If the current is from A to B, the displacement is in the direction of the electromagnetic force ϕ, that is, from a through the galvanometer to b, in the case of iron, cobalt, and zinc; and in the contrary direction, that is to say, from b through the galvanometer to a, with gold, nickel, and bismuth. With platinum and lead there seems to be no effect.

Let C be the strength of the original current, c the strength of the current in the galvanometer, F the intensity of the field: if V_a and V_b are the potentials at the points a and b, and R the resistance between these two points, we have from Ohm's law—

$$E = V_a - V_b = cR.$$

Experiment shows that if e is the thickness of the plate, and G a constant, the law of the phenomenon is given by the formula

$$E = G\frac{CF}{e}.$$

The following are the values of the constant G in $C. G. S.$ units according to v. Ettingshausen:—

Bismuth	−	10·1
Nickel	−	·024
Gold	−	·00071
Zinc	+	·0004
Cobalt	+	·0046
Iron	+	·0113
Antimony	+	·192
Tellurium	+	530·

The sign + represents the cases in which the current is in the direction of the electromagnetic force.

When bismuth is heated its coefficient falls until at about 10 degrees from its melting point its value is only one-sixth of the value at ordinary temperatures; and at the melting point itself it falls to sensibly zero. For the magnetic metals there is a sudden fall in the value of the coefficient in the neighbourhood of their magnetic critical points.

The phenomenon is due to a modification of the lines of electric force under the influence of the magnetic field. In the absence of this field, if the plate is homogeneous, and has the same thickness throughout, the current at a small distance from the electrodes A and B is uniformly distributed; the lines of flow are parallel to AB, and the equipotential lines are perpendicular to them (Fig. 253).

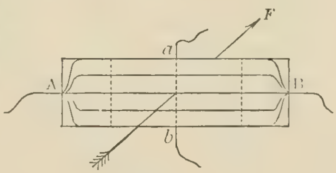


FIG. 253.

But when the magnetic field is set up, a transverse electric field is produced. The resultant field is obtained by superposing this on the original electric field which drives the current. All the equipotential lines are thus rotated through a small angle; this angle has the same value at all parts of the sheet at which the current flows in parallel lines; hence the equipotential lines no longer meet the edge at right angles. Since the current cannot have a component at right angles to the edge, its direction remains unchanged—at least in the absence of the electrodes a and b. When these are present and are connected together, the transverse electromotive force drives a current between them, and this is the current observed by Hall.

Under the same circumstances the resistance of bismuth increases. This increase of resistance may serve to measure the intensity of the magnetic field.

The proportionate change in resistance (i.e. dR/R) is given for various strengths of field F in the following table :—

F	dR/R	F	dR/R
2660	+·052	8300	+·338
3930	+·106	9380	+·395
5130	+·171	10530	+·460
6290	+·232		

Under the same circumstances antimony shows a small increase, nickel and iron a small decrease in resistance.

293.* Reflection of Light from Magnets.—In 1877 Kerr observed that when plane-polarised light is reflected from the pole of a magnet it becomes elliptically polarised, the longest axis of the ellipse making a small angle with the initial plane of polarisation. When the incidence is normal to the reflecting surface the light can be extinguished by an analyser suitably placed so long as the magnet is not excited. When, however, the magnet is excited the field grows bright; but the light can be again nearly extinguished by rotating the analyser. Kerr found that the direction in which the analyser must be turned is opposite to that of the current producing the magnetisation. Righi, repeating Kerr's experiments with more powerful magnets, succeeded in obtaining a rotation of about half a degree. A piece of gold leaf placed in close contact with the pole entirely eliminates the effect, thus showing that this must be due to a modification of the iron and not to a change in the properties of the air in the immediate neighbourhood of the pole. Nickel and cobalt produce a similar rotation in the same direction as with iron. Du Bois found for the ratio of the rotation in minutes of arc to the intensity of magnetisation the values −·0138 for iron, −·016 for nickel, and −·0198 for cobalt: the negative sign indicating that the rotation and magnetising current are opposite one to the other. For magnetite the corresponding ratio is +·012. The magnitude of the effect in all cases depends upon the colour of the light. For iron and cobalt Du Bois found an increased rotation with increased wave-length. Nickel shows a minimum, and magnetite a maximum, both in the yellow.

When the incident light is inclined to the normal the effects are more complicated. The explanations usually given of these phenomena require that a transverse electric force be called into play by the incident light; but no definite indication has yet been

given of the mechanism by which this electric force originates. A transverse electric force appears also as the cause of the Hall effect; but beyond this fact there does not seem to be any connection between the two phenomena. For example, the Hall effect is exceedingly large in the case of bismuth; but this metal does not exhibit even a trace of the effects we have just described.

CHAPTER XXVI

ELECTROMAGNETIC INDUCTION

294. Induction Currents.—Whenever the flux of magnetic induction through a closed circuit is changed, the circuit becomes the seat of a temporary current which lasts as long as the variation of the flux. Such a current is called an *induction current* or *induced current.* Induction currents were discovered by Faraday in 1831, and we shall here describe some of his fundamental experiments.

295. Induction by Currents.—A closed circuit, AB (Fig. 254), contains a galvanometer, G; a second circuit, parallel to the first for a great part of its length, is connected with a battery, and can be closed or broken, or the current can be reversed at pleasure. The former will be called the *induced* or *secondary* circuit, and the latter the *inducing* or *primary* circuit.

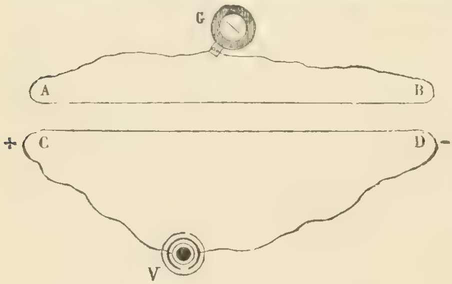


FIG. 254.

Every time the battery circuit is closed, the induced circuit AB is traversed by a momentary current in a contrary or *inverse* direction to that in the parallel portion CD of the inducing circuit. Every time the inducing circuit is broken, the secondary is traversed by a momentary current in the same direction as the inducing current, that is, by a *direct* current.

Nothing is observable in the induced circuit so long as a steady current passes through the inducing circuit, and so long as it remains at the same distance. But when the wire CD is brought near AB, or when the current in CD is increased, an *inverse* induced current is produced in AB; while if the wire CD is moved away or the strength of its current is decreased, a *direct* induced current is produced.

In short, a current which begins to flow, or increases in strength, or is brought nearer, produces in an adjacent circuit an *inverse* induced current; a current which ceases, or diminishes in strength, or is moved away, produces a *direct* induced current. These currents, which are produced by the influence of other currents, were spoken of by Faraday as currents due to *volta-electric* induction.

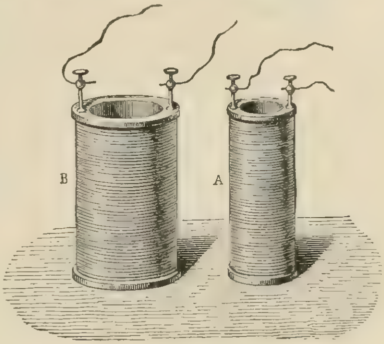


FIG. 255.

The effects are stronger the nearer are the two wires, and the greater the length of the parallel portions. The experiment is ordinarily made with two coils of insulated wire, one of which (Fig. 255) can be placed inside the other, like A and B.

296. Induction by Magnets.—Suppose the coil B (Fig. 255) to be connected with a galvanometer: if a bar magnet is inserted in it, the effects are the same as though a second coil, A, carrying a current, had been put in in place of the magnet. In order to find the direction of the induced current, we may suppose the magnet replaced by the equivalent electromagnetic cylinder (**266**). In like manner any increase in the strength of the magnet produces an inverse current, and any diminution a direct current. This action Faraday called *magneto-electric induction.*

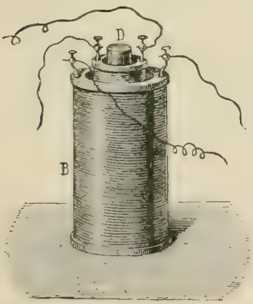


FIG. 256.

Both effects are obtained *simultaneously* and with far greater strength by placing a *core* of soft iron, D, in the inducing coil (Fig. 256). When the current is made, the cylinder of soft iron is magnetised, and the two actions, of the coil and of the magnet, which are evidently in the same direction, are added together. They are also added when the current is broken.

297. Induction by the Action of the Earth.—If a closed circuit be displaced or deformed in the magnetic field of the earth so that the number of lines of induction passing through it is altered, an induced current is produced, the direction of the current depending upon whether the change of the number of lines of

induction through the circuit is positive or negative. Thus if a coil, held with its axis in the line of dip, and connected by long wires with a galvanometer, is quickly turned through an angle of 180° about a line perpendicular to the axis, the galvanometer needle is deflected.

298. Self-Induction — Extra Current. — Lastly, Faraday showed that any variation in the strength of a current produces in the circuit itself, in which the current passes, an induction current, which is superposed on the principal current, and always opposes the actual variation of strength, thus lessening the rate of rise of an increasing current and the rate of fall of one that is decreasing. This phenomenon is called the *induction of a current on itself*, or, more briefly, *self-induction*, and the resulting current is called an *extra current*. The effect is especially marked in circuits which contain coils or electro-magnets. In this case the spark, on breaking the circuit, is much stronger and louder than the spark on closing the circuit, and if the body is interposed in the circuit, the extra current may produce strong physiological effects. Faraday used the following method for showing the extra current produced on making and breaking the circuit:—

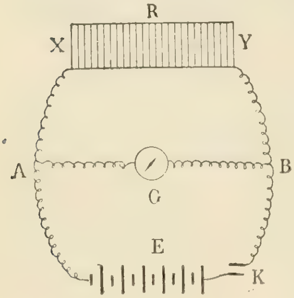


FIG. 257.

A coil, R (Fig. 257), is connected with a battery, E, and a key, K, by which the circuit can be closed or opened, and a galvanometer, G, is connected in multiple arc with the coil, so that the battery-current divides between it and the coil. Let α be the position of equilibrium of the needle under the influence of the current which passes through the galvanometer when the circuit is closed at K, and a stationary condition is established. The needle is kept at the angle α by means of a stop (Fig. 258), which prevents it from going back to zero when the circuit is interrupted. On again closing the circuit, the needle is deflected beyond α, because the "extra current" delays the growth of the current in R, and at first more than the normal proportion flows along AGB.

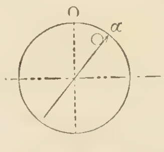


FIG. 258.

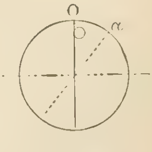


FIG. 259.

Next, the stop is placed at O (Fig. 259), so as to keep the needle at 0°, while a steady current is again made to traverse the circuit; when the circuit is now broken at K, the needle is

deflected in the opposite direction, because the current in the coil continues for an instant to flow after the battery-circuit has been broken. If the original directions were AXYB and AGB, the current continuing along XY, after K is open, causes a current through the galvanometer from B to A (**303**).

299. Characteristics of Induction Currents. — All the phenomena of induction have the common characteristic that they correspond to a modification of the magnetic field enclosed by the induced circuit, whether this field be due to currents or to magnets, or to a current in the circuit itself. It remains to establish the numerical laws of these currents.

We will first consider currents due to a displacement of the circuit.

Almost immediately after the discovery of induced currents, Lenz gave a very simple rule for their direction, which is this: *the direction of an induced current is always such that by its electromagnetic action it tends to oppose the displacement.*

In investigating the conditions which determine the *strength* of an induction current, we may remark, in the first place, that the existence of such a current implies the existence of an induced *electromotive force*, on which the strength of the current depends in the way expressed by Ohm's law. All that is necessary here, therefore, is to examine the conditions determining the magnitude of the electromotive force.

Going back to (**134**), we know that, when work is done at the expense of the energy of an electric current, an inverse electromotive force is generated, the magnitude of which is expressed by the equation

$$e = \frac{W}{C},$$

where W is the work done in unit of time as the result of a current of strength C. If W is measured in watts, and C in amperes, e is given by this equation in volts.

We have seen further (**267**), that when a conductor carrying a current moves in a magnetic field so as to cut lines of induction, work is done equal to the product of the strength of the current into the number of lines of induction cut through; or, more generally, that when the flux of magnetic induction through the area of a circuit alters, work is done equal to the product of the strength of the current into the increment of the magnetic flux.

Apply these considerations to a case such as that represented

in Fig. 260, where AB and A′B′ are parallel metal rails connected by a cross-bar CC′, which can slide along them at right angles to its own length, and P is a battery by which a current can be sent through the rails and cross-bar. Suppose the plane of the rails and bar to be perpendicular to the direction of a magnetic field of intensity H; let the direction of the current be from A to C and C′ to A′ in the figure, and the direction of the magnetic field to be as indicated by the arrow H. This is also the direction of the magnetic induction. Then the electromagnetic forces will tend to enlarge the area of the figure, so as to make it enclose a greater number of lines of induction, and the sliding bar will move towards BB′. Let ϕ be the force tending to displace the bar: while it moves through a small distance ds, the work done will be $dW = \phi ds$. But (**267**) we have also $dW = CdN$, if C is the strength of current and dN the increment of magnetic flux through the circuit caused by the displacement. Repeating the reasoning of (**134**), we may write the following equation, which states that the energy received by the circuit in an element of time, dt, in consequence of the electromotive force E of the battery, is equal to the energy expended in generating heat and in doing work:

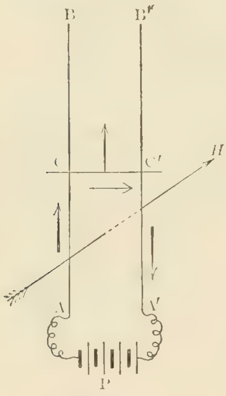


FIG. 260.

$$ECdt = C^2Rdt + CdN.$$

If the sliding bar were not allowed to move, that is, if no work were done, the strength of the current would be C_o, and the above equation would become

$$EC_odt = C_o^2Rdt.$$

It is evident that C is not the same as C_o, and, as the resistance is the same in both cases, a change in the strength of the current can be due only to a change in the effective electromotive force of the circuit—in other words, to the generation of an additional electromotive force e. The effective electromotive force is the algebraic sum of the electromotive forces of the circuit, and by Ohm's law we may write

$$E + e = CR.$$

But if we divide the first of the above equations by C and by dt, we get

$$E - \frac{dN}{dt} = CR,$$

and therefore

$$e = -\frac{dN}{dt} = -Hl\frac{ds}{dt} = -Hlv^*$$

if the medium surrounding the wire is of unit permeability (*e.g.* air). If the medium surrounding the wire is of permeability μ, the above expression must be multiplied by μ.

It is to be noted that the induced electromotive force e is independent of the strength of the current, and therefore has the same value even if there is no other electromotive force, and therefore no current except that due to induction. It depends only on the rate of increase of the magnetic flux through the circuit. This, in the case supposed, is equal to the strength of the magnetic field H multiplied by the distance l between the rails and by the velocity, $v = \frac{ds}{dt}$, with which the bar moves.

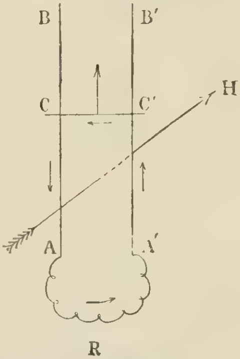


FIG. 261.

The negative sign in the expression for e receives its interpretation from Lenz's law, of which, in fact, it is the expression. It indicates that the electromotive force, due to an increase of the magnetic flux in a given direction, would, if it acted alone in a circuit, produce a current causing a magnetic flux in the opposite direction. Thus if, in the case already considered, the battery is replaced by a conducting wire R (Fig. 261), motion of the cross-bar from A towards B will produce a current in the direction of the arrows, that is, opposite to the battery current in the previous case. The current is inverted if the motion is from B towards A. These effects can be rendered visible by including a galvanometer between A and A′.

300. General Law of Induction—Electromotive Force.— The general law of induction is only a generalisation of the foregoing. Whether the circuit moves across the lines of field, or whether the lines of field move in reference to the circuit, whether or not the electromagnetic forces do work, that is to say, whether there is or is not a displacement of their points of application, each element of the circuit which in the time dt cuts a number of lines of induction dN, is the seat of an electromotive force,

$$de = -\frac{dN}{dt},$$

* We are here neglecting the effect due to change of self-induction of the circuit (301). In strictness, the value of $\frac{dN}{dt}$ is $Hl\frac{ds}{dt} + \frac{d(LC)}{ds} \cdot \frac{ds}{dt}$.

whose direction is from foot to head of an observer who, looking along the lines of induction, sees them pass from right to left.

All these electromotive forces add up algebraically, like those of a number of galvanic cells connected in the same circuit.

If the conductor does not form a closed circuit, the two electricities are accumulated at the opposite ends until a difference of potential is produced equal to the algebraic sum of the elementary electromotive forces.

If the circuit is closed, the total electromotive force is likewise the algebraic sum of the electromotive forces corresponding to the different elements: and since the algebraic sum of the lines of induction cut by all the elements is equal to the variation dN of the product of the flux of induction enclosed into the number of times the circuit surrounds it, the electromotive force at the time t is

$$e = -\frac{dN}{dt}.$$

If R is the resistance of the circuit, this electromotive force produces at the moment in question a current of strength

$$C = \frac{e}{R} = -\frac{1}{R}\frac{dN}{dt}.$$

The direction of the current is such that its axis is in the opposite direction to the flux if dN is positive, and in the same direction if dN is negative.

301. Coefficient of Self-Induction, or Self-Inductance.—It has been shown (**252** *et seq.*) that the existence of an electric current in a conductor implies the existence of magnetic field in the neighbouring space. Indeed, in order to ascertain the existence, direction, and strength of a current, we usually examine, not the conductor itself, but the magnetic properties of the space near it. Whatever the ultimate nature of the phenomenon which we recognise and speak of as an electric current may be, it is an incomplete view of it to confine our attention to the process going on in the conducting circuit; it involves also, as an essential part, the existence of the correlative magnetic field.

Every complete circuit forms a closed curve, and the corresponding lines of magnetic induction, which also form closed curves, are linked through it (see, for example, Figs. 219, 220). Hence, irrespective of any other magnet or circuit, there is, through the area of every circuit carrying a current, a flux of magnetic induction depending on the current in the circuit. As already mentioned (**252**), the magnetic field due to a current remains similar to itself whatever the strength of the current, and

so also does the induction, but the magnitude of both at any point of the field is proportional to the current. Hence the total flux through the circuit is proportional to the current, and may be represented by

$$N = LC,$$

where L is a factor depending on the geometrical characters of the circuit,—its shape, the area enclosed, and the number of times the current goes round it,—and on the magnetic permeability of the surrounding medium.

The coefficient L is called the *coefficient of self-induction*, or the *self-inductance* of the circuit. It may be defined as the magnetic flux which, in consequence of a current of unit strength in the circuit, is linked with the circuit. It is a quantity of the same kind as the mutual inductance (**267**) of two circuits.

302. Calculation of Self-Inductance.—To calculate the value of this constant for any circuit requires a knowledge of the distribution of magnetic induction over the area of the circuit; and, in general, to ascertain this requires higher mathematical treatment than falls within the range of this book. In a few cases, however, the calculation is easy, and we will give three such as illustrations.

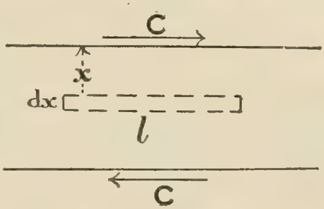


Fig. 262.

(i.) *Two thin parallel wires of infinite length* (Fig. 262). Radii, a, distance between axes $= d$.

The magnetic induction at any point is the algebraic sum of the values due to the two wires. Hence at a distance x from one wire it is

$$B = \frac{2C\mu}{x} + \frac{2C\mu}{d-x}.$$

The total induction passing through an elementary area of length l and width dx is $Bldx$. Hence the self-inductance, L, of a length, l, of the circuit is

$$L = \int_a^{d-a} \frac{Bldx}{C} = \mu \int_a^{d-a} \left(\frac{2}{x} + \frac{2}{d-x}\right) ldx$$

$$= 2l\mu\left[\log_e \frac{d-a}{a} - \log_e \frac{a}{d-a}\right]$$

$$= 4l\mu \log_e \frac{d-a}{a}.$$

We have here only taken account of the induction in the space between the wires: there is induction also in the wires themselves, and this must be taken into account in a rigorous solution. If the wires are thin compared with their distance apart, the error made in leaving it out is small.

(ii.) *A concentric cable.* Radius of inner conductor, a, internal radius of outer conductor, b.

The magnetic induction at all points at distance r from the axis is $\frac{2\mu C}{r}$.

Total induction enclosed by a length, l, of the conductors is

$$\int_a^b \frac{2\mu C}{r} l\,dr = 2\mu C l \log_e \frac{b}{a}$$

Hence the self-inductance of unit length is

$$L = 2\mu \log_e \frac{b}{a}.$$

Referring to (**53**), it will be seen that the *capacity* of unit length of such a cable is

$$K \Big/ 2 \log_e \frac{b}{a}.$$

Hence in this case capacity × self-inductance $= \mu K$. This result can be shown to be general. If both are expressed in electromagnetic measure, K has not the same value as when the capacity is expressed in electrostatic measure. This subject is treated fully in a later chapter (Chapter XXXI.).

(iii.) *An anchor ring.* In (**281**) it was shown that the total induction through such a ring is approximately $4\pi n_1 \mu S C$; since this passes through each of the turns of wire, and there are $n_1 l$ turns in all, l being the mean circumference, the self-inductance is

$$4\pi n_1^2 l \mu S.$$

This is also the self-inductance of a length, l, of an electromagnetic cylinder (**280**), except for portions near its ends.

303. Electromotive Force due to Self-Induction.—If we connect what has been said in the last paragraph with the result arrived at in (**299**) as to induced electromotive force, we see that the self-induction of a circuit must give rise, in the case of any

variation of the current, to an electromotive force tending to oppose the variation. In fact, the equations

$$e = -\frac{dN}{dt} \text{ and } N = LC$$

give, when L is constant,

$$e = -\frac{d(LC)}{dt} = -L\frac{dC}{dt}.$$

Hence when $\frac{dC}{dt}$ is positive, that is, when the current is increasing, there is a negative electromotive force due to self-induction; and when $\frac{dC}{dt}$ is negative, or the current is decreasing, the electromotive force of self-induction is positive. Whenever the strength of the current is constant, $\frac{dC}{dt} = 0$, and the electromotive force of self-induction is nothing.

These results contain the proper interpretation of the phenomenon of the "extra-current" (**298**).

304. Work Spent in Establishing a Current—Intrinsic Energy of a Circuit.—If there were no such thing as self-induction, an electromotive force, E, applied to a circuit of resistance R, would instantly give rise to a current of strength $C = E/R$. The electromotive force of self-induction, however, causes the establishment of a current to be a more or less gradual process. It is true that in many cases it is so rapid as to need delicate observation to detect that it is not instantaneous, but in cases where the self-inductance has a high value, the growth of the current occupies a very appreciable time. This point is discussed more particularly in (**318**).

During the variable period at the commencement of a current, more energy is given out by the battery than appears in the conductor in the form of heat or otherwise. The excess is consequently stored up in the field, and goes to increase the energy of the circuit. Take again the equation

$$ECdt = C^2Rdt + CdN,$$

already employed in (**299**). In the paragraph referred to, dN was an increment of magnetic flux through the circuit due to a deformation of the circuit in an independent magnetic field; in the present case, we take it to stand for an increment of the magnetic flux due to an increase in the strength of the current in a circuit of constant size and shape. Expressing the total flux by

means of the coefficient of self-induction, we have $N = LC$, and therefore $dN = LdC$. Hence we may write

$$ECdt - C^2Rdt = LCdC.$$

In this expression, $LCdC$ clearly appears as the excess of the energy given out by the battery over that converted into heat (comp. **322**).

If we put W for the total energy stored up while the strength of the current changes from an initial value, C_o, to a final value, C, we have

$$W = \int_{C_0}^{C} LCdC = \tfrac{1}{2}L(C^2 - C_o^2).$$

If the initial value $C_o = 0$, this gives

$$W = \tfrac{1}{2}LC^2$$

as the value of the work expended in establishing the current, or that of the intrinsic energy of the circuit.

305. Analogy between Magnetic Energy and Electrostatic Energy.—Using, as before, N for the magnetic flux, we have the three equivalent expressions—

$$W = \tfrac{1}{2}LC^2 = \tfrac{1}{2}NC = \tfrac{1}{2}\frac{N^2}{L},$$

which may be compared with the three forms in which the energy of an electrostatic field (**55**) may be expressed, namely,

$$W = \tfrac{1}{2}C(V - V')^2 = \tfrac{1}{2}Q(V - V') = \tfrac{1}{2}\frac{Q^2}{C}.$$

It is to be remembered that, in the latter case, C stands for electrostatic capacity.

The analogy between the energy of a conducting circuit and that of an electrostatic field goes beyond the mere parallelism of the formulæ, for in both cases the surrounding medium plays an important part.

The seat of the magnetic energy of an electric current is the medium surrounding the current, including that portion of the wire which surrounds the particular element of current which is under investigation. In other words, the seat of the magnetic energy is the whole of the region in which the corresponding magnetic field exists. In the case of electrostatic energy the conducting boundaries of the field affect the energy only by virtue of their geometric characters, while the dielectric medium affects it by virtue of its specific inductive capacity. The difference in the electromagnetic case arises from the distribution of the current

throughout the whole cross-section of the conductor: the magnetic energy for a given current depends in part upon the permeability of the surrounding dielectric and in part upon the permeability of the conductor itself.

In either case. the distribution of energy in the field might be mapped or represented graphically, if we suppose drawn the tubes of induction and equipotential surfaces. These, by their mutual intersections, divide the field into a number of compartments or cells, each of which is the seat of the same quantity of energy (**66**). As the electrostatic charge, in the one case, or the strength of the current in the other, increases, we must imagine more energy flowing out from the conducting surfaces into the medium, and the number of cells increasing. They become smaller and more closely crowded together; that is to say, more energy is accumulated in a given space. As the charge, or the strength of the current, diminishes, energy returns from the medium into the conductors, the cells expanding and decreasing in number till all are gone.

If f is the intensity of electrostatic force at any part of a field, and K the dielectric coefficient, the energy in a portion of the field, whose volume, dv, is so small that the force may be taken as constant throughout, is

$$\frac{f^2K}{8\pi}dv.$$

The corresponding expression in the magnetic case is

$$\frac{F^2\mu}{8\pi}dv,$$

where F is the intensity of magnetic field, and μ the magnetic permeability of the medium (comp. **350**).

306. Transfer of Electric Energy.—In the electromagnetic case both these kinds of energy exist together. For, besides the existence of a magnetic field, another necessary condition of the production of a current by a galvanic battery, or any equivalent arrangement, is a difference of potentials between different parts of the circuit, and this difference of potentials implies the existence of electrostatic energy, which might be represented by tubes of electric induction and equipotential surfaces. If the current is of constant strength, the potential remains constant at each point of the circuit and of the field, and consequently the equipotential surfaces are stationary, but each of them intersects the surface of the conductor at the part where its potential has the corresponding value. The tubes of electric force, which are necessarily ortho-

gonal to the equipotential surfaces, consequently meet the surface of the conductor obliquely; they cannot, therefore (**33**) be in equilibrium. Their ends run along the conductor towards the side on which they meet the surface at an acute angle, and at the same time they, as it were, sweep the energy into the conductor, where it appears as heat. The result would be the almost instantaneous disappearance of all energy, were it not for the battery, which pours energy into the field as fast as it is absorbed into the conductor. Part of the process going on in a closed circuit is thus a flow of energy from the battery across the field into the conducting wire: this corresponds to the generation of heat at the rate C^2R.

This flow or transfer of electric energy across the field is accompanied by the existence of magnetic field, and therefore of magnetic energy in the field. The lines of magnetic field coincide with the lines in which the surfaces of equal electrostatic potential and of electrostatic force intersect each other. So long as the flow of electric energy remains constant, the distribution of magnetic field and energy remains stationary. A constant electromagnetic field thus accompanies and requires a constant rate of transfer and dissipation of electric energy. For a given form and size of circuit, the difference between what is called a good and a bad conductor may, from our present point of view, be stated thus: with a good conductor an electromagnetic field of given intensity can be maintained with a smaller rate of dissipation of electric energy than with a bad conductor.

The lines of electric force and equipotential lines are shown in Fig. 263 for the case of a circuit consisting of a broad and long strip of conductor bent twice at right angles, the distant ends being supposed connected to a battery. The magnetic lines are of course perpendicular to the plane of the paper. It should be noted that the lines of electric force become more crowded in regions more remote from the bends. That this must be the case may be seen by considering that the difference of potential between opposite points of the strip is greater for more remote points, and since the distance between the strips is constant, the *average* force must increase proportionally.

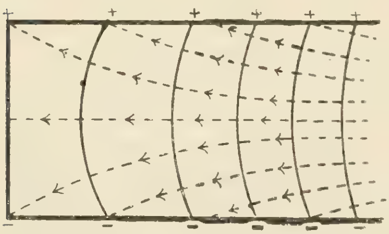


FIG. 263.

Each part of the surface of the conductor is in general charged.

For inside the conductor the electric force is wholly parallel to the boundary, while outside it this is not the case. Thus there is an abrupt change in the value of the normal component of electric induction, and this corresponds to the charge upon the surface (**44**, **50**). If the mid-point of the cross-conductor is kept at potential zero by being earth-connected, the charges are symmetrically arranged positive on one half and negative on the other, as shown in the figure, and the surface-density increases numerically *pari passu* with the electric induction.

307. Energy of Two Circuits.—If two circuits, each carrying a current, are in the presence of each other, the total energy is the sum of the intrinsic energies of the two circuits and of the energy due to their co-existence. If C and C' are the strengths of the two currents, L and L', the two coefficients of self-induction, and M the coefficient of mutual induction, the whole energy is

$$W = \tfrac{1}{2}LC^2 + MCC' + \tfrac{1}{2}L'C'^2.$$

308. Quantity of Electricity.—The quantity dQ, of electricity put in motion in the circuit during the time, dt, by an induced current is

$$dQ = Cdt = -\frac{dN}{R}.$$

The total quantity, Q, corresponding to a finite variation of flux, is the sum of all such expressions. If we consider a variation in which both the initial and final values of the strength of the current are zero, the variations of the flux due to the current itself give a sum zero, and any resultant change of flux is due to the external field. If this be called $N_2 - N_1$, we have

$$Q = \frac{N_1 - N_2}{R},$$

consequently *the total quantity of electricity put in motion by an induced current, the limiting values of which are zero, is equal to the quotient of the total variation of magnetic flux through the circuit by the resistance of the circuit.*

It depends neither on the time the variation has occupied nor on the manner in which it has taken place. The induced flow, Q, is to be reckoned positive or negative in accordance with what has been already said in (**300**).

CHAPTER XXVII

SPECIAL CASES OF INDUCTION

309. Applications.—We proceed now to discuss the application of the general principles pointed out in the last chapter to some important special cases.

Take first the case in which, passing from a stationary condition where the flux through a circuit is N_1 to another stationary condition in which the flux is N_2, we wish to determine the quantity of electricity put in motion. The result of the last paragraph applies at once. If the variation takes place very quickly, the quantity, Q, of electricity traverses the circuit in a momentary current or a sudden rush, and may be easily measured (**372**).

A second equally simple case is that in which the variation of flux takes place uniformly. The circuit is the seat of a constant electromotive force, the value of which is given directly by the formula arrived at in (**300**), namely, $e = -\frac{dN}{dt}$, where $\frac{dN}{dt}$ is the rate of increase of flux.

A more complicated problem is that of determining the strength of the current at each instant during the variable period. This is always given by the equation $ECdt = C^2Rdt + CdN$, but the solution of problems of this kind often involves mathematical difficulties that we cannot attempt to deal with. We must, therefore, in some cases only give the results.

310. Momentary Current due to the Displacement of a Circuit—Movable Circuit in a Uniform Field.—Consider a closed circuit of any given shape; let A be its effective area, which we will suppose is plane. If the circuit is in a uniform field of induction, B, and its plane, which at first is perpendicular to the field, is turned half round, the total variation of flux is $-2AB$, and if R is the total resistance of the circuit, the quantity of electricity put in motion is

$$Q = \frac{2AB}{R}.$$

Suppose the case of the earth's field and the medium to be air. If the rotation is about a *vertical* axis, the horizontal component, H, alone comes into play. The circuit, at first perpendicular to the meridian, being turned half round, we have

$$Q_1 = \frac{2AH}{R}.$$

If the rotation is about a *horizontal* axis, the plane at first being horizontal, we need only consider the vertical component, Z, and we have

$$Q_2 = \frac{2AZ}{R}.$$

Combining these two equations we get (240)

$$\frac{Q_2}{Q_1} = \frac{Z}{H} = \tan I.$$

This method was used by Weber to determine the magnetic dip. The apparatus (Fig. 264) employed for this purpose is known as *Weber's Inclinometer*.

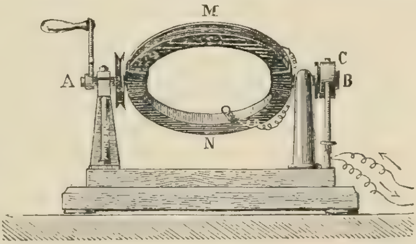


FIG. 264.

311. Measurement of a Magnetic Field.—By means of the same method, and using a very small coil movable about an axis in its plane, the intensity of a variable field in air may be measured. The plane of the coil, at first perpendicular to the direction of the field, is suddenly turned through 180°, and the resulting current is measured by the angle of throw of the galvanometer (372). Thus, if F is the intensity of the field, and R the total resistance

$$F = \frac{RQ}{2A}.$$

A measurement of the surface of the coil may be dispensed with by observing the effect of suddenly reversing the coil in a field of known intensity, that of the earth, for example.

312. Measurement of the Normal Component at the Surface of a Magnet.—Suppose that a small coil is applied to the surface of a magnet, as a proof plane might be, and that it is then turned half round, that is, through an angle of 180°. The

current will give twice the flux which traverses the coil, and dividing by the effective surface, we should have the mean value of the normal component for the portion of the surface covered by the coil, and this, as we have seen, is equal to the normal component of *induction* just inside.

Another method may be adopted. Suppose that the bar is cylindrical (Fig. 265), and is surrounded by a ring connected with a galvanometer. If the ring is moved from M to M′, the deflection of the galvanometer measures the flux cut during the displacement. This flux divided by the area of the cylindrical surface comprised between M and M′ gives the mean value of the component of magnetic field normal to the surface of the magnet.

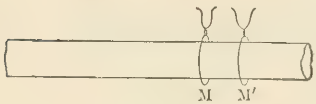


Fig. 265.

If the ring were pulled off the bar and were carried to an infinite distance, the current would measure the total flux from the part of the bar beyond the point M. If the ring fits closely round the cylinder, this is, by Gauss's theorem (**193**), the flux which traverses the section M of the magnet.

313. Induced Current produced by a Variation of the External Field—Case of Two Coils.—Suppose the circuit stationary and the variation of magnetic flux to be produced in the external field. Let N be the flux linked with the circuit; if this flux is suddenly withdrawn, we have an induced flow, $Q=\frac{N}{R}$; if the flux is restored, there is an equal flow in the opposite direction; if the direction of the flux is reversed without changing its value, we get twice the flow,

$$Q=2\frac{N}{R}.$$

This is the case of a current induced in a closed circuit, A, by breaking, making, or reversing the current in a neighbouring circuit, B. Let C be the strength of the steady current in B, and M the value of the flux due to B which would pass through A if B were traversed by unit current, or, in other words, let M be the mutual inductance of the two circuits (**267**), then $N=MC$, and the value of the induced flow, corresponding either to breaking or making the inducing circuit B, is

$$Q=\pm\frac{MC}{R}.$$

In both cases the same quantity of electricity is put in motion; but the law according to which the intensity varies is not the same. The time occupied by the cessation of the inducing current, even when it is prolonged by the spark (320), is usually less than that which corresponds to making. Hence the electromotive force of the direct current is usually much greater than that of the inverse current, but its duration is less.

314. Mutual Inductance.—Suppose a long uniformly wound

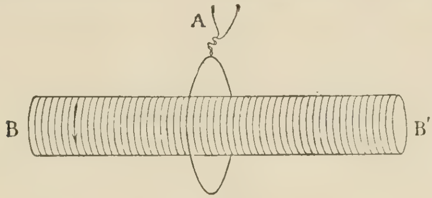


FIG. 266.

coil, BB′, containing n_1 turns per unit length, area of section A, to be surrounded about the middle (Fig. 266) by another coil, A, composed of n turns of wire.

The flux in the interior of the coil BB′ when the current is unity is $4\pi n_1 \mu S$ (280), and as this is enclosed n times by the coil A, the coefficient of mutual induction for this special case is

$$M = 4\pi n n_1 \mu A.$$

315. Determination of Coefficients of Magnetisation.—Suppose that the coil BB′ contains a long cylinder of soft iron, NS,

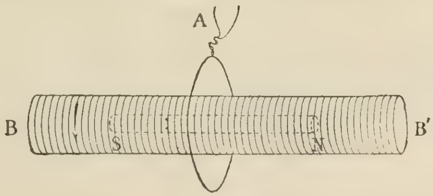


FIG. 267.

placed parallel to the axis (Fig. 267). Let A' be the section of the cylinder, and C the strength of the current in the coil BB′.

The intensity of the field inside the coil is $F = 4\pi n_1 C$, and the induction in the interior of the cylinder $B = \mu F$. The total flux N, which traverses the n turns of the coil A, is

$$N = nF(A - A' + \mu A').$$

If the current is reversed, the induced current will be proportional to twice N, and from this μ can be deduced.

If the coil BB′ is wound directly on the iron rod, we have $A = A'$, and therefore

$$N = nBA = n\mu FA = 4\pi n_1 n\mu AC.$$

The result is the same whatever be the form and size of the windings of the coil A, provided that neither the inner coil nor the core produce any external flux. This condition is strictly fulfilled with an annular coil (**266**), and very approximately with a coil and core which both project a long way on each side of the outer coil. In all cases this coil, A, if wound directly on the core, will measure the flux of induction which traverses the corresponding section.

316. Constant Electromotive Force.—Let us return to the case considered in (**299**). If the sliding bar move with uniform velocity, v, and if l is the distance between the rails, and H the component of the field perpendicular to the plane of the rails, we have already seen that the electromotive force is

$$e = lvH,$$

and, if R is the resistance of the circuit, the strength of the induced current is

$$C = \frac{e}{R} = \frac{lH}{R}v.$$

The direction of the current is that shown by the arrows in Fig. 268, it being assumed that the direction of the field is from front to back, and that the bar moves away from AA′.

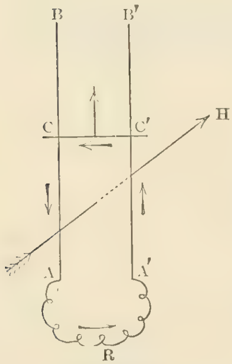


FIG. 268.

For unit field and a velocity of 1 centimetre per second, the electromotive force per centimetre in length of the bar is one C.G.S. unit or 10^{-8} volts. A bar a metre long, moving at right angles to its own length and to the vertical component of the earth's field with a uniform velocity of 20 metres per second, would give an electromotive force for $Z = 0{\cdot}438$ (**244**), of

$$100 \times 0{\cdot}438 \times 2000 = 87600 \text{ C.G.S.}$$
$$= 8{\cdot}76 \times 10^{-4} \text{ volts.}$$

317. Faraday's Disc.—A second example of a uniform electromotive force is furnished by Faraday's disc (**275**). Suppose that the two brushes, one of which presses against the circumference of the disc, and the other against the axis, are connected by a wire, and that the disc is made to rotate uniformly (Fig. 269).

Let a be the radius of the disc; ω the angular velocity, and H the magnetic field outside the disc, parallel to the axis; both of these being in the directions represented by the arrows. Each radius of the disc has a mean velocity perpendicular to its length of $\frac{1}{2}a\omega$, its length is a, and the induction perpendicular to the plane of the motion is μH where μ is the permeability of the medium surrounding the disc. There is therefore in *each* radius an electromotive force whose magnitude is given by the product of these three quantities (**299**), or $\frac{1}{2}a^2\omega\mu H$, and whose direction is from the centre to the circumference of the disc. If the brushes are not present this electromotive force produces merely a temporary current which charges the circumferential regions positively, the axial regions negatively, until the potential difference thereby set up exactly counteracts it. But if the brushes and connecting wire are present the flow takes place continuously and constitutes a current of magnitude, $C = \frac{1}{2}\frac{a^2\omega\mu H}{R}$ where R is the resistance of the wire and disc.

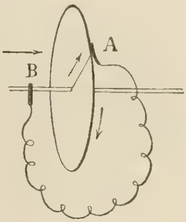


FIG. 269.

If T is the time occupied by a single revolution, and S is the surface of the disc, we have $\omega T = 2\pi$ and $S = \pi a^2$: the above expression therefore becomes

$$C = \frac{S\mu H}{RT}.$$

Suppose that the disc has a surface of one square metre, and let it revolve in air at the rate of 10 turns per second about a horizontal axis in the magnetic meridian; then if $H = 0{\cdot}2$, the electromotive force equals

$$E = 10^4 \times 0{\cdot}2 \times 10 = 2 \times 10^4 \text{ C.G.S.} = 2 \times 10^{-4} \text{ volt.}$$

If the circuit had a resistance of only 10^{-4} ohm, the strength of the current would be 2 amperes.

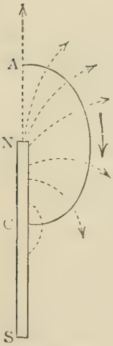


FIG. 270.

In like manner, if an arc of any form (Fig. 270) be made to turn uniformly about the north pole of a magnet, the strength of the pole being s, the arc will, in each unit of time, cut a flux equal to $2s\omega$ (**271**), and in a circuit of resistance, R, will give a constant current of strength

$$C = \frac{2\omega s}{R}.$$

Let us suppose that at A and at C the arc is in contact by means of two brushes with the surface of the magnet, or with a conducting surface rigidly connected therewith ; the same current will be produced going from C to A by the arc, whether the arc is moved about the magnet in the direction of the hands of a watch, or, the arc being stationary, the magnet is moved about its axis in the opposite direction. This latter effect is often referred to as *unipolar induction*.

318. Variable Period—Establishment of a Current.—Let E be the total electromotive force of the battery, R the total resistance, L the self-inductance of the circuit, and $C_0 = E/R$ the ultimate strength of the current when it has become uniform. While the current is increasing in strength from 0 to C_0, the strength at each instant satisfies the equation (**304**)—

$$E = CR + L\frac{dC}{dt}.$$

Dividing throughout by R, we may write this

$$\frac{E}{R} - C = \frac{L}{R} \cdot \frac{dC}{dt},$$

$$= -\frac{L}{R} \cdot \frac{d\left(\frac{E}{R} - C\right)}{dt},$$

since E and R are both constant. Rearranging this expression we get

$$\frac{d\left(\frac{E}{R} - C\right)}{\frac{E}{R} - C} = -\frac{R}{L}dt,$$

where the left-hand side is the differential of $\log_e\left(\frac{E}{R} - C\right)$. If we count time from the instant of closing the circuit, the strength of the current $= 0$ when $t = 0$. Putting C for the current after any assigned interval of time, t, and integrating on the left between the limits 0 and C, and on the right between 0 and t, we get

$$\log_e\left(\frac{E}{R} - C\right) - \log_e\frac{E}{R} = -\frac{R}{L}t\,;$$

or, passing from logarithms to exponentials and rearranging the expression—

$$C = \frac{E}{R}\left(1 - e^{-\frac{R}{L}t}\right) = C_0\left(1 - e^{-\frac{t}{\lambda}}\right),$$

where e is the base of natural logarithms, the never-ending decimal 2·71828. . . The self-inductance and resistance of the circuit are permanent characteristics of it, and their quotient L/R determines the behaviour of the circuit in all cases of varying currents. It is commonly called the *time-constant* of the circuit, and we shall denote it by the symbol λ, as in the last formula.

This formula shows that when t is great, that is, when the circuit has been closed for a long time, the strength of the current is equal to $C_o = E/R$, for e raised to a very high negative power is sensibly equal to nothing. In order to trace the growth of the current, we may consider what fraction of its ultimate strength, C_o, it has acquired at the end of various intervals from making contact: for instance, suppose t to have in succession the values $\lambda, 2\lambda, 3\lambda$ . . . . and let C_1, C_2, C_3, ... denote the corresponding strengths of the current. From the above equation we obtain

$$\frac{C_1}{C_o} = 1 - \frac{1}{e} = 0·6321. \qquad \frac{C_3}{C_o} = 1 - \frac{1}{e^3} = ·9502.$$

$$\frac{C_2}{C_o} = 1 - \frac{1}{e^2} = ·8647. \qquad \frac{C_4}{C_o} = 1 - \frac{1}{e^4} = ·9817.$$

According to the formula, the ratio C/C_o never becomes $= 1$ exactly in any finite time, but in most cases it attains so nearly this value in a very short time, usually less than a second, unless the circuit includes the coils of electromagnets, that the subsequent change is imperceptible.

319. Cessation of a Current.—Suppose that, when a current has attained a steady strength, the electromotive force is withdrawn from the circuit, the resistance, however, remaining unchanged. The equation from which we started in (**318**) will still apply if we give to E the value corresponding to the conditions of the case, namely, 0. We then get

$$CR + L\frac{dC}{dt} = 0.$$

Dividing by C and L and rearranging the terms

$$\frac{dC}{C} = -\frac{R}{L}dt.$$

We again count time from the instant when the current begins to change; consequently, for $t = 0$ we have the current $= C_o = E/R$. Integrating from C_o to C, and from 0 to t, we get

$$\log_e \frac{C}{C_o} = -\frac{R}{L}t = -\frac{t}{\lambda},$$

which is equivalent to

$$C = C_o e^{-\frac{t}{\lambda}}.$$

If, as before, we put C_1, C_2 . . . for the strength of the current at intervals of time equal to *once*, *twice*, ... the time-constant, λ, after the electromotive force has ceased to act, we get

$$\frac{C_1}{C_o} = \frac{1}{e} = 0{\cdot}3679. \qquad \frac{C_3}{C_o} = \frac{1}{e^3} = 0{\cdot}0498.$$

$$\frac{C_2}{C_o} = \frac{1}{e^2} = {\cdot}1353. \qquad \frac{C_4}{C_o} = \frac{1}{e^4} = {\cdot}0183.$$

The gradual growth and cessation of a current in a circuit of constant self-inductance are represented graphically in Fig. 271.

The case here supposed might be furnished by a circuit in which the current was due to the motion of a Faraday's disc (**317**) or other equivalent arrangement: if the motion of the disc were

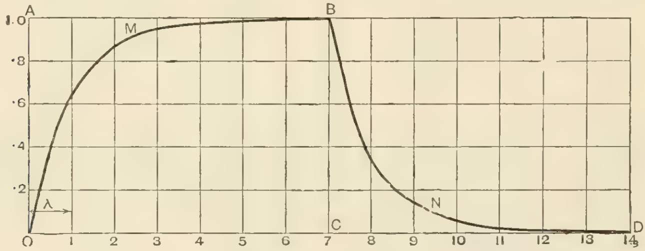


FIG. 271.

suddenly stopped, the impressed electromotive force would cease to act, but the resistance would be unaltered, and the current would die out according to the law stated. The more usual case of stopping a current is by increasing the resistance to a practically infinite amount while the electromotive force continues to act. The change of resistance may occur very quickly, but it is never absolutely sudden, and often, even after the metallic circuit has been interrupted, conductivity is maintained for a short but perceptible time by a spark caused in a manner that we shall explain immediately.

If the resistance were suddenly increased from R to R_1, the current would change from the initial value $C_0 = E/R$ to $C_1 = E/R_1$, its value at any instant during the period of change being given by the formula

$$C = C_1 + (C_o - C_1)\, e^{-\frac{t}{\lambda_1}},$$

where $\lambda_1 = L/R_1$ is the time-constant corresponding to the new

resistance R_1. If the change of resistance takes place gradually, the resulting variation of the current is much more complicated; but, other things equal, it is obvious that the more rapidly the resistance of a circuit is increased, the more rapidly the strength of the current will diminish.

The total quantity flowing in any interval is of course given by the integral $\int C dt$ taken between proper limits. For example, if the law of variation is that expressed by the formula $C = \frac{E}{R}(1 - e^{-\frac{R}{L}t})$, the quantity passing in the interval of time $t_2 - t_1$ is

$$Q = \int_{t_1}^{t_2} \frac{E}{R}(1 - e^{-\frac{R}{L}t})\,dt = \frac{E}{R}(t_2 - t_1) - \frac{E}{R}\int_{t_1}^{t_2} e^{-\frac{R}{L}t}\,dt$$

$$= \frac{E}{R}(t_2 - t_1) - \frac{EL}{R^2}(e^{-\frac{R}{L}t_1} - e^{-\frac{R}{L}t_2}).$$

The first term of the right of this expression represents the quantity that would have passed in the given time in the absence of self-induction.

320. Spark on Breaking Circuit.—A rapid fall of current strength, however, has an effect that we have now to consider. We have seen (**300**) that the electromotive force of induction is always numerically equal to the rate of change of the magnetic flux through the circuit; therefore, in the case we are now considering, the electromotive force of induction is

$$\epsilon = -L\frac{dC}{dt}.$$

When the strength of the current is falling, dC/dt is negative, and therefore the resulting electromotive force is positive,—that is, it acts in the direction of the existing current, which it tends to maintain.

By Ohm's law, the strength of the current at any instant multiplied by the resistance of the circuit gives the effective electromotive force; consequently we have $\epsilon = CR_1$, and transforming the expression by substituting for C its value already found, we may write

$$\epsilon = E(1 - e^{-\frac{R_1}{L}t}) + E\frac{R_1}{R}e^{-\frac{R_1}{L}t}.$$

If t is very small, $e^{-\frac{R_1}{L}t}$ will not differ much from unity even when R_1 is great. Therefore, at the first instant after the resistance has been suddenly increased from the value R to R_1, the electromotive force tending to maintain the current is approximately represented by

$$\epsilon = E\frac{R_1}{R},$$

and may therefore be many times as great as the primary electromotive force, E, of the circuit.

This shows us the cause of the spark often seen on breaking the circuit of an electromagnet. In such cases the induced electromotive force tending to maintain the current is sufficient to cause a discharge through the small space of air which at the first instant separates the ends of the metallic circuit, and the incandescent particles of metal and air which constitute the spark act for an instant as a conductor, and a momentary electric arc is established.

321. Quantity of the Extra Current.—The electromotive force due to self-induction affects not only the strength of the current at each instant while it is rising or falling, but also, as a consequence, the quantity of electricity that traverses the circuit in any given time. This quantity is the algebraic sum of the quantity that would pass in the time in question if self-induction had no effect, and of that which would pass in the same time if the electromotive force of self-induction acted by itself. This latter quantity is easily found by applying the result given in (**308**), namely, that the quantity of electricity traversing a circuit in consequence of induction is equal to the change of magnetic flux linked through the circuit divided by the resistance. In the case of self-induction, the change of magnetic flux due to a change in the strength of the current from C_1 to C_2 is $L(C_2 - C_1)$, and the quantity of electricity is

$$Q = -\frac{L(C_2 - C_1)}{R}.$$

Suppose the current to be rising according to the law discussed in (**318**), and that C_1 and C_2 are the strengths after intervals t_1 and t_2 respectively from the instant at which the electromotive force began to act. We then have

$$C_1 = C_o\,(1 - e^{-\frac{t_1}{\lambda}}) \text{ and } C_2 = C_o\,(1 - e^{-\frac{t_2}{\lambda}}),$$

and for the difference,

$$C_2 - C_1 = C_o\,(e^{-\frac{t_1}{\lambda}} - e^{-\frac{t_2}{\lambda}}).$$

Consequently the total quantity of the extra-current in the interval from t_1 to t_2 is

$$Q = -\frac{LC_o}{R}\left(e^{-\frac{t_1}{\lambda}} - e^{-\frac{t_2}{\lambda}}\right) = -\frac{LE}{R^2}\left(e^{-\frac{t_1}{\lambda}} - e^{-\frac{t_2}{\lambda}}\right).$$

which is the same result as that found in (**319**) for the flow due to self-induction.

As an important special case, suppose $t_1 = 0$ and t_2 very great; then $e^{-\frac{t_1}{\lambda}} = 1$ and $e^{-\frac{t_2}{\lambda}} = 0$. In this case, therefore, we get

$$Q = -\frac{LE}{R^2} = -\lambda C_o.$$

The quantity of electricity which would have traversed the circuit in the same time in the absence of self-induction is $C_o t_2$. Consequently, the actual quantity which passes in any long interval t from the instant at which the electromotive force begins to act is

$$Q = C_o (t - \lambda).$$

The product λC_o gives the total quantity of the "extra-current." In Fig. 271 the curve OMB represents the growth of the current, and BND the way in which it dies away when the electromotive force ceases. Time is represented by the abscissæ in terms of the time-constant λ, taken as unity. The point O denotes the instant of applying the electromotive force, and C the instant at which it is withdrawn. The ultimate strength which the current would attain is represented by the ordinate OA; after the lapse of seven times the time-constant, the strength of the current differs from this by less than one part in a thousand, and it would be impossible to represent any further increase on a diagram of this size. The extra-current on making contact is represented by the area between OMB and the horizontal line through A; similarly, the extra-current corresponding to the cessation of the electromotive force is represented by the area between the curve BND, which represents the dying out of the current, and the axis of time. This curve is the same as that for the increasing current, but its position is reversed, as represented in the figure. After the withdrawal of the primary electromotive force, the electromotive force of self-induction is the only one acting in the circuit, so that, in this case, the whole quantity that passes is given by the formula

$$Q' = -\lambda C_o \left(e^{-\frac{t_2}{\lambda}} - e^{-\frac{t_1}{\lambda}}\right),$$

where t_1 and t_2 are both counted from the instant at which the primary electromotive force ceases to act. In the figure the value of Q' is represented by the area below the curve BN which stands on the part of the base corresponding to the interval of time $t_2 - t_1$.

If we put $t_1 = 0$, and make t_2 very great, the above formula becomes

$$Q' = \lambda C_o \text{ or } Q' = -Q.$$

Hence, when the resistance of the circuit remains constant, the total quantity of the extra current due to the cessation of the primary is equal and opposite to that of the extra-current due to the commencement of the primary, or the sum of the two is zero. This is also true in a more general form which we may state as follows: If a given electromotive force is applied to a circuit of invariable resistance for any given period and then withdrawn, the total quantity of the resulting current is the same as if the circuit had no self-induction. In such a case, self-induction affects the rate of rise or fall of the current, but not the total quantity.

322. Energy of the Extra-Current.—The work done against the electromotive force of self-induction, while an elementary quantity of electricity dQ traverses the circuit, is equal to the product of dQ into the electromotive force $L\frac{dC}{dt}$; but $dQ = Cdt$, so that the work done in the element of time dt is

$$dW = LCdC,$$

and the work done while the strength of the current varies from nothing to any given value C is the integral of this expression, or

$$W = \tfrac{1}{2}LC^2,$$

as already shown in (**304**).

As was there pointed out, this work is spent by the battery in imparting magnetic energy to the medium surrounding the conducting circuit. Properly speaking, it has not passed out from the conductor into the medium: it rather represents energy which, though it has left the battery, has not yet reached the external conductor. When the primary electromotive force ceases to act, the energy of the field flows in upon the conductor, causing the temporary continuation of the current which we recognise as the extra-current, and being converted into heat.

From this point of view we see that a circuit of great self-inductance is a circuit in which a relatively large quantity of energy is poured into the field from the battery while the strength of the current is rising to a given value.

323. Self-Induction in Parallel Circuits.—If two conductors whose self-inductances are L_1 and L_2, and whose resistances are R_1 and R_2, are employed to connect the same two points of a circuit, we know (**119**) that the total current ultimately divides between them in the inverse ratio of their respective resistances; but this proportion does not hold good while the currents are varying in strength. Seeing that the conductors are in contact

at each end, the total electromotive acting in each of them is the same: this is expressed by the equation

$$c_1R_1 + L_1\frac{dc_1}{dt} = c_2R_2 + L_2\frac{dc_2}{dt}.$$

When the currents have become steady, the second term on each side vanishes, and we get $c_1R_1 = c_2R_2$, the relation stated above. But at the first instant, neither c_1 nor c_2 has as yet reached a finite value, and the equation becomes

$$L_1\frac{dc_1}{dt} = L_2\frac{dc_2}{dt};$$

that is to say, the currents in the two branches *begin* to rise at rates which are inversely proportional to the self-inductances and independent of the resistances. As the currents increase, their relative strengths are influenced more and more by the resistance, and less and less by the self-inductance of the two conductors.

In the case of a current which rises in strength from any given value (which may be zero) to some higher value, and then falls to the same strength again, like that resulting from the discharge of a condenser, or from the closing or opening of a neighbouring circuit, the effect of self-induction during the increase of strength is, in each branch of a divided circuit, equal and opposite to that during decrease. Consequently the ultimate quantity of electricity traversing each branch in such a case is the same as if there were no self-induction, and is determined by the resistances only.

324. Induction in Conductors which are not Linear.—Any variation of magnetic flux produces induction currents not only in linear conductors, like wires, but also in conductors which are not linear. Gambey, as long ago as 1824, observed that the oscillations of a bar magnet are rapidly damped when the magnet is placed over a plate of copper. This phenomenon was studied by Arago, and was at first attributed to a particular form of magnetism developed by motion, and called *magnetism of rotation*. It was only properly explained after Faraday's discovery of induction.

The relative displacement of a magnet and of a metal plate determines induction currents. Let us suppose that lines are drawn on a surface representing at each point the direction of the current; these lines are evidently closed curves, some of them enclosing others, but without ever intersecting. On the other hand, the space between two infinitely near lines may be regarded as a closed linear current, which is equivalent to a magnetic shell of the same contour (**255**).

Let the surface of the conductor be thus subdivided into infinitely narrow belts by lines of flow, and let each of these belts be replaced by the corresponding magnetic shell; all the currents which rotate about a point are equivalent to a complex shell, the strength of which at each point is evidently the sum of the strengths of the superposed shells. Fig. 272 represents the lines of flow in the case of an unlimited plate, moving from left to right below the south pole of a magnet whose projection would be at the centre of the figure.

There is a flux of magnetic induction directed upwards through the plate, and this, though constant as regards space, is increasing relatively to the plate in the part which is on the left of the central line, and decreasing in the part to the right of the central line. Consequently (**299, 300**), the part of the plate on the left becomes the seat of closed currents circulating like the hands of a watch, and the part on the right the seat of closed currents circulating in the opposite direction, as indicated in the figure. The former are equivalent to a number of magnetic shells superposed with their negative (south) faces upward, and the latter to similar shells with their positive faces upward. The mutual forces exerted between the circulating currents (or the equivalent mag-

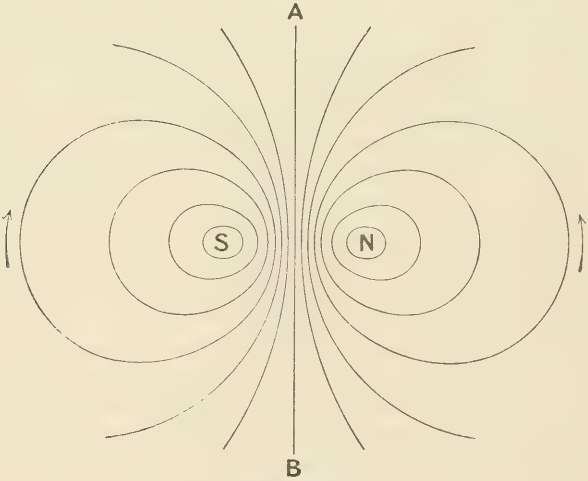


FIG. 272.

netic shells) and the fixed pole are such as to oppose the existing motion of the plate.

The experiment may be easily made by means of a disc which is made to rotate between the poles of an electromagnet (Fig. 273). The disc, which before the passage of the current turns easily, offers a considerable resistance as soon as the electromagnet is excited. The effect is greater the greater the conductivity of the disc, but it is almost wholly destroyed if the continuity of the material is broken by a number of radial saw-cuts (Fig. 274).

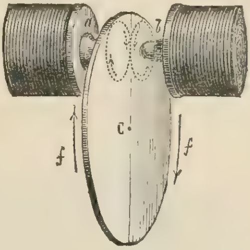


FIG. 273.

In like manner, if a cube of copper, C, is suspended between the two poles, B and A, of an electromagnet by a twisted thread, the cube, if left to itself, spins round rapidly, but it stops the moment the current is turned on (Fig. 275).

It is clear that a pole which can rotate about an axis concentric

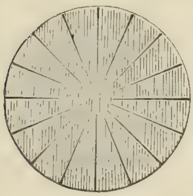


FIG. 274.

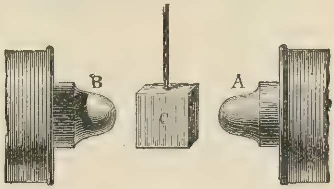


FIG. 275.

with the disc must follow the motion of the disc. This was Arago's first experiment: a magnetic needle is suspended horizontally above a copper disc mounted on the vertical axis of a whirling table; when the disc is made to rotate, the needle follows its motion, though more slowly.

325. Damping the Vibrations of Magnets.—The phenomenon in the form originally observed by Gambey is used for deadening or damping the oscillations of compass-needles and galvanometer-needles.

The forces called into play act as though there were a frictional resistance to the relative motion of the needle and the conductor, the effect at each instant being proportional to the strength of the current and to the relative velocity.

326. Foucault Currents.—The energy absorbed is found again in the form of heat developed by the current in accordance with Joule's law. Foucault, who first observed this, made the experi-

ment in a very striking form. By means of a system of toothed wheels worked by a handle, a stout disc of pure copper, D, is made to rotate between the branches, AB, of a powerful electromagnet (Fig. 276).

As soon as the current passes, a considerable effort is required to keep up the motion of the disc, and this soon rises to a very high temperature. This experiment furnishes thus a beautiful illustration of the transformation of mechanical work into heat.

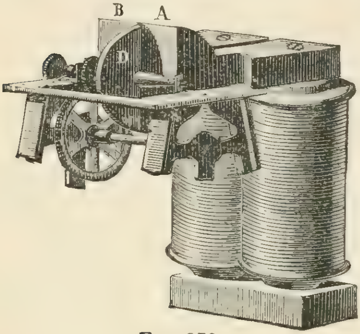


Fig. 276.

The term *Foucault currents* is applied to the induced currents produced in the core of an electromagnet whenever the strength of the current in the surrounding wire varies.

The heat which results from this cause is considerable when the wire is traversed by a rapid succession of currents either in the same or in opposite directions.

The currents which cause this heating, and the loss of work which is the consequence, are greatly diminished by constructing the core of a bundle of separate wires, or of a number of thin plates which are insulated from each other by varnish or by paper, and are arranged parallel to the axis, and therefore perpendicular to the windings, and therefore to the direction in which induced currents tend to be set up.

As these currents are thus suppressed, any heating effects which still remain are due to the work of magnetisation or hysteresis (**212**).

327. Electromagnetic Screens.—Let us suppose that a pole is rapidly brought near an infinite conducting surface. The currents induced in the plate produce a magnetic flux which, for all points behind the plate in respect to the pole, is in the opposite direction to that from the pole. Theory shows that if the plate had no resistance, the magnetic forces at any point would be always equal and in opposite directions, that there would therefore be no inductive action behind the plate, which would thus play the part of an *absolute screen*. Experiment shows, in fact, that the magnetic force is lessened, and the more so, the better the plate conducts, and the more rapidly the changes of magnetic induction occur.

CHAPTER XXVIII

SPECIAL CASES OF INDUCTION

ALTERNATING CURRENTS AND ELECTRICAL OSCILLATIONS

328. Currents varying Harmonically.—Consider the case of a coil wound in a narrow groove, and rotating uniformly in air about a vertical diameter in a uniform magnetic field like that of the earth (Fig. 277). Let H be the horizontal component of the magnetic field, and S the effective area enclosed by the coil—that is, the average area enclosed by one turn of wire multiplied by the number of turns—then, when the plane of the coil is perpendicular to the meridian, the magnetic flux through the coil is HS; and, when the coil has been turned through an angle α from this position, or when its axis makes an angle α with the meridian, the magnetic flux is $N = HS \cos \alpha$.

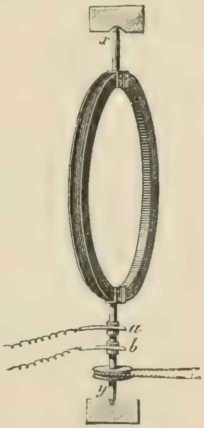


FIG. 277.

If ω be the velocity of rotation, and T the period, or the duration of one complete revolution, we have at time t, measured from an instant at which the axis of the coil is in the meridian,

$$\alpha = \omega t = 2\pi \frac{t}{T}.$$

The electromotive force due to the variation of the flux through the coil is given at each instant by the corresponding value of $-\frac{dN}{dt}$. Hence, seeing that H and S are constant, we may write

$$E = -HS\frac{d \cos\alpha}{dt} = HS \sin\alpha \frac{d\alpha}{dt} = \omega HS \sin\omega t = \frac{2\pi}{T} HS \sin \frac{2\pi t}{T}.$$

The magnetic flux through the coil varies most rapidly when its instantaneous value is zero. this is when the plane of the coil is

parallel to the meridian, and therefore a is a right angle. At this instant, therefore, the induced electromotive force is a maximum, and if we denote this by E_0, we have for the electromotive force at any time t

$$E = E_0 \sin a = E_0 \sin \omega t = E_0 \sin \frac{2\pi t}{T}.$$

In such a case the electromotive force is said to vary *harmonically.* A variation of this kind is expressed graphically by the ordinates of the sign-curve OAB (Fig. 278). In this figure the distance OB represents half the period T, and the maximum ordinate represents E_0, the *amplitude.*

If self-induction produced no effect, the current would be proportional at every instant to the existing electromotive force, or

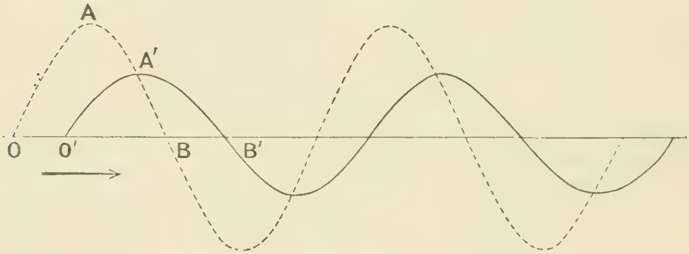


FIG. 278.

$C = E/R$. The maximum strength of current would be E_0/R, and the current at any instant would be

$$C = \frac{E_0}{R} \sin \frac{2\pi t}{T},$$

that is, the current would also vary harmonically in the same period as the electromotive force, and might be represented by a similar sine-curve, having the same period and having zero value at the same instants as the curve OAB.

329. Effect of Self-Induction.—From what has been said already (**318, 319**) as to the way in which self-induction retards the rise or fall of the current when a steady electromotive force is suddenly applied or removed, it is easy to infer that, in the case we are now considering, the changes of current-strength will lag behind those of electromotive force, and that the maximum strength of current due to a rapidly alternating electromotive force will be smaller than that which an equal electromotive force would produce if it alternated more slowly. We will accordingly assume that the strength of the current, when the effects of self-induction are appreciable, still varies harmonically in the same period as the electromotive force, but that the maximum value is

less than E/R, and that it is retarded in *phase* relatively to the electromotive force. This assumption, which is confirmed by experiment, is expressed by the equation

$$C = a\frac{E_o}{R} \sin 2\pi\left(\frac{t}{T} - \phi\right) \quad . \quad . \quad . \quad . \quad (1),$$

where a is a proper fraction and ϕ represents the retardation of phase. We have now to investigate the conditions on which these two quantities depend. If the dotted curve OAB (Fig. 278) represents the current without self-induction, the continuous curve $O'A'B'$ may represent it when the effect of self-induction is appreciable.

To simplify notation we will write

$$C_o = aE_o/R, \quad \omega = 2\pi/T, \quad \text{and } p = 2\pi\phi.$$

The last equation then becomes

$$C = C_o \sin(\omega t - p),$$

and the electromotive force of self-induction is

$$L\frac{dC}{dt} = LC_o\omega \cos(\omega t - p).$$

Now the effective electromotive force at any instant is the excess of the applied electromotive E over the electromotive force of self-induction, and it is equal to the strength of the current at the same instant multiplied by the resistance of the circuit; or, in symbols,

$$E - L\frac{dC}{dt} = CR.$$

Re-writing this equation with the values already adopted, we get

$$E_o \sin \omega t = C_o R \sin(\omega t - p) + LC_o\omega \cos(\omega t - p) \quad . \quad . \quad . \quad (2).$$

If we put $t = 0$, the applied electromotive force vanishes, while for $t = \frac{1}{4}T$ it has a maximum value $= E_o$. In these two cases, equation (2) takes the respective forms

$$0 = -R \sin p + L\omega \cos p \quad . \quad . \quad . \quad . \quad (3),$$

and

$$\frac{E_o}{C_o} = R \cos p + L\omega \sin p \quad . \quad . \quad . \quad . \quad (4).$$

From (3) we get at once

$$\tan p = \frac{L}{R}\omega = \lambda\omega; \quad . \quad . \quad . \quad . \quad (5),$$

while, by squaring (3) and (4) and adding, we get

$$E_o^2 = C_o^2 (R^2 + L^2\omega^2),$$

or

$$C_o = \frac{E_o}{R} \cdot \frac{1}{\sqrt{1 + \lambda^2\omega^2}} = \frac{E_o}{R} \cos p; \quad . \quad . \quad . \quad . \quad (6);$$

whence
$$a = \frac{1}{\sqrt{1+\lambda^2\omega^2}} = \cos p.$$

Using these results in the original expression for the current-strength (equation 1), we have

$$C = \frac{E_o}{R\sqrt{1+\lambda^2\omega^2}} \sin(\omega t - p) \quad . \quad . \quad . \quad . \quad (7a).$$
$$= \frac{E_o}{\sqrt{R^2+L^2\omega^2}} \sin(\omega t - p) \quad . \quad . \quad . \quad . \quad (7b).$$
$$= \frac{E_o \cos p}{R} \sin(\omega t - p) \quad . \quad . \quad . \quad . \quad . \quad (7c).$$
$$= \frac{E_o}{R^2+L^2\omega^2}(R\sin\omega t - L\omega\cos\omega t) \quad . \quad (7d).$$

It thus appears that, besides causing the phase of the current to lag behind that of the electromotive force, the effect of self-induction is equivalent to dividing the electromotive force by $\sqrt{1+\lambda^2\omega^2}$, or to multiplying it by $\cos p$, or again to increasing the resistance in the ratio $R : \sqrt{R^2+L^2\omega^2}$. The quantity last written, which forms the denominator of the expression ($7b$), is sometimes called the apparent resistance, or the *impĕdance* of the circuit, while $L\omega$ is called the *reactance*. If a right-angled triangle be drawn with the sides about the right angle proportional to R and $L\omega$ respectively, the hypothenuse will represent the impedance. At slow speeds of rotation, or when the self-inductance is small, this reduces to the simple resistance R; as the speed and self-inductance increase it becomes more and more nearly independent of the resistance, and approaches to $L\omega$.

330. Geometrical Representation.—The problem we are considering admits of a very simple geometrical treatment. Let O (Fig. 279) be the projection of the axis of rotation on the horizontal plane, which we take as the plane of the figure, OH the direction of the field, and OR the direction at a given instant of the plane of the coil. We suppose the direction of rotation to be that of the hands of a clock.

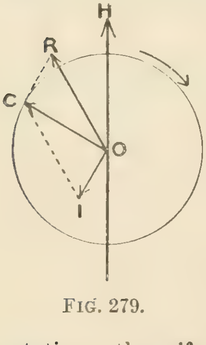


FIG. 279.

Let the length OR represent the maximum value of the electromotive force due to the field, that is $E_o = \omega HS$; draw OC so as to make an angle with OR equal to the retardation p or $2\pi\phi$ measured in the opposite direction to the rotation; then if RC be drawn perpendicular to OC, the length intercepted, or OC = OR $\cos p$, represents the maximum effective

electromotive force. Lastly, draw OI so as to complete the parallelogram ORCI, the angle COI being a right angle again measured in the opposite direction to the rotation.

Now let the whole figure except the line OH rotate in its own plane about the point O, and consider the projection at any instant of the three straight lines, OC, OR, OI, upon the direction OH of the field. The projection of OR represents the instantaneous value of the impressed electromotive force due to the field; the projection of OI represents the electromotive force of self-induction; and that of OC, which is equal to the difference of the other two, gives the effective electromotive force, that is to say, the product of the strength of the current at the instant considered, into the true resistance R of the circuit.

It is evident that the phase of the electromotive force of self-induction must be one-quarter of a vibration behind that of the effective electromotive force, or what comes to the same thing, behind that of the current, since the maxima of one concur with the zero-values of the other, and reciprocally. This is expressed in Fig. 279 by the angle COI being an angle of 90°.

331. Average and Effective Strength of a Current.—The average strength of a variable current, for an interval of time t, is the quotient by t of the total quantity of electricity which has traversed any given section of the circuit in this time.

If the current is represented by a curve as a function of the time, the quantity of electricity, Cdt, which corresponds to the interval of time MM′ (Fig. 280) is represented by the area of the infinitely narrow quadrilateral MPP′M′, and that which corresponds to a finite time AB by the area below the curve comprised between the corresponding two ordinates AA′ and BB′. If this area is divided by the time AB, we have the *average strength of current*, or, more shortly, the *average current*.

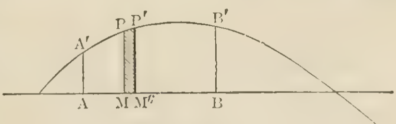


FIG. 280.

It is manifest that if the current varies harmonically, the average current corresponding to an entire period is zero; and the same applies to any interval of time divided into two equal parts by an instant at which the sign of the current is reversed. The mean strength for a half period from zero to zero is

$$C_m = \frac{C_0}{\frac{1}{2}T}\int_0^{\frac{1}{2}T} \sin \omega t dt = \frac{2C_0}{\pi},$$

C_0 being the maximum current.

If the ordinates of the curve (Fig. 280), instead of being equal to the currents, were taken equal to the squares of the currents at each instant, the area AA′B′B multiplied by the resistance R of the circuit, would represent the energy transformed into heat in the time t. The quotient of this area by t would be the mean square of the current, and the square root of the quotient would give the strength of the constant current, which would develop the same quantity of heat in the circuit.

This is a quantity of great practical importance, and is frequently referred to as the *root of mean-square current* (R.M.S. current). It might be called the mean working strength. We shall denote it by C_e. It can be measured directly by the electro-dynamometer, or, still better, by the electrometer (**83**). With a current varying harmonically, the root of mean-square strength for any whole number of half-periods is

$$C_e = \frac{C_o}{\sqrt{\frac{1}{2}T}}\sqrt{\int_o^{\frac{1}{2}T}\sin^2\omega t dt} = \frac{C_o}{\sqrt{2}},$$

that is, it is equal to the maximum divided by $\sqrt{2}$.

The square root of the mean square of the electromotive force is in like manner

$$E_e = \frac{E_o}{\sqrt{2}}.$$

Dividing E_e by C_e we have the impedance (**329**) R_a. Accordingly the impedance may be defined as the factor by which the root of mean-square current must be multiplied to give the root of mean-square electromotive force.

332. Work done by an Alternating Current.—We have seen (**125, 126**) that the *power* of an electric circuit, or the rate of doing work by the current, is equal to the product of the strength of the current into the electromotive force. In the case of alternating currents, both electromotive force and current-strength vary from one moment to the next, but at a particular instant we may write for the power

$$P = EC = E_oC_o \sin \omega t \sin(\omega t - p),$$
$$= \tfrac{1}{2}E_oC_o[\cos p - \cos(2\omega t - p)].$$

Now, during half a period, that is, while t changes to $t + \frac{1}{2}T$, the angle $2\omega t - p = \frac{4\pi t}{T} - p$ changes by 2π, and its cosine goes through all possible values, positive and negative, the mean value

being nothing. Hence, the average rate of doing work, during any space of time which is a multiple of the half-period, is

$$P = \tfrac{1}{2} E_o C_o \cos p = E_e C_e \cos p \quad . \quad . \quad . \quad . \quad (8)$$

Hence *the power of an alternating current is equal to the product of the root of mean-square electromotive force and root of mean-square current into the cosine of the difference of phase.*

Referring to (**329**), we see that $C_o \cos p = E_o \cos^2 p / R$, and that $\cos^2 p = \frac{R^2}{R^2 + L^2\omega^2}$; hence we get

$$P = \tfrac{1}{2} E_o^2 \frac{R}{R^2 + L^2\omega^2}.$$

The multiplier of $\frac{1}{2}E_o^2$ in this expression is a maximum for a given speed of rotation (ω constant), when

$$R = L\omega = \frac{2\pi L}{T}.$$

In this case the rate of working is

$$P = \frac{E_o^2}{4R},$$

and

$$\cos^2 p = \tfrac{1}{2}, \text{ or } p = 45^\circ;$$

that is, the difference of phase between electromotive force and current is $\frac{1}{8}$ of a complete period.

333. Case of Soft Iron.—Suppose a soft iron core surrounded by a coil of wire between the ends of which a difference of potential is maintained which varies as the ordinate of a sine curve.

In order to simplify the problem, we will assume that the magnetic induction is at each instant proportional to the existing strength of the current. This simplification, which is justifiable as a first approximation, amounts to disregarding hysteresis and the variations of permeability; in other words, it amounts to replacing the closed curve, which represents a complete cycle of magnetisation, by a simple straight line passing through the origin (Fig. 180).

If n is the number of turns, and R the magnetic reluctance (**283**), we have at each instant for the relation between the strength, C, of the current and the total flux of induction, N,

$$\mathrm{R}N = 4\pi n C.$$

Consequently, supposing the strength of current to be unity, and

remembering that the flux is encircled n times by the wire, we get for L, the self-inductance,

$$L = \frac{4\pi n^2}{\mathrm{R}} = \frac{4\pi n^2 S\mu}{l};$$

and by putting this value into the formulas (5), (6), and (7) of (**329**), we get the retardation of phase and the strength of current.

The high value of μ causes the self-inductance to be great, and consequently, for high frequencies, the retardation is nearly 90°, and therefore $\cos p$ differs but little from zero, and formulas (6) (**329**) and (8) (**332**) show that the current and rate of doing work are very small.

A coil with a soft iron core placed in a circuit carrying a rapidly alternating current has thus the curious property of *choking* the current, but without the expenditure of energy which would take place with a simple resistance.

334. Action on an Adjacent Circuit.—A harmonically varying current causes a harmonically varying electromotive force to act on an adjacent circuit, and therefore gives rise to a current which is also harmonic, and of the same period as itself, but which differs in phase. The difference of phase can never be less than $\frac{\pi}{2}$, nor greater than π. For the secondary electromotive force has a retardation of $\frac{\pi}{2}$ in respect of the primary current, and self-induction produces a further lag comprised between 0 and $\frac{\pi}{2}$.

Professor Elihu Thomson has shown that, under these conditions, there is always repulsion between the two circuits. Fig. 281 represents one of his experiments. An electromagnet is traversed by a powerful alternating current: when a metal ring is brought near one end and left to itself, it is strongly repelled.

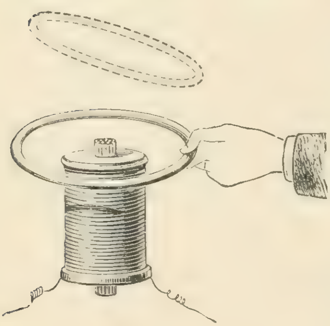


FIG. 281.

The electrodynamic force at each instant is proportional to the product of the strength of the current equivalent to the electromagnet and that induced by it in the ring. From Ampère's law, the two currents attract if they are in the same direction, and repel if they are opposite.

In Figs. 282 and 283, two sine curves, A and A′, have been traced with the same period, but with amplitudes which are as 4 : 1. The ordinates of the curve B are taken equal to the products of the corresponding ordinates of A and A′, and are drawn above the line if the product is positive, and below

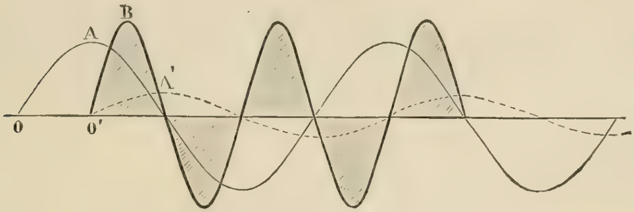


FIG. 282.

if it is negative. When the difference of phase is equal to $\frac{\pi}{2}$ (Fig. 282), the attraction and repulsion are equal both in intensity and in duration, and the resulting effect is therefore nothing. But if the difference of phase exceeds $\frac{\pi}{2}$, and it always does exceed this value, repulsion preponderates over attraction (Fig. 283), and the more so the greater the difference of phase.

Professor E. Thomson has planned several arrangements in

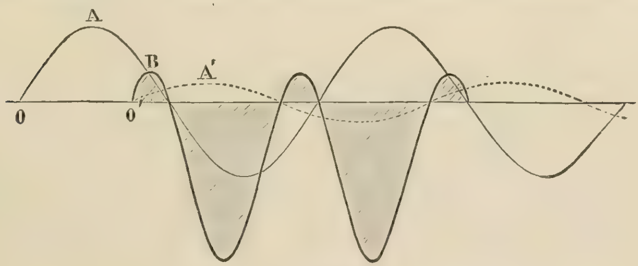


FIG. 283.

which the effect of repulsion may give rise to a continuous rotatory motion. In that shown in Fig. 284, a copper disc, B, mounted on a spindle, rotates rapidly when it is placed excentrically in reference to the alternating pole, and when another disc, A, partly screens the first. If the disc A is itself movable, it will turn in a contrary direction to the disc B.

335. Rotating Fields.—Let two coils be placed with their

centres coincident but their planes at right angles to one another. If alternating currents of the same frequency and amplitude be sent through both, their phases differing by 90°, the resultant magnetic field at the common centre remains of constant magnitude, but its direction rotates at a uniform rate. For if $H \cos \omega t$ and $H \sin \omega t$ be the two fields at right angles, the resultant R is given by the square root of the sum of their squares (Fig. 285); that is,

$$\begin{aligned} R &= \sqrt{H^2 \sin^2 \omega t + H^2 \cos^2 \omega t} \\ &= H, \end{aligned}$$

and its direction θ is given by

$$\tan \theta = H \sin \omega t \,/\, H \cos \omega t = \tan \omega t,$$

consequently $\theta = \omega t$, *i.e.* it increases at a uniform rate. Let now another coil be mounted so that its plane is at some instant at

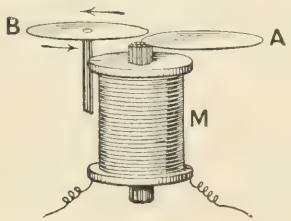


FIG. 284.

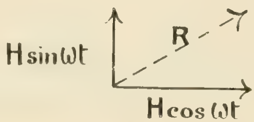


FIG. 285.

right angles to the resultant field. If this coil be held fixed, the lines of induction threading it will change in number as the field rotates, decreasing from a positive maximum, when the axis of the field coincides with the axis of the coil, to zero when the axis of the field has turned through a right angle; increasing to a negative maximum, when the field has turned through 180°; then passing through zero again for three right angles; and returning to the original value when the field has made a complete rotation. The induction varies in the same direction during the first half of the rotation of the field and in the opposite direction during the second half, the rate of variation being greatest when the value of the induction itself passes through zero and vanishing as the induction goes through its maximum value, positive or negative. The consequence is an alternating electromotive force in the coil proportional at each instant to the rate of variation of the flux of induction through the coil. It is evident that this electromotive force varies harmonically in a period equal to the period of rotation of the field;

in fact, the conditions are virtually the same with a stationary coil in a rotating field as with a rotating coil in a stationary field. Accordingly, the conclusions arrived at in (**329**) can be at once applied to the present case, and we see that the coil will be traversed by a current varying harmonically in a period equal to the period of rotation of the field. If A is the effective area enclosed by the coil, and t time reckoned from an instant at which the direction of the field coincided with that of the axis of the coil, the strength of the current is

$$C = \frac{AH\omega}{\sqrt{R^2 + L^2\omega^2}} \sin(\omega t - p),$$

where p, whose value is given by

$$\sin p = L\omega / \sqrt{R^2 + L^2\omega^2} \quad \text{or} \quad \cos p = R / \sqrt{R^2 + L^2\omega^2},$$

expresses the retardation of the phase of the current relatively to the electromotive force, or (what comes to the same thing) relatively to the rotation of the field.

The coil traversed by this current is equivalent to a magnetic shell of moment AC (**255**), and is consequently subject to a couple always tending to make its axis coincide in direction with the axis of the field. At a given instant, when the angle between the two directions is ωt, the moment of this couple is

$$\begin{aligned} G &= ACH \sin \omega t, \\ &= \frac{A^2H^2\omega}{\sqrt{R^2 + L^2\omega^2}} \sin \omega t \,.\, \sin(\omega t - p), \\ &= \frac{A^2H^2\omega}{R^2 + L^2\omega^2} (R \sin \omega t - L\omega \cos \omega t) \sin \omega t. \end{aligned}$$

The average value of G during half a revolution of the field from $\omega t = 0$ to $\omega t = \pi$ is easily seen to be

$$G = \frac{A^2H^2R\omega}{2(R^2 + L^2\omega^2)},$$

since the average of $\sin^2\omega t = \frac{1}{2}$ and that of $\sin\omega t \cos\omega t = 0$.

So far we have supposed the coil to be held at rest, but now let it be free to revolve in obedience to the turning couple the magnitude of which we have just investigated. Suppose its angular velocity to be ψ; then the angular velocity of the field relatively to the coil becomes $\omega - \psi$ and the results arrived at above, on the supposition that the coil was at rest, will still apply if we substi-

tute this value for ω. This gives for the average turning moment acting on the rotating coil

$$G = \frac{A^2H^2R(\omega - \psi)}{2[R^2 + L^2(\omega - \psi)^2]}.$$

This has its maximum value when $\omega - \psi = R/L$, or $\psi = \omega - R/L$. Assuming this value, we get

$$G_{\text{max.}} = \frac{A^2H^2}{4L}.$$

Motors, whose action depends on the principle here indicated, are now made on a large scale.

336. Distribution of Alternating Currents in Conductors.—In the case of alternating currents the impedance may greatly exceed the resistance; moreover this latter is itself increased by the fact that the current is not distributed uniformly throughout the whole section of the conductor, but is concentrated near the surface.

Experiment shows, in fact, that a current at starting is first formed on the surface, and that it only gradually, though usually very quickly, reaches the deeper layers. If its direction is reversed at very short intervals, the current never fully reaches the centre. The case may be compared with that of an infinitely long tube filled with liquid, which is rapidly moved to and fro lengthwise. If the liquid is a perfect fluid, it will remain at rest; if it is viscous, the layers in contact with the sides of the tube will share the motion of the tube, but the inner layers will oscillate with an amplitude which decreases, and with a retardation of phase which increases, from the periphery to the axis.

The results in the electrical case are in full agreement with what has been already stated (see especially **305**, **306**, and **322**) as to the seat of electric and magnetic energy. It has been pointed out that the energy of an electric current enters the conductor from the surrounding non-conducting medium. It is a necessary consequence of this that it arrives first at the outer layers, and that its penetration to the inside, though a rapid, is not an instantaneous process.

What we recognise as an electric current is, in one aspect, the process by which energy, whether electric or magnetic, passes from the field and generates heat in the conductors. Whenever the result of a current in a conductor would be a diminution of the energy of the field, such a current takes place. If energy is poured into the field in any way as fast as it is removed by the conductors,

the condition known as that of a steady current is set up, but in all cases the ultimate distribution of the current among the available conductors is such that the total energy is a minimum. The electric energy corresponding to a given strength of current is proportional to the difference of potentials between the ends of the conductor, or, what comes to the same thing under the conditions specified, to the resistance—and this is least if the current makes use of the whole cross-section, distributing itself so that the current-density is uniform. On the other hand, the magnetic energy is least when the current is confined to the outer skin of the conductor; for it is easily proved, by an adaptation of the reasoning employed in (**13, 20**) in relation to spheres, or by the method of (**266**), that a current distributed uniformly on the surface of a circular cylinder has no magnetic field at an internal point, and that at external points it acts as though it were concentrated in the axis. It follows that the external magnetic field is the same whether the current is confined to the surface of a conductor, or uniformly distributed through the cross-section; whereas, in the former case the magnetic field inside the conductor itself is nothing, and in the latter case the internal magnetic field is proportional to the distance from the axis. The magnetic energy of the conductor itself is greater the greater the magnetic permeability, and hence the effect in question is more marked with iron than with copper.

For a frequency of 80 periods per second, the virtual increase of resistance due to concentration of the current near the surface is 1 per cent. in a copper wire 1 centimetre in diameter, and 8 per cent. in a wire of 2 centimetres diameter. For large diameters the conducting power of a cylindrical conductor increases almost as the perimeter instead of as the section. Hence we have the practical conclusion, that for very strong alternating currents tubes should be used, and not solid conductors.

337. Influence of Capacity.—If a condenser is connected in circuit with a coil revolving in a magnetic field, as described in (**328**), or with any other equivalent source of alternating electromotive force, it acquires alternate positive and negative charges, the difference of potentials of the coatings varying harmonically, and the period being the period of revolution of the coil. The difference of potentials of the coatings is equivalent to an electromotive force opposing that due to the rotation of the coil. Calling the potential-difference at any instant V, the strength of current at the same instant C, and the capacity of the condenser S, the charge is SV, and its rate of change, $S\frac{dV}{dt}$, is equal to the

strength of the current. Writing, as before, $E_0 \sin\omega t$ for the instantaneous electromotive force, we have

$$E_0 \sin\omega t = CR + L\frac{dC}{dt} + V.$$

To express that V varies periodically in the same period as the electromotive force, we may write

$$V = V_0 \sin(\omega t - r),$$

where V_0 is the maximum value and r is a constant determining the difference of phase between the electromotive force and the charge of the condenser. From this expression for V we get for the strength of current

$$C = S\frac{dV}{dt} = S\omega V_0 \cos(\omega t - r),$$

and for the rate of variation of the current

$$\frac{dC}{dt} = -S\omega^2 V_0 \sin(\omega t - r).$$

Using these values in the original equation, we may write it as follows :—

$$E_0 \sin\omega t = [RS\omega \cos(\omega t - r) + (1 - LS\omega^2)\sin(\omega t - r)]V_0.$$

If we now make $\omega t - r$ first $= 0$, and then $= \frac{\pi}{2}$, this gives

$$E_0 \sin r = RS\omega V_0 \quad \text{and} \quad E_0 \cos r = (1 - SL\omega^2)V_0$$

respectively.

Hence, the retardation of phase r is given by

$$\tan r = \frac{RS\omega}{1 - LS\omega^2}.$$

Squaring the two foregoing expressions, adding them, and taking the square-root of the sum, we have

$$V_0 = \frac{E_0}{\sqrt{R^2S^2\omega^2 + (1 - LS\omega^2)^2}},$$

and for the potential difference at any time t,

$$V=\frac{E_0 \sin(\omega t - r)}{\sqrt{R^2S^2\omega^2+(1-LS\omega^2)^2}}.$$

The simultaneous strength of current is given by

$$C=S\frac{dV}{dt}=\frac{E_0 \cos(\omega t - r)}{\sqrt{R^2+\left(\frac{1}{S\omega}-L\omega\right)^2}}.$$

The denominator of this expression is the impedance of the circuit and $L\omega-\frac{1}{S\omega}$ is the reactance.

338. Electrical Oscillations.—When the surfaces of a charged condenser are suddenly connected by a conductor of sufficiently small resistance, the electric energy is converted into heat, but the process is, comparatively speaking, a gradual one, and involves the charging of the condenser a large number of times in succession, and alternately in opposite directions; the energy represented by each charge being a constant fraction of that of the preceding one. In fact, a series of electric oscillations of gradually decreasing amplitude is set up.

In order to follow out the theory of the process, we will, in the first instance, suppose the resistance of the conductor to be inappreciable. Denoting the charge of a condenser by Q, and its capacity by S, the electrostatic energy is $\frac{1}{2}Q^2/S$ (**65**). When the coatings are connected by a conductor of self-inductance, L, the electrostatic energy disappears, giving rise to a current and an amount of electromagnetic energy represented (**304**) by $\frac{1}{2}LC^2$. This in its turn dies away, reproducing the original amount of electrostatic energy in the form of a reversed charge of the condenser. This, as soon as it is formed, begins to diminish again, giving rise to an inverse current and reproducing electromagnetic energy; and so the process goes on, the original energy of the condenser being transformed into the electromagnetic and electrostatic forms alternately.

From the principle of conservation of energy we conclude that at any stage of the process, the total energy, electrostatic and electromagnetic together, is constant, or

$$\frac{1}{2}\frac{Q^2}{S}+\frac{1}{2}LC^2=\text{constant}.$$

Differentiating with respect to time, we get

$$\frac{Q}{S}\frac{dQ}{dt}+LC\frac{dC}{dt}=0,$$

which simply means that an increase of one kind of energy is accompanied by an equal decrease of the other kind. But $C=dQ/dt$, and therefore $dC/dt=d^2Q/dt^2$: consequently, the last equation may be written

$$\frac{d^2Q}{d^2t}=-\frac{1}{LS}\cdot Q.$$

Let us compare this with the formula that expresses the acceleration of a particle of mass, m, which, when displaced from a position of stable equilibrium, is urged back by a force proportional to the displacement, x, namely, the formula

$$\frac{d^2x}{dt^2}=-\frac{f}{m}x,$$

where f stands for the restoring force corresponding to unit displacement. This equation states that the acceleration is proportional to displacement and in the opposite direction. We know that, under the conditions here specified, the motion of the particle is simple harmonic, and may be represented by

$$x=x_0\cos\omega t,$$

if x_0 is the amplitude and t denotes time reckoned from an instant when the particle is at the positive extremity of its path. The velocity is

$$v=\frac{dx}{dt}=-\omega x_0\sin\omega t,$$

and the acceleration

$$\frac{d^2x}{dt^2}=-\omega^2x_0\cos\omega t=-\omega^2x.$$

Equating this with the expression for d^2x/dt^2 given above, we get

$$\omega^2=\frac{f}{m},$$

or, for the periodic time, we have

$$T=\frac{2\pi}{\omega}=2\pi\sqrt{\frac{m}{f}}.$$

All these mechanical results have strict analogues in the electrical case. The charge, Q, of a condenser may be taken as a measure of the displacement from the condition of electrical

equilibrium; $\frac{dQ}{dt}$ then becomes the rate of change of electric displacement, and might be called electric velocity; and $\frac{d^2Q}{dt^2}$ becomes acceleration of electric displacement. The expression we obtained above for this quantity shows it to be directly proportional to the electric displacement, Q. Hence, following the mechanical analogy, and putting Q_0 for the initial charge of the condenser, we may write

$$Q = Q_0 \cos \omega t;$$

$$C = \frac{dQ}{dt} = -\omega Q_0 \sin \omega t;$$

$$\frac{d^2Q}{dt^2} = -\omega^2 Q_0 \cos \omega t = -\omega^2 Q.$$

From this, by the original expression for $\frac{d^2Q}{dt^2}$, we get

$$\omega^2 = \frac{1}{LS}, \text{ or } T = 2\pi\sqrt{LS}.$$

339. Comparison with Experiment.—These equations represent the discharge of a condenser, under the conditions stated, as giving rise to an endless series of oscillations of the same amplitude taking place in the periodic time $T = 2\pi\sqrt{LS}$.

For two reasons, however, these results do not accurately agree with what takes place in real cases. On the one hand, the true seat of the changes which we have spoken of as electrostatic and electromagnetic oscillations is not the conductor connecting the surfaces of the condenser, but the dielectric medium of the condenser and that by which the conductor is surrounded. Any change of condition, such as those accompanying electric charge and discharge, produced at one part of this medium, is propagated with a definite velocity to more and more distant parts, and consequently a series of electrical alternations, such as we have been discussing, gives rise to a series of waves which travel outwards in all directions into the surrounding space. Electric energy is thus radiated away from the oscillating system, and the energy of the latter is consequently diminished in proportion. The process is closely analogous to the gradual loss of energy by a vibrating tuning-fork consequent upon its radiation of sound-waves into the air.

We shall return presently (**343**) to the subject of electric radiation.

Another respect in which the conditions assumed in (**338**) differ

from those of actual experiment is that they take no account of the resistance of the conductor. In certain cases this resistance may be so small that the formulæ we have arrived at represent the results with fair accuracy, but it can never be got rid of altogether. The existence of resistance causes a certain amount of electric energy to be converted into heat at every oscillation, just as friction does in the case of ordinary mechanical vibrations. The electric energy which is thus expended in an element of time dt is, in agreement with Joule's law,

$$RC^2 dt = R\left(\frac{dQ}{dt}\right)^2 dt.$$

Adding this term to the equation from which we started in **(338)**, we get after simplification,

$$\frac{d^2Q}{dt^2} + \frac{R}{L} \cdot \frac{dQ}{dt} + \frac{Q}{LS} = 0.$$

This equation may be taken as a statement in mathematical language that the rate of generation of electromagnetic energy, together with the rate at which electric energy is expended in generating heat, is equal to the rate of decrease of electrostatic energy; or, again, as a statement that the electromotive force due to self-induction, together with that required to maintain the existing current, is equal and opposite to the difference of potentials between the surfaces of the condenser. The discussion of the equation would involve more elaborate mathematical processes than we can enter upon: we must, therefore, be content with merely stating the most important conclusions to which it leads. It will assist the reader if we point out that the equation is of the same form as that for small motions of a simple pendulum in a resisting medium. If the medium is very viscous, the pendulum may return from any displaced position without oscillations; on the other hand, if the resistance is negligible, as in the case already considered, the motion is oscillatory.

In like manner, in the electrical case, the character of the discharge depends essentially on the relative magnitudes of the constants in the equation. When the discharge is oscillatory the period is given by the equation

$$T = \frac{2\pi LS}{\sqrt{LS - \frac{1}{4}R^2S^2}},$$

which, as R diminishes, approaches more and more nearly the value $T = 2\pi\sqrt{LS}$ previously obtained. As R increases, the

period increases, and becomes infinite when the two terms under the root-sign are equal to one another. If R is made still greater (the other quantities remaining constant), the expression under the root-sign becomes negative, and T becomes imaginary. The real solution is then non-oscillatory, the charge decreasing in a continuous manner. Ultimately, when R is made so great that the first term under the root-sign is negligible compared with the second, the charge decreases continuously by a constant fraction of its existing amount at any instant in a given interval of time. The equation, in fact, may then be written

$$R\frac{dQ}{dt}+\frac{Q}{S}=0.$$

or

$$\frac{dQ}{Q}=-\frac{dt}{RS}$$

whence by integration

$$\log_e\frac{Q}{Q_0}=-\frac{t}{RS}$$

or

$$Q=Q_0e^{-\frac{t}{RS}}$$

where Q_0 is the initial charge, and e is the base of natural logarithms.

This equation implies that Q falls off to $\frac{1}{e}$ of its initial value in the time RS; hence this product, which plays the same part as the ratio L/R in a simple inductive circuit, may be called the time-constant for a simple condenser circuit in which the induction is negligible.

The general character of an oscillatory discharge is indicated in Fig. 286. The ordinates of the sinuous curve represent successive values of the electric charge. The periodic time is denoted by the distance along the axis ot between two points where the curve cuts it in the same direction. In the discharge of a Leyden jar, the oscillations may occupy from 10^{-4} to 10^{-8} of a second, according to the size of the jar and the nature of the discharging conductor. The intermittent character of the discharge can be directly observed by examining it by reflection in a rapidly revolving mirror, or by photographing it by means of a moving camera.

The limiting continuous discharge which is approached as R is increased would be represented graphically by a curve such as BND (Fig. 271, **319**), the time-constant, $\nu=RS$, being indicated by one of the horizontal divisions.

As an example, take the case of a Leyden jar of diameter 10

centimetres, with tinfoil coatings extending to a height of 12·5 centimetres, and glass of thickness 0·2 centimetre. If the inductive capacity of the glass were six times that of air, the capacity of the jar would be 1125 K, if K denotes the inductive capacity of air. Suppose the jar discharged by connecting the coatings by a copper wire 1 metre long and of 0·2 centimetre radius, bent into an approximate circle. If the specific resistance of the copper be taken as 1600, the resistance, R, of the wire would be $1{\cdot}273 \times 10^6\mu$. The self-inductance, L, would be about $941{\cdot}3\mu$. In both these formulæ μ denotes the magnetic permeability of air. The period of oscillation, derived from the simple formula,

$$T = 2\pi\sqrt{LS},$$

becomes $6466\sqrt{K\mu}$. Now, although there is no means of ascertain-

FIG. 286.

ing with certainty the separate values of K and μ, we know that the product denotes the reciprocal of the square of a velocity, which for air is 3×10^{10} centimetres per second (see **351** and **411**). We have thus finally the value

$$T = \frac{6466}{3 \times 10^{10}} = 2{\cdot}155 \times 10^{-7}.$$

The period of oscillation given by the more complete formula is not appreciably different. The amplitude decreases in the ratio $e : 1 = 2{\cdot}72 : 1$, in the time $2L/R = 1{\cdot}479 \times 10^{-3}$ seconds; or the amplitude decreases in the ratio named in the time occupied by $\dfrac{1{\cdot}479 \times 10^{-3}}{2{\cdot}155 \times 10^{-7}} = 6863$ oscillations. It is as though the amplitude of vibration of a tuning-fork sounding "middle C" (256 vibrations

per second) decreased in the ratio 2·72 : 1 in a little less than half a minute (26·8 seconds).

339.* Electric Resonance.—Let us now return to the case considered in **337**, that, namely, of a condenser of capacity of S the coatings of which are connected by a conducting circuit, of self-inductance L and resistance R, with a source of electromotive force varying harmonically in the period $2\pi/\omega$. We saw that the difference of potentials between the surfaces of the condenser and the current in the circuit both vary harmonically with the same period as the electromotive force, being represented respectively by—

$$V = \frac{E_o}{\sqrt{R^2S^2\omega^2 + (1 - LS\omega^2)^2}} \sin(\omega t - r),$$

and

$$C = \frac{E_o S\omega}{\sqrt{R^2S^2\omega^2 + (1 - LS\omega^2)^2}} \cos(\omega t - r).$$

Since the difference of potentials and the current are proportional respectively to the sine and cosine of the same angle, they must differ in phase by one-quarter of a complete oscillation. The condenser has its maximum charge when the strength of the current is momentarily zero and the current is changing sign from positive to negative or *vice versâ*. The difference of potentials between the surfaces of the condenser assists the inverse current, and this continues to increase until the condenser is discharged. The current has now its greatest (inverse) value and begins to charge the condenser in the inverse way. As the inverse charge increases, the resulting difference of potentials increases, causing the current to decrease and to become zero again when the inverse charge has attained its greatest value. Thus during a quarter period, while the current falls from a positive maximum to zero, the charge of the condenser rises from nothing to a positive maximum; during the next quarter period the current changes from zero to a negative maximum, and the charge of the condenser falls from the positive maximum to nothing; during the next two quarter periods the changes of current-strength and of charge are the same as before, but with signs reversed. The maximum charge of the condenser, positive or negative, is thus the quantity conveyed by the mean current, $C_m = 2C_o/\pi$ (**331**), in a quarter period, or $\frac{1}{2}C_oT/\pi = C_o/\omega$, and the maximum difference of potentials is the ratio of charge to capacity, or $V_o = \dfrac{C_o}{S\omega}$.

Referring now to the values given above for V and C, we see from (**338**) and (**339**) that the product LS which occurs in the denominator of both of them is equal to $T^2/4\pi^2$, if T is the period of the free electric oscillations which would occur in the condenser-circuit if the square of its resistance were inconsiderable in comparison with the quotient $4L/S$. If we put $T_1 = 2\pi/\omega$ for the

period of the impressed electromotive force, we may write the expression for the maximum difference of potentials as follows—

$$V_o = \frac{E_o}{\sqrt{R^2S^2\omega^2 + \left(1 - \frac{T^2}{T_1^2}\right)^2}}.$$

From this it appears that, if $T_1 = T$, accurately or approximately, the last term of the denominator becomes negligible and we get

$$V_o = \frac{E_o}{RS\omega} = E_o \frac{L\omega}{R}.$$

Under the same conditions, tan r, of which the value is given by $RS\omega/(1 - LS\omega^2)$, becomes very great (since $LS\omega^2 = 1$), whence it follows that r itself is very nearly a right angle. Consequently, we get

$$V = \frac{E_o}{RS\omega}\cos\omega t \text{ and } C = \frac{E_o}{R}\sin\omega t = \frac{E}{R}.$$

That is, the current is at each instant in phase with the electromotive force and is the same as if the circuit included neither self-inductance nor capacity.

The phenomena we have been considering may be regarded as a combination of those discussed in **329**, due to a harmonically varying electromotive force in an inductive circuit, with those due to the discharge of a condenser through a similar circuit discussed in **338, 339**. In the latter case, the vibrations die out in proportion as energy is dissipated as heat or by radiation. In the absence of such dissipation, or if energy were supplied from without as fast as it is dissipated, vibrations of constant amplitude would be maintained indefinitely. On the other hand, if energy be supplied faster than it is dissipated, the amplitude of the vibrations increases, until the greater rate of dissipation corresponding to the greater amplitude results in the steady state of oscillation discussed above.

When the period of an alternating electromotive force applied to a condenser is nearly the same as the free period of the vibrations of the condenser-circuit, the maximum difference of potentials may, if the resistance is small, become much greater than the source of electromotive force would give on open circuit.

The phenomenon is closely analogous to the familiar case of a pendulum being thrown into strong vibrations by the application of a succession of minute impulses following each other periodically at the rate of the free swings of the pendulum, so that each impulse increases somewhat the existing motion resulting from those that have preceded, until the energy dissipated at each swing through friction or viscous resistance is equal to the energy imparted. Another well-known case of the same kind is that of acoustic *resonance* in which an elastic body capable of vibrating in

a definite period, such as a tuning-fork, or a column of air in an organ-pipe, is caused to give audible vibrations by the cumulative effect of small periodic impulses following each other at intervals equal to the free period of vibration of the elastic body.

340. Effects of an Instantaneous Discharge—Lodge's Experiments.—Very sudden discharges give rise to phenomena which, at first sight, seem inconsistent with familiar electrical experience, but which are explained when we consider that, in the case of rapidly varying currents, the properties of a conductor depend essentially upon its self-inductance, and only to a small extent upon its resistance. These effects may be compared with those produced by the sudden application of mechanical forces. A block of gun-cotton placed on a steel plate burns quietly if it is lighted with a match; it smashes the plate if it is exploded by fulminating powder. In both these cases the same quantity of gas is developed, and the same energy put into play; but in the second case the explosion is so sudden that the resistance of the air becomes comparable with that of steel.

Many of these effects have long been known, but they have been set in their proper light only by the experiments of Sir Oliver Lodge.

1. The conductor by which a charged Leyden jar or battery is discharged is capable of giving a strong spark to any conductor brought near it; this is what is called a *lateral discharge*. Hence arises the necessity, which has long been known, of having glass handles to discharging rods (**61**).

If the two branches of the discharger are connected by a wire, covered with silk, bent backwards and forwards several times, sparks pass not only at the points where the wire is connected but, notwithstanding the insulation, wherever the loops of the wire cross each other.

2. If a long copper wire 7 to 8 millimetres in diameter is used for discharging the jar, and if this wire is led round a large room, not only does it everywhere give lateral discharges, but gas or water pipes connected with it give sparks to any conductor brought near them, whether insulated or not. If taken at the end of a gas jet, these sparks ignite the gas as it issues.

The induction on all adjacent bodies is often so strong that they become capable of giving sparks without being connected with the discharging wire.

3. Two Leyden jars, M and N (Fig. 287), are joined in cascade with the two poles, A, of a Holtz machine, for instance; the two outer coatings are connected by a continuous wire, L, and with two spark knobs, B. Whenever a spark passes at A, a spark also

passes at B, if the distance of the two knobs is not too great. The limiting distance depends on the wire L, but it may be greater than that of the knobs A. The greater part of the discharge of the external armatures, a discharge which takes place suddenly and *without preparation* as in the case of the discharge at A, passes through the air in preference to the wire L, although the resistance of the air-break may be many million times as great.

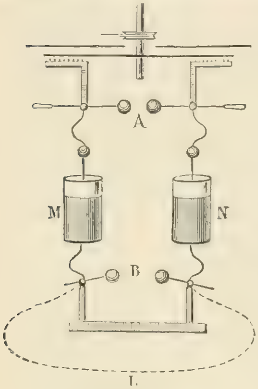


FIG. 287.

4. The internal coating of a Leyden jar is connected with one of the poles of the machine, the other coating being connected by a long conducting wire L, with the second pole (Fig. 288). Every time a spark passes at A, sparks are seen to strike between the two coatings along the surface of the glass, and the jar is *self-discharged*. When the difference of potential between the two coatings has reached the value which corresponds to the striking distance, a powerful spark passes between the two knobs; this sets up for a short time *a path* of feeble resistance, across which electricity oscillates from one armature to another. When the spark ceases, the path closes suddenly, and the moving electricity strikes across the gap between the two coatings by an action comparable with that of the hydraulic ram.

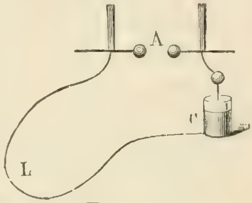


FIG. 288.

341. Propagation along a Cable.—Hitherto we have considered the total current to be the same at all parts of a conducting circuit. This is strictly true only when the current is steady. Each element of a wire acts as a capacity, and, in consequence, at all parts of a circuit action is taking place similar to that which in the preceding paragraphs we have supposed to be localised in a *condenser; i.e.* the current which flows into any element is in general different from that which is simultaneously leaving it at the other end. In order to illustrate this we will take a circuit consisting of a long straight wire with a coaxal cylindrical tube forming the return, *i.e.* a concentric cable, and we will suppose that an electromotive force varying as $\sin \omega t$ is applied at one end. Each small portion of this double conductor acts, as we have said, as a condenser, that is to say, the inner and outer portions are in general oppositely charged, and to an amount

proportional to the difference of potential between them. This difference of potential and accompanying charge vary from point to point along the length of the cable, and the object of the following discussion is to obtain the law of their variation.

Consider any elementary length dx of the cable, and recognise that *at any one time* the current may have different values at different points, and let $\frac{\partial C}{\partial x}$ be the rate of change of the current with change in distance from any origin measured along the cable. Then if C is the current entering the element, the *simultaneous* value of the current leaving the element is $C+\frac{\partial C}{\partial x}dx$. The excess of that entering over that leaving is therefore $-\frac{\partial C}{\partial x}dx$, and this must measure the rate of accumulation of charge within it. Let the potential difference between the inner and outer conductors in the same region be V, and let S denote the capacity *per unit length;* the charge on the element is therefore $Sdx.V$ and its rate of increase with time is $Sdx.\frac{\partial V}{\partial t}$; hence

$$-\frac{\partial C}{\partial x}dx = Sdx.\frac{\partial V}{\partial t},$$

$$-\frac{\partial C}{\partial x} = S\frac{\partial V}{\partial t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

Again, let r be the resistance per unit length of the cable (this will be shared between direct and return portions of the unit length), and let L be the self-inductance also per unit length. Then we may, as in **329**, equate $Ldx.\frac{\partial C}{\partial t} + rdx.C$ to the impressed electromotive force between the ends of the element—that is to $-\frac{\partial V}{\partial x}dx$. Hence

$$-\frac{\partial V}{\partial x} = rC + L\frac{\partial C}{\partial t} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

In order to fully understand the meaning of these equations (1) and (2), it must be noted that in (1) $\frac{\partial C}{\partial x}$ is the rate of increase of the current with distance, *the time being supposed constant;* while $\frac{\partial V}{\partial t}$ is the rate of increase of V with time *for a constant position on*

the cable; that is to say, that each of these expressions is what is called a *partial* differential coefficient. Similar statements apply to $\frac{\partial V}{\partial x}$ and $\frac{\partial C}{\partial t}$ in (2).

If V is rapidly alternating—as in telephonic circuits—the last term becomes increasingly important, and we will suppose that r is so small that the average value of rC is negligible compared with the average value of $L\frac{\partial C}{\partial t}$. It is then possible to satisfy both of these equations by assuming that V (and C) vary in a simple and periodic manner, not only in respect to time but in respect to position also; in other words, the disturbance which satisfies the equations propagates itself along the cable, in the same kind of way as that in which a wave is propagated along an elastic cord. An equation to such a wave is

$$V = V_o \sin\left(\frac{2\pi t}{T} - \frac{2\pi x}{\lambda}\right) = V_o \sin(\omega t - qx),$$

where T is the period and λ is the wave-length; for if t increases by the amount T, while x remains constant, the quantity in brackets increases by 2π, and its sine—and therefore V—passes through a complete cycle and returns to its initial value; a complete cycle is also passed through if x increases by λ while t remains constant. The characteristic of such a wave is that any particular value of the disturbance denoted by V is found at succeeding instants at increasing distances from the starting point—that is, it travels onward, and the velocity at which it travels is constant. For in order to maintain the sine term constant, x must increase with t at the rate λ/T (or ω/q). When the above equation represents V, it is necessary, in order to satisfy both (1) and (2), to take C as fluctuating according to the same law, so that we may write

$$C = C_o \sin(\omega t - qx).$$

It is also necessary that a certain relation shall exist between q and ω; this we may find as follows: Obtain the several differential coefficients which occur in equations (1) and (2) from the equations for V and C:

$$\frac{\partial V}{\partial t} = V_o\omega \cos(\omega t - qx) \qquad \frac{\partial C}{\partial t} = C_o\omega \cos(\omega t - qx)$$

$$\frac{\partial V}{\partial x} = -V_o q \cos(\omega t - qx) \qquad \frac{\partial C}{\partial x} = -C_o q \cos(\omega t - qx)$$

Insert these in (1) and (2) leaving out the common factor $\cos(\omega t - qx)$,

$$C_0 q = V_0 S\omega$$
$$V_0 q = LC_0\omega;$$

whence $q^2 = LS\omega^2$, which is the desired relation. The velocity of propagation is

$$v = \frac{\omega}{q} = \sqrt{1/LS}.$$

It should be noted that q may have a negative value and the velocity would then be negative, that is, the wave would be travelling backward. Such a wave arises when the disturbance reaches the far end of the cable—in other words, a reflected wave is there set up. For simplicity we shall suppose the cable to be indefinitely long, so that only the forward wave is present.

To find the value of the velocity of propagation we require to know the product LS; it has been shown in (**302**) that this product equals μK, where μ = the magnetic permeability and K the dielectric constant of the medium between the conductors—both being in electromagnetic units. It is shown in Chapter XXXI. that this product is such that the velocity equals 3×10^{10} cm. per second if the intervening medium is air.

Thus waves of electric disturbance travel along a cable such as we are discussing with the velocity of light in air.

If the medium is not air, the proper values of μ and K must be inserted; it is noteworthy that the velocity depends upon the properties of the medium and not upon those of the conductor, at least when the resistance of this is negligible. If the resistance of the conductors may be neglected, as we have supposed, the current fluctuates between the same extreme values at all points along the cable. The effect of the resistance is to cause the maximum value of current to decrease as x increases. The physical reason of this is that, instead of the energy which travels with the wave remaining as electromagnetic energy, part of it is dissipated into heat, in accordance with Joule's law, in the conductor. The law of variation is that the amplitude of the current (*i.e.* C_0) falls off in geometric progression as the distance increases in arithmetic progression. Fig. 286, therefore, represents the condition of things if the abscissæ are taken as distances (x) and the ordinates are the amplitudes of the current (C_0).

342. Hertz's Experiments.—Hertz obtained very rapid oscillations by means of a discharger or *vibrator* formed of two

spheres or of two plates, AA′, connected by a nearly continuous brass rod, C (Fig. 289).

If the two spheres are suddenly raised to the potentials $+V$ and $-V$, and the system is left to itself, equilibrium is attained after a series of oscillations the period of which is determined by the formula already given (**338**),

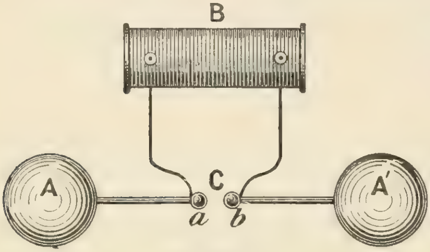


FIG. 289.

$$T = 2\pi\sqrt{SL},$$

which, according to the dimensions of the apparatus, may vary between 1 and 30 thousand millionths of a second.

To get an instantaneous charge of the vibrator, a break is left at C between the two parts of the connecting rod, and these are provided with two small knobs which are connected with the poles of a powerful induction coil, B (**437**).

When induction takes place on the secondary wire of the coil, the two arms of the vibrator which form the ends of the wire are raised to different potentials, and at the same instant a spark strikes between the knobs a and b. This, as was explained above, *makes a path* for the oscillations, and the vibrator now discharges itself independently, as if it were insulated from the coil. The same succession of phenomena takes place at each to and fro motion of the secondary current of the coil. During these oscillations the energy, just as in the discharge of a Leyden jar, passes alternately from the electromagnetic to the electrostatic form. The magnetic lines consist in circles round the connecting rods, while the lines of electric force and induction stretch between the two branches; each of these sets of lines vanishes in turn: the magnetic, when the balls are at their maximum difference of potential; the electric, when their difference of potential is zero.

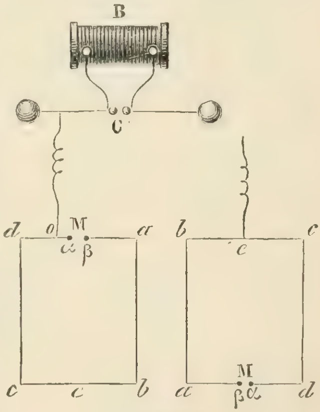


FIG. 290.

If a wire be attached to one of the connecting rods (Fig. 290), then at each oscillation it picks up part of the disturbance, which

then travels along it as a wave. It has already been said (**341**) that the velocity of propagation of energy through the ether is very great (3×10^{10} cm. per second), but since the period of the oscillations can be made exceedingly small, it is possible for the wave length to be small; and the presence of this undulatory phenomenon can then be demonstrated by means of apparatus of moderate size.

Thus if a conductor, such as *abcd* (Fig. 290), with a break at M, is connected by a wire of any length with one of the branches of a vibrator, sparks are obtained at M if the point of attachment, *o*, is unsymmetrical with respect to the interval; they disappear if the point of attachment is exactly at *e*. In this case the two parts into which the wave divides arrive at the same time at the knobs α and β, and cannot, therefore, produce the difference of potential which is necessary for a spark. On the other hand, the spark is stronger the nearer the point of attachment is to one of the knobs. Experiment shows, moreover, that there is a certain size of the circuit for which the spark is a maximum; it is that for which the time which the waves take to traverse the wire is equal to half the free period of oscillation of the vibrator; the impulse increases at each oscillation, and the circuit acts like a *resonator* (**339***).

343. Propagation of Vibratory Motion.—It is unnecessary to directly connect the rectangle with the vibrator by a wire; the latter when present serves merely as a guide directing the flow of energy; without it the energy radiates outwards in all directions from the oscillator, and as it passes over the rectangle some of it is absorbed, exciting a current in it as before. When the vibrator works well, sparks can be taken from every piece of metal, large or small, insulated or not, contained in the room where the vibrator is, or in the adjacent ones. They strike between two coins or two keys which are brought near together; they can be taken from water or gas pipes when the point of a penknife is presented. Dr. Hertz used a wire bent in the form of a circle, one end provided with a small knob, and the other with a point whose distance from the knob could be varied at will (Fig. 291). The intensity of the action at any point is indicated by the length and brightness of the spark. The effect is greatest for a particular size of the circuit, namely, that which causes it to act as a resonator.

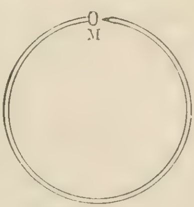


FIG. 291.

The length of the spark in the secondary depends upon the position in which it is placed. If its plane be as in Fig. 292, *a*,

and with its spark gap in any position in the circle, sparking will occur, but it will cease if the plane be placed in the perpendicular position, b. The reason of this will be clear if it be noticed that, in the first case, the circle is so placed as to catch the greatest possible number of lines of magnetic induction, and there will be a high electromotive force in it in consequence of their rapid reversal; while, in the second case, the lines do not thread the circle and hence no electromotive force is to be expected from the reversal. Sparking may, however, also be caused by the variations of the electrostatic field; thus if the circle be placed in position, c, with its spark gap outside the plane of the paper, sparks may occur, and are then due solely to this cause.

Near the vibrator, the resonator gives sparks of 7 to 8 millimetres; and at distances of 15 to 20 metres sparks of only a few hundredths of a millimetre, but still visible. A stone or brick wall or a door does not stop the action; a metal surface, according to its thickness and conducting power, reflects the vibrations, and acts like a more or less perfect screen.

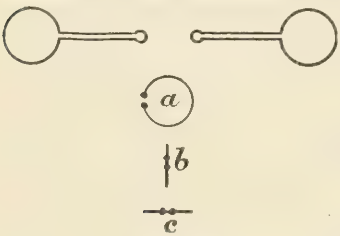


FIG. 292.

The question arises, whether this action is instantaneously transmitted in the medium, or whether it has a finite velocity. In the first case, all points should be at the same moment in the same phase as the source; in the second, the propagation must take place in the form of waves. If V is the uniform velocity of propagation and T the period, the wave-length,

$$\lambda = VT,$$

is the distance through which the effect is propagated during a complete vibration. At points distant by λ, 2λ, 3λ . . . the electrical disturbance is the same at each instant as that of the source, except as to amplitude; at points distant by $\frac{1}{2}\lambda$, $\frac{3}{2}\lambda$, $\frac{5}{2}\lambda$, the motion is in the contrary direction.[1]

In order to render these waves manifest Hertz changed them by reflection into stationary vibrations. It is known that if waves

[1] Let us imagine two wires stretched parallel to each other, one belonging to a primary and the other to a secondary circuit. If the former is traversed by an alternating current, and the latter is at a distance from the inductor equal to one wave-length, it will be acted upon in the same way as if it were in contact, except as regards intensity. But if it is at a distance of half a wave-length, the induced electromotive force will at each instant be in the opposite

are reflected from a plane surface, the reflected waves interfering with the direct waves give stationary vibrations separated by fixed nodes. In order to see this we need only draw a sine curve and fold it back on itself about any line perpendicular to its axis (Fig. 293) (comp. **354**). The points where the ordinates of the two curves are equal, are always, if we do not take account of the sign, at distances 0, $\frac{1}{4}\lambda$, $\frac{2}{4}\lambda$ . . . from the reflecting surface. If the reflection takes place with change of sign, the distances 0, $\frac{1}{2}\lambda$, λ, $\frac{3}{2}\lambda$ correspond to the nodes, the distances $\frac{1}{4}\lambda$, $\frac{3}{4}\lambda$ . . . to the loops.

This is exactly what takes place when waves from an electrical vibrator strike against a metal wall parallel with the vibrator. There is a node at the wall, and others follow at equal distances. For a vibrator whose period is thirty thousand-millionths of a second, the interval between the nodes is about 5 metres. The wave-length in this case is 10 metres, which makes the velocity of

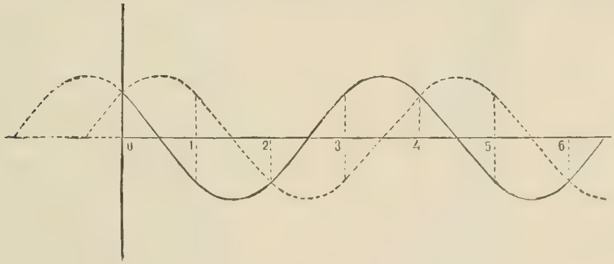


FIG. 293.

propagation 300,000 kilometres; that is to say, exactly the velocity of light.

344. Lecher Wires.—One of the readiest ways of investigating these undulatory effects is to make them excite stationary oscillations in long conductors. The arrangement is represented in Fig 294. A and B are the Hertzian oscillators, and *a*, *b*, are two smaller plates placed near and parallel to them. To these smaller plates, long parallel wires are attached. The waves spreading out from the oscillating charges in A and B are picked up in part by the plates, *a*, *b*, and run along between the wires till they reach the

direction to that which would be produced close to the primary conductor. In such a case, the usual laws as to the direction of the induced currents would be reversed. The experimental proof of this fact would be the most direct proof that the propagation occupies time; but if the velocity were that of light, the wave-length of an alternating current of 100 periods in a second would be 3000 kilometres, and the two wires would have to be placed 1500 kilometres apart.

distant end at which reflection occurs. The direct and reflected systems form stationary waves (as in organ pipes) and the positions of the nodes may be obtained by placing a vacuum tube as a bridge across the wires. If it be placed at the nodes it does not glow; at intermediate places it glows if suitably exhausted. The distance between two nodes is half a wave-length. Besides the fundamental wave, harmonics may usually be detected.

The dotted lines in the figure (Fig. 294) show approximately the state of things at the end of a quarter of a period from the commencement of the first oscillation; the state when stationary waves are fully set up is shown in like manner in Fig. 295. In this figure the directions in which the lines of electric force have just been moving are indicated by arrows. In both cases the long conductors are shown connected together by a wire at their ends; when this is the case the end is necessarily a node, for the connecting wire maintains the difference of potential there permanently zero—at least if its resistance may be neglected.

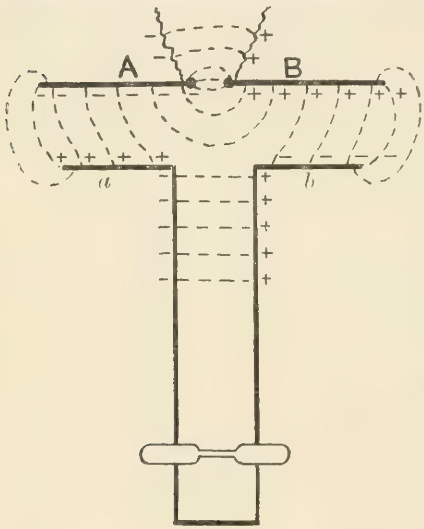


FIG. 294.

345. Rays of Electrical Force.—True rays of electrical force may be obtained by placing a small vibrator along the focal-line of a parabolic cylinder (Fig. 296).

The field in which the phenomenon can be perceived, and in which sparks can be obtained with the resonator, is bounded by two vertical planes which pass through the edges of the mirror, and are parallel to the axis of the parabola. This forms a *parallel electric beam*, identical with the luminous beam which a source of light would give if put in the place of the vibrator.

If this beam is received upon a mirror identical with the first, the well-known experiment of the two conjugate mirrors may be repeated, and it may be shown that the vibratory motion is concentrated on the focal line of the second mirror.

The ray may also be reflected at a vertical plane, and the angle of reflection proved to be equal to the angle of incidence.

The radiation can be transmitted through a prism with vertical edges; in this case it is deflected towards the base of the prism, and the refractive index of the substance for the wave-length used can be deduced from the amount of deviation.

A flat grating formed of parallel copper wires intercepts the beam if the wires are parallel to the vibrations, and allows them to pass if they are perpendicular. It thus acts like a tourmaline on a ray of polarised light.

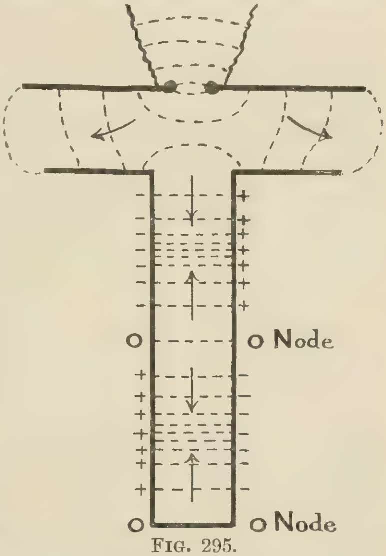


FIG. 295.

These resemblances between electric waves and light were early detected; at the present time there is scarcely any simple optical fact which has not its experimentally proved electrical analogue. For example, many varieties of optical interference and diffraction experiments have been imitated (Fresnel's biprism and mirrors, diffraction grating, Michelson's interferential refractometer). In regard to polarisation, the ellipsoid of elasticity for electric waves has been determined in the case of selenite by measuring the refraction through prisms cut in different manners.

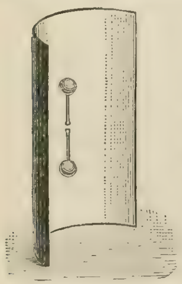


FIG. 296.

Nicol's prisms have been constructed of sulphur, a plate of ebonite taking the place of Canada balsam; and with their aid the lines of extinction for a cleaved slice of selenite have been ascertained. Though these lines do not agree with those obtained by optical methods, the difference is simply one of those which throw fresh light on a subject; for it turns out that one of the extinction lines is practically parallel to a secondary cleavage line of the crystal, which is not the case optically; and thus it would seem that the behaviour of selenite to long waves is much more nearly related to its crystallographic form than are

its optical properties. It is true that neither natural nor magnetic rotation of the plane of polarisation (**290**) has been observed; but since optical experiments show that both decrease in amount with increase of wave-length this is not surprising. In general the analogy between the motion produced in the ether by electrical oscillations and by luminous phenomena appears complete, and we are led to the conclusion that they only differ as to their respective periods of vibration, and therefore in their wave-lengths. The wave-length of visible rays is about 0·00005 centimetre; that of thermal rays, as observed by Rubens, is 0·006 centimetre; the smallest wave-length which has hitherto been observed for electrical rays is about 4 mm., that is to say, 8000 times as great as for the visible rays.

346. Propagation along Wires.—A straight insulated wire fixed perpendicularly to the centre of a metal plate placed close to one of the knobs of the discharger is traversed by waves which, reflected from the end of the wire, give rise to stationary vibrations. The distance from one node to another is constant whatever be the

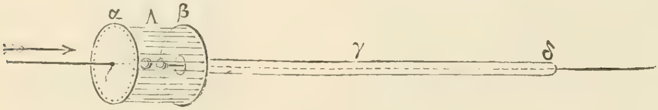


FIG. 297.

nature of the wire, and this value is the same as for air. It follows from this that the propagation takes place through the air, and not through the wire.

We might infer from the extreme rapidity of the oscillations that the phenomenon does not penetrate beyond the surface of the wire (**336**). Hertz demonstrated this directly by the following arrangement (Fig. 297). The wire is cut at A, and the gap is enclosed in a kind of cage made of metal wires stretched between two discs, α and β. The disc α is in contact with the wire; the disc β is supported by a metal tube $\gamma\delta$, which surrounds the wire, but does not touch it. As the waves arrive in the direction of the arrow, there is no spark at A if the tube is connected at δ with the wire; the electrical action stops at the outer surface. The sparks reappear, however, if the tube is insulated at δ; in this case the oscillations travel through the dielectric between the wire and the inner surface of the tube.

347. Coherers.—When a tube containing iron filings is interposed in a battery circuit, although it may be non-conducting to start with, it becomes suddenly conducting when an electric spark

passes in the neighbourhood; a gentle tap, however, is usually sufficient to restore the previously non-conducting state. This phenomenon was first observed by Calzecchi-Onesti in Italy (1884), and afterwards by Branly in France (1890). A similar phenomenon had been observed by Sir Oliver Lodge (1889), who found that when two metallic knobs were placed so nearly in contact that they just failed to allow a feeble current to pass, communication can be established by the passage of a spark between them.

The name *coherer* is given to any arrangement exhibiting this phenomenon.

The explanation of the action is not yet complete. It is certain that under the action of the exciting spark the filings are caused to cling together; indeed, long chains of such adherent particles can be obtained; and, according to Lodge, this coherence is the main, if not the sole cause of the diminution of resistance. On the other hand, with some substances (peroxide of lead, magnesium and potassium in paraffin oil) the spark causes an *increase* in resistance, especially for high-voltage currents, and although this effect may be owing to secondary actions, this does not yet seem to have been proved definitely.

Owing to their sensitiveness to electric disturbances coherers are largely employed in experiments on electromagnetic waves; indeed, most of the quasi-optical experiments which have been referred to (**345**) were made with the use of coherers as detectors.

Sir Oliver Lodge has also devised a detector which depends on the fact that, if a small pool of mercury be divided by a greasy knife, the two portions remain distinct, unless a difference of potential of a fraction of a volt is produced between them, when they immediately coalesce. In this detector a greasy steel wheel is maintained in constant rotation with its edge dipping in a pool of mercury. Every incident wave reunites the mercury, which is at once automatically redivided. No tapping is required—indeed, the apparatus has the great advantage that it does not respond to mechanical tremors.

348. Magnetic Detectors.—The behaviour of the ordinary coherers is, however, rather fickle, and it seems likely that they will be completely displaced by devices similar to the last or to the *magnetic detector* of Rutherford, especially where quantitative results are desired. The action of this detector depends on the fact that if a piece of iron be magnetised to saturation, a magnetic field acting in the sense of its existing magnetisation produces no sensible change of magnetic condition; whereas a reverse field produces in general a marked decrease of magnetisation. Hence a small

bundle of strongly magnetised iron wire placed inside a small coil of insulated wire serves as a detector of electromagnetic waves. When electric waves fall upon the coil they generate in it alternating currents which are gradually damped out, and these partially demagnetise the iron. The amount of demagnetisation —which can be measured by a magnetometer—is a measure of the intensity of the electric waves which produce it. The apparatus is made much more sensitive when only qualitative results are desired by magnetising the bundle of wires to a degree corresponding to a point on the steep part of the magnetisation curve.

349. The Electromagnetic Field.—The mutual relations between electrical and magnetic phenomena, and especially phenomena such as those discussed in the latter part of the present chapter, afford strong confirmation of the view, already referred to several times in the course of this work, that electric and magnetic forces are the result and manifestation of a special state, or rather states, of the dielectric medium surrounding the bodies on which the forces are exerted.

At any part of a dielectric medium, electric and magnetic fields may exist either separately or simultaneously, and in the latter case the effects of one are not affected by the existence of the other. Otherwise expressed, an electric or magnetic field may exist independently of each other, or they may be superposed without the direction or intensity of either being modified by the coexistence of the other.

But this mutual independence of the two fields applies only to the steady state of each. The movement of lines of electric force is accompanied by magnetic field, which at any point is perpendicular to the plane containing the tangent to the line of electric force through that point and the direction of motion; and, reciprocally, the movement of lines of magnetic field is accompanied by electric force at right angles to the direction of motion and to that of the magnetic field.

A complete theory of electricity and magnetism would require as its basis such a conception of the structure of the various media as would afford a mechanical explanation of the properties of electric and magnetic fields and of their mutual relations. Various more or less satisfactory attempts to form a conception of this kind have been made, but, without discussing them, we may say that the evidence seems conclusive that the phenomena we are studying have their origin in special conditions of the medium occupying the field surrounding conductors, although we cannot say definitely what these conditions are.

Electric and magnetic fields can exist, not only within ordinary matter in any of its forms, solid, liquid, or gaseous, but also in what we are accustomed to speak of as a perfect vacuum. We must suppose, therefore, that the electromagnetic medium is something different from ordinary matter, existing where matter is not, and, indeed, extending throughout space to the farthest limits of which we have any cognisance.

Even where ordinary matter exists, say atmospheric air, its mechanical properties are not such as can explain the electric and magnetic fields which exist in it. We must think of the required medium as existing throughout space, and retaining everywhere essentially the same properties, whether the space is otherwise vacuous or occupied by any kind of matter. When an electric field is filled with air, glass, or any other dielectric, we must suppose that the properties of the field depend essentially on the electromagnetic medium coexisting with the ordinary matter, though the properties of the medium are modified quantitatively by the nature of the ordinary matter with which it is associated.

350. Electricity and Optics.—In order to explain the propagation of light across stellar space, and to account for the properties of light as ascertained by experiment, men of science have long recognised the necessity of supposing some medium, the so-called "luminiferous ether," possessing mechanical properties different from those met with in ordinary matter, to exist throughout all transparent substances and to extend through space.

After imagining one intangible, all-pervading ether, in order to explain optical phenomena, it seems highly artificial and complicated to imagine a second intangible medium to account for the phenomena of electricity and magnetism. It turns out, however, that these two hypotheses mutually support each other; that, in fact, the mechanical properties which must be attributed to the ether, in order to explain the transmission and properties of light, are just those which the electromagnetic medium must possess in order to produce the effects of electricity and magnetism. We may say, therefore, that the required electromagnetic medium has been found in the long-admitted luminiferous ether.

It appears, indeed, that in studying optics and electricity we are really investigating the properties of the ether from two different sides. In optics, we are dealing with effects due to vibrations of enormous rapidity;[1] whereas in electricity we observe the effects of very much slower vibrations, or of steady,

[1] About 600 million million vibrations per second for the mean rays of the spectrum.

non-vibratory conditions of the ether. Such at least seems to be the fair inference from what is known so far.

Without attempting to give a complete account of the evidence in favour of the statements we have just made, we will endeavour to explain the nature of some of the chief steps in the argument in favour of regarding electrical and optical phenomena as different manifestations of the same agency.

The fundamental property that must be ascribed to the luminiferous ether is that of receiving and transmitting energy in the form of vibrations; hence it must have inertia and elasticity, and thus be capable of possessing energy either in the kinetic or potential form. The kinetic energy of a portion of the ether possessing inertia m, and moving with velocity c, is of course $=\frac{1}{2}mc^2$. Comparing the ether with an ordinary elastic material possessing the same properties in all directions, we may express the potential energy of a given portion of it as follows:—Consider a rectangular block of an elastic material having one face, call it M, held fast, while a force f is applied uniformly all over the parallel face N. For definiteness, suppose f to be parallel to M and N, and to another pair of the faces of the block. If a is the area of one of the faces M, the intensity of the applied force, or the *stress*, tending to deform the block is $p=f/a$. The result will be that every point of the face N will move relatively to M in the direction of the force through a distance λ, say, producing a *strain* (in this case a shear) $s=\lambda/l$, if we put l for the distance between the surfaces M and N. The applied force f will be proportional to the displacement λ which it produces, and during its application work will be done equal to the product of its average value into the final value of the displacement; this is easily seen to be

$$\tfrac{1}{2}f\lambda=\tfrac{1}{2}pa\,.\,ls=\tfrac{1}{2}ps\,.\,v,$$

where $v=a\,l$ represents the volume of the elastic block.

The work done in producing the strain is stored up as potential energy in the strained material, and from the above it appears that the energy per unit volume of a uniformly strained elastic substance is equal to half the product of the stress into the strain. The criterion of perfect elasticity is that the ratio of stress to strain is constant, and this constant ratio is taken as the measure of the elasticity. Denoting it by N, we may write the potential energy of unit volume of an elastic substance in the three alternative forms

$$\tfrac{1}{2}ps=\tfrac{1}{2}\frac{p^2}{N}=\tfrac{1}{2}Ns^2.$$

In the case of electrical energy, it seems natural to regard the energy of an electrostatic field as potential, and the energy corresponding to the movement of lines of force, that is, the magnetic energy of a current, as kinetic. We will consider first electrostatic energy. If the charge of a field of capacity S is Q, and the difference of potential between the boundaries is $V-V'$, the total energy may be written

$$\tfrac{1}{2}(V-V')Q=\tfrac{1}{2}(V-V')^2S=\tfrac{1}{2}\frac{Q^2}{S}.$$

To obtain expressions for the energy of unit volume, we will consider the energy of a portion of a field bounded by area a of two parallel plane surfaces at a distance l from each other. The energy in this case is

$$\tfrac{1}{2}(V-V')^2\frac{aK}{4\pi l}=\tfrac{1}{2}\left(\frac{V-V'}{l}\right)^2\cdot\frac{K}{4\pi}v,$$

where v is the volume of the field $=al$. If we put f for the intensity of electric force in the field, or $(V-V')/l$, and express the charge as $Q=\sigma a=Da$, where σ is surface-density and D is electrostatic induction (**44**), we get the three following expressions for the energy of unit volume of the field—

$$\tfrac{1}{2}fD=\tfrac{1}{2}\cdot\frac{f^2K}{4\pi}=\tfrac{1}{2}\cdot\frac{4\pi D^2}{K}.$$

The electric force f per unit area of the field, and D the electrostatic induction, may be looked upon as representing respectively electric stress and electric strain, and their ratio, $\dfrac{4\pi}{K}$, then appears as the electric elasticity of the field. From this point of view, the three shapes in which we have written the expression for electric energy per unit volume correspond exactly with the three forms of the expression for elastic energy per unit volume.

As we have said, if the energy of an electric field is assimilated to the potential energy of elastic strain, the energy of a current must be of the nature of kinetic energy, and must, therefore, be expressible in a form comparable with $\tfrac{1}{2}mc^2$, where c is a velocity. An electric current implies the motion of lines of force (**306**), and the strength of a current is an expression for the number of lines which pass a given section of the circuit in unit of time. We may, therefore, take c as equal to or proportional to C. It remains to consider what we are to take as representing the inertia m. To investigate this point we will again choose a case that admits of

easy calculation. Consider the energy of a length l of an electromagnetic cylinder (266) of cross-section a, having n turns of wire per unit length, through which a current of strength C is flowing. The energy is (304) $\frac{1}{2}LC^2$, where L, the self-inductance, is the magnetic flux through the circuit multiplied into the electric flux round the circuit, both for the case of unit current. This gives (280)

$$L = a \,.\, 4\pi n\mu \,.\, nl,$$

μ being the magnetic permeability of the medium inside the coil. The energy may, therefore, be written

$$\tfrac{1}{2}LC^2 = \tfrac{1}{2}al \,.\, 4\pi\mu \,.\, (nC)^2.$$

Here al is the volume of the electromagnetic field considered. The energy of unit volume is therefore

$$\tfrac{1}{2} \,.\, 4\pi\mu \,.\, (nC)^2,$$

where nC is the effective electric flux round unit length, and may be taken as the measure of electric velocity. In this case $4\pi\mu$ appears as the electric mass or inertia of unit-volume, or more simply as the electric density of the medium.

We may sum up the results of this discussion by saying that electrostatic energy and electromagnetic energy may be expressed as respectively the potential and the kinetic energy of the medium occupying the field, if $\frac{4\pi}{K}$ be taken as a measure of the electric elasticity of the medium, and $4\pi\mu$ as a measure of its electric density.

351. Dielectric Constant and Refractive Index.—We began by recalling that the luminiferous ether, in order to fulfil its function of transmitting light, must possess elasticity and density. The velocity with which vibrations travel through an elastic medium is equal to the square root of the ratio of the elasticity to the density of the medium. If, then, the values of the elasticity and density that we have arrived at for the medium concerned in electric and magnetic phenomena are the values of the elasticity and density of the luminiferous ether, we should expect the velocity of light to be equal to

$$\sqrt{\frac{4\pi}{K} \div 4\pi\mu} = \frac{1}{\sqrt{K\mu}}.$$

Now, although, as has been said, no means is known of measuring the separate values of K and μ, the product $K\mu$ is measurable in various ways, and many experiments agree in indicating for

its value for air $1 \div 9 \times 10^{20} \left(\frac{\text{centim.}}{\text{second}}\right)^2$ (411). The square root of the reciprocal of this, or 3×10^{10} centimetres per second, is almost exactly the mean of the best determinations of the velocity of light and electromagnetic waves in air.

For other transparent media, the value of the magnetic permeability, μ, does not differ to any important extent from its value for air, therefore the difference of the velocity of light in different media must, if our theory is right, depend upon differences of specific inductive capacity, K. That is, if u is the velocity of light in air and u' its velocity in another medium, and K and K' the corresponding inductive capacities, we ought to have

$$\frac{u}{u'} = \sqrt{\frac{K'}{K}}.$$

The numbers that are recorded as the values of the specific inductive capacity for various substances are really comparative values referred to air, and therefore give at once the ratio K'/K. On the other hand, the undulatory theory of light indicates that the ratio u/u' of the velocities in air and any other medium is the *index of refraction*, n, of light from air into this medium. We thus come to the further conclusion that the *dielectric coefficient* of a transparent substance (that is, its relative specific inductive capacity compared with air) should be equal to the *square of the index of refraction* of light from air into that substance.

In considering this statement it should be carefully borne in mind that this correspondence must be expected to hold only when both the constants (K and n) are measured for waves of the same frequency. The great disparity between the wave-lengths of light (or even of radiant heat) and electric waves makes the direct comparison impracticable in general, and in seeking for evidence of correspondence some extrapolation formula must be employed. But here a great difficulty presents itself; it is not safe to assume that the refractive index varies in a continuous manner with change in wave-length. Even within the range of the visible spectrum substances are known (*e.g.* iodine vapour, fuchsine) whose refractive power varies abnormally, being greater in the red than in the green. This abnormality, which is exceptional in the visible spectrum, is found to be general when larger ranges of wave-length are examined. Thus for quartz the index for the visible rays is about 1·5, and it decreases to about 1·3 for rays whose wave-length is about ·0008 cm.; but for rays of wave-length ·0056 cm. it has *risen* to 2·2. Now the dielectric coefficient of

quartz for electromagnetic waves has been found to be 4·6, the square root of which is 2·12. Thus we see that for this substance, on which an extended series of measurements has been made, agreement is found to be at least possible, although a hasty examination of the visible spectrum would lead one to conclude that agreement was impossible.

In some cases, however, the refractive index has been obtained for electromagnetic waves themselves. Thus, Lampa has shown by direct measurement of the index for waves 8 mm., 6 mm., and 4 mm. long, that the corresponding refractive indices of water are 8·97, 9·40, 9·50, the squares of which numbers lie between 80 and 90; and this is about the value of the dielectric constant obtained by other observers for long waves. Thus, again, the theory receives a confirmation which is unexpected when the index of water for visible rays (1·33) is alone regarded. In fact, where the theory has been thoroughly tested, such good agreement is found that we may reasonably suppose that any apparent discordance in other cases arises merely from the scantiness of our experimental data.

352. Dielectric Currents.—In the foregoing part of this chapter the aim has been to present the reader with a general account of electric waves and their propagation. Maxwell, as the result of a brilliant generalisation, had predicted their possibility as early as 1865. We shall now give an account of this generalisation and of the argument on which his expectation was based.

We have seen that, when an electric current divides along two paths at a point in a circuit, the algebraic sum of the currents flowing to the point is zero (**118**). Consider now the case shown in Fig. 298, where a condenser has a resistance in parallel with it. If the current C is constant the potential difference between the terminals of the condenser is constant, and so also is its charge: the whole current, therefore, flows along the parallel conductor. If the current C change in any way this is no longer the case; for any change will cause a variation of the potential difference between the armatures of the condenser, and consequently the charge of the condenser will vary also. If Q is the instantaneous value of this charge, and c_1 the current in the parallel resistance, the following relation is now satisfied at the point A—

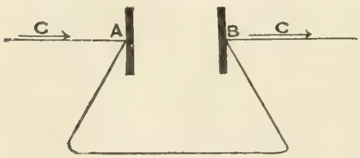


FIG. 298.

$$C = c_1 + \frac{dQ}{dt}.$$

Since a corresponding equation holds at B the result is the same—as far as Kirchhoff's law is concerned—as if a current of value, dQ/dt, were flowing from A to B *through the condenser*. Now there is no current there in the ordinary sense, but there is a change in progress in the dielectric, viz., the change in the value of the electric induction there. The value of this induction at any point varies proportionately to the charge on the boundaries. If we imagine that there is an actual current in the dielectric equal at any point to $\frac{dD}{dt}$ where D is the electric induction there, and flowing along the lines of induction, then the tubes of induction will also represent tubes of flow of this current, and these will be continuous with the tubes of flow in the conductor; Kirchhoff's law will then apply to every part of the circuit. This in itself would be of little importance. Maxwell, however, went further, and assumed that the current above defined *possesses the same magnetic properties as the currents in conductors*. This assumption is indeed the chief peculiarity of the theory given in his treatise. He called the supposed current a *displacement current*; in order, however, not to tacitly imply any particular theory of the nature of the actions which go on in the electric field we shall refer to it as the *dielectric current*. No experimental evidence of its existence was obtained in Maxwell's lifetime (1831–1879); but the phenomena which have been described in this chapter abundantly justify us in believing in its reality.

In discussing the flow of current along a cable (**341**) we stated that the difference of the currents simultaneously entering and leaving any element was represented by the rate of charging of the condenser, of which the element is one coating. In accordance with the definition just made this will be otherwise described by saying that the total current entering the element is equal to the total current leaving it, the dielectric current being in each case included.

353. Propagation of Plane Waves.—In order to illustrate the application of the above principle to cases where no conductor is present, and where in consequence the current must be wholly a dielectric one, we shall consider the case of the propagation of a plane electromagnetic wave. In such a wave the electric force, F, and magnetic field, H, are at right angles to one another and also at right angles to the direction of the propagation—that is to say, they lie in the plane of the wave-front. They have uniform values for all points in a wave-front, but their values change in

passing from point to point in the direction of propagation. If we cut out a small piece of the medium lying between two successive positions of the wave-front, the values of these quantities may be represented as in the following diagram (Fig. 299), the relative directions having been chosen from comparison with an ordinary conducting circuit (Fig. 263).

Since by assumption the electromagnetic laws are to be the

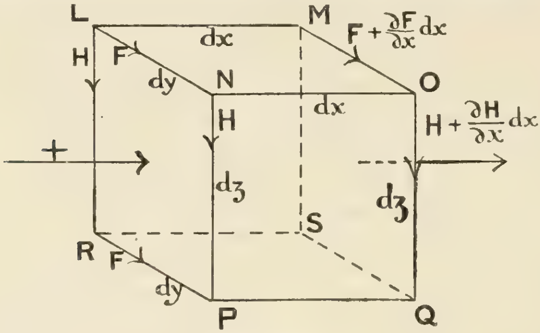


FIG. 299.

same as we have previously shown to be true for conducting circuits, we shall first apply Faraday's law of induced electromotive force to the rectangle LMON, viz.:—Total electromotive force in clockwise direction round the circuit equals rate of decrease of the total induction downward through it. But the total electromotive force is equal to the line-integral of the electric force round the circuit. Hence

$$\left[F+\frac{\partial F}{\partial x}dx\right]dy - Fdy = -\frac{\partial N}{\partial t} = -\frac{\partial}{\partial t}(\mu H)dxdy,$$

which if μ is constant becomes

$$\frac{\partial F}{\partial x} = -\mu\frac{\partial H}{\partial t} \quad . \quad . \quad . \quad . \quad (1).$$

Again (265), the line-integral of magnetic field round a closed curve $=4\pi\times$ total current through it. Consider the rectangle NOQP: the current through it is wholly a dielectric current (since no conductor is present), and its total value is $-\frac{K}{4\pi}\frac{\partial F}{\partial t}dxdz$ entering the face from the near side; the line-integral of magnetic field is wholly contributed by the two side paths OQ and PN, and has for value

$$\left(H+\frac{\partial H}{\partial x}dx\right)dz - Hdz.$$

Equating these two expressions and dividing by $dxdz$ we obtain

$$\frac{\partial H}{\partial x} = -K\frac{\partial F}{\partial t} \quad . \quad . \quad . \quad . \quad (2).$$

Equations (1) and (2) are the fundamental equations for a plane wave. We have come across similar equations before in connection with a cable (**341**), and may at once write down the corresponding solutions, merely making the necessary change from symbol to symbol. Hence, for a forwardly propagated simple wave

$$\begin{aligned} H &= KvF \\ F &= \mu vH \end{aligned} \qquad v = \sqrt{\frac{1}{\mu K}}.$$

For a backwardly propagated wave the same equations hold if v be given an equal negative value. Thus waves are propagated through the dielectric with the same velocity as that which we proved (**341**) for the propagation of a current-wave in a conductor of negligible resistance. Since the forward flow of energy is in each case in the dielectric, and the conductor when present merely serves to degrade some of the energy into heat, diverting it out of the dielectric to that end, we see that according to this theory the two cases are essentially identical.

354. Reflection of Plane Waves.—If the wave be incident normally upon a plane conductor, we should expect reflection to occur. The simplest case is that in which the conductor has practically perfect conductivity: in this case the electric force in the forward wave must at the mirror be reduced to a permanently zero value by the reflected wave, for no electric stress could exist in such a conductor. Let F_1 and H_1 be values for the incident wave at the plane of the conductor, and F_2 and H_2 for the reflected wave, then since $F_1 = \mu v H_1$, and $F_2 = -\mu v H_2$

$$F_1 + F_2 = \mu v H_1 - \mu v H_2 = 0,$$

or

$$H_1 = H_2.$$

Hence, while the resultant electric force is there brought to zero the magnetic field, $H_1 + H_2$, is doubled. In other words, while the electric force is reflected with reversal of phase, the magnetic field is reflected without any alteration. Students of sound will recall that when sound is reflected from the open end of an organ pipe, the same statements can be made with regard to the excess of pressure and velocity of the particles respectively.

In order to determine the values of F and H at other points, it

is necessary to superpose two wave-trains travelling in opposite directions, and of such relative phases as to satisfy the values already determined (comp. Fig. 293, **343**). In other words, if we write

$$F = A \sin(\omega t - qx) + B \sin(\omega t + qx + e),$$

and take the mirror as origin for x; then, since $F = 0$ when $x = 0$, we have to satisfy the equation

$$A \sin \omega t + B \sin(\omega t + e) = 0,$$

and this can be permanently satisfied only if

$$B = -A \quad \text{and} \quad e = 0;$$

whence

$$F = A\,[\sin(\omega t - qx) - \sin(\omega t + qx)],$$
$$= -2A \cos \omega t \sin qx.$$

This equation represents what is called a stationary undulation. The characteristic of such an undulation is that there are positions for which F is permanently zero, viz., wherever $\sin qx = 0$; these are called nodes. Between each two nodes there is a position where the range of variation of F is a maximum, viz., wherever $\sin qx = 1$; these are called antinodes. The wave in an organ pipe or vibrating stretched string when sounding is of this type.

In order to find the corresponding values of H we must make use of equation 2, which gives

$$\frac{\partial H}{\partial x} = -K\frac{\partial F}{\partial t} = +2AK\omega \sin \omega t \sin qx,$$

whence by integration

$$H = \int 2AK\omega \sin \omega t \sin qx\, dx$$
$$= -\frac{2AK\omega}{q} \sin \omega t \cos qx$$

where the integration constant is ignored, since it would be non-periodic with respect to x. This is similar to the equation for F, but the positions of the nodes and internodes are interchanged. It may be pointed out that in an organ pipe the excess pressure is permanently zero at the positions where the particle-velocity undergoes maximum variations, and *vice versâ;* the two phenomena are strictly analogous.

These considerations will make clearer the description of the experiments in **343**.

354.* Reflection at the Interface between two Dielectrics. —When the second medium is a dielectric as well as the first the wave transmitted through the surface must be taken into account. Let the constants of the second medium be μ', κ', v'. Then, still restricting the problem to the case of normal incidence, we have the following relations:—

Forward wave in first medium . .	$F_1 = \mu v H_1$
Reflected wave in first medium . .	$F_2 = -\mu v H_2$
Forward wave in second medium .	$F_3 = \mu' v' H_3$

The wave in the second medium will go on unchanged until another interface is reached when reflection will again occur and the new wave which thus arises will ultimately strike the first surface, again setting up new waves of transmission and reflection. This additional complication will not enter for a long time if the second medium is very thick; we will suppose the thickness to be practically infinite and that the complication does not enter at all. The three waves referred to above, then represent the whole phenomenon. These waves are not independent, for the usual boundary conditions must be satisfied by them. These conditions give

$$F_1 + F_2 = F_3$$

$$H_1 + H_2 = H_3$$

that is, the tangential components of electric and magnetic forces are continuous across the boundary. Inserting the values of H_1, H_2, and H_3 in terms of F_1, F_2, F_3 and eliminating F_3 and F_2 in turn, there result the equations

$$F_2 = \frac{\mu' v' - \mu v}{\mu' v' + \mu v} F_1,$$

$$F_3 = \frac{2\mu' v'}{\mu' v' + \mu v} F_1.$$

These equations give the values of the reflected and transmitted waves in terms of the incident wave. They indicate that the waves fluctuate in simple proportion one to another, the transmitted and incident waves being in the same phase at any instant, whereas the reflected and incident waves are in the same or in opposite phases according as $\mu' v'$ is greater or less than μv.

*

If $\mu = \mu'$ as is usually the case, at least to a considerable degree of accuracy, these equations become

$$F_2 = \frac{v' - v}{v' + v} F_1,$$

$$F_3 = \frac{2v'}{v' + v} F_1.$$

Denoting the densities of the electrostatic energy by E_1, E_2, and E_3 we have

$$\frac{E_2}{E_1} = \frac{\kappa F_2{}^2}{\kappa F_1{}^2} = \left(\frac{v' - v}{v' + v}\right)^2,$$

$$\frac{E_3}{E_1} = \frac{\kappa' F_3{}^2}{\kappa F_1{}^2} = \left(\frac{2v}{v^1 + v}\right)^2,$$

remembering that

$$\frac{\kappa'}{\kappa} = \frac{v^2}{v'^2}.$$

Precisely the same relations exist between the densities of the magnetic energy in the respective waves.

Now the intensity of radiation is measured by the energy passing in unit time through unit area drawn parallel to the wave front. This amount of energy is calculated by multiplying the energy-density by the velocity of propagation. Hence we have, denoting the intensities by I_1, I_2, and I_3,

$$\frac{I_2}{I_1} = \frac{E_2 v}{E_1 v} = \left(\frac{v' - v}{v' + v}\right)^2 = \left(\frac{n - 1}{n + 1}\right)^2,$$

$$\frac{I_3}{I_1} = \frac{E_3 v'}{E_1 v} = \frac{4vv'}{(v' + v)^2} = \frac{4n}{(n + 1)^2}$$

where n is written for v/v' and thus stands for the refractive index for the radiation in passing from the first to the second medium.

It should be observed that $I_2 + I_3 = I_1$; or, in words, the whole of the electrostatic energy which reaches the interface in unit time is equal to the sum of the two amounts which leave it in the same time.

These expressions agree with those obtained by Fresnel from a non-electrical theory of light.

355. Electric Convection.—When an electrically charged body moves, the lines of electric induction in its neighbourhood must also move; and in general, therefore, the value of the electric induction at any point must vary. This is equivalent to saying that there is a dielectric current associated with the movement, and therefore also a corresponding magnetic field. Experiment has shown that a magnetic field results, even when the motion takes place in such a way as not to involve a variation in the electric induction. This action was detected by Rowland, who caused a disc of ebonite (whose face had been gilt) to rotate rapidly between two discs of glass which were also gilt on the faces nearest the ebonite disc. These fixed discs were intended to shield a magnet suspended immediately above from electrostatic action, and also from the mechanical action of currents of air produced by the rotating disc. When the ebonite disc was positively charged, a deflection of the needle took place in the same direction as it would have been under the action of an electric current flowing parallel to the motion of each point; on the other hand, the displacement took place in the opposite sense when the disc was charged negatively. The action occurred both when the ebonite disc was completely gilt and also when the gilding was removed along radial sectors.

The effect obtained was very small, but seemed to be definite. Its magnitude was consistent with assuming that an electric charge of e coulombs moving with velocity v centimetres per second, produces the same magnetic effects at external points as one centimetre of a current of ev amperes.

The equivalent current is known as a *convection current.*

355.* Calculation of the Corresponding Magnetic Field.—To illustrate the calculation of the magnetic field which surrounds a moving charged body we will take the case of a charge on a very small body moving with velocity, v, in a straight line. Consider any circle, of radius a, whose centre is on the line of motion and whose plane is perpendicular thereto and is at distance x from the charge (Fig. 299A). The total flux of electric induction which threads the area enclosed by the circle at any moment is the solid angle subtended by the circle $\times e/4\pi$, or $\frac{e}{2}(1-\cos\theta)$. As e moves from left to right the solid angle, and the plane angle θ, increase; and therefore the total flux also increases, the rate of increase

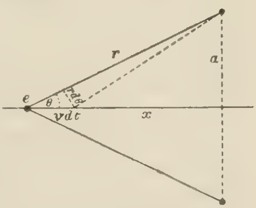


FIG. 299A.

being $\frac{e}{2}\sin\theta\frac{d\theta}{dt}$. But $\frac{rd\theta}{vdt}=\frac{a}{r}$; whence $\frac{d\theta}{dt}=\frac{av}{r^2}$; thus the total

dielectric current (*i.e.* the sum of the currents through each part of the area) is equal to

$$\frac{eav \sin\theta}{2r^2}.$$

The magnetic field corresponding to this current being at right angles to this, it may be represented by circles concentric to the line of motion; and since the line integral of the magnetic field round any one of these circles is equal to $4\pi \times$ total dielectric current enclosed we have

$$H.2\pi a = 4\pi\left(\frac{eav \sin\theta}{2r^2}\right)$$

or

$$H = \frac{ev. \sin\theta}{r^2}.$$

Comparing this with Laplace's rule (**261**), it is seen that ev takes the place of the expression Cds; that is, it is equivalent to an elementary length ds of a current C. We have supposed in the above that the electric induction is the same as if the charge were at rest. But since at any point in space the magnetic field changes with time there will be an induced electric field surrounding the charge; and this will alter the distribution of the lines of the resultant electric induction. Consequently the magnetic field will itself be modified to an extent which increases as greater speeds are considered. But for ordinary velocities this reaction is negligible.

Such an element of current with its associated field of electric induction may be taken as a realisable current element as distinct from the hypothetical one imagined by Laplace. This mode of regarding it is very important in connection with that view of the electric current according to which it is regarded as consisting of moving, minute, charged particles. Each such particle in the conductor produces an electrostatic field whether it is at rest or in motion; but during motion it produces also a magnetic field; and the totality of electromagnetic effects produced is the resultant of those due to the several particles. In a conducting wire these particles consist of positive and negative bodies in nearly equal amount at each place as long as the difference of potential between different parts of the conductor remains moderate and the conductor is uncharged on the whole; thus the resultant electrostatic effects may be only small. But if there are n moving electrons per unit volume and the motion in the case of the positively charged ones in the direction of x is in excess by v cms. per second, or, in the case of negatively charged ones, is in defect in the same direction by this amount on the average: then, through any unit area perpendicular to the direction of x there will be a current, C, equal to

nev where e is the average charge on a positive particle. In a length dx of the conductor the number of moving particles is ndx; and, $Cdx = (ndxe)v$. This group of moving charges with its accompanying magnetic field, calculated according to the method of this section, constitutes what is known as a rational (or realisable) current element.

355. Energy of a Moving Charge.**—The total energy corresponding to the slow motion of a charge can be easily calculated for the case in which it resides on a sphere. Let b be the radius of the sphere. Consider first a spherical shell, of radius r and thickness dr, concentric with the moving charged sphere, and on this take a circular belt of radius a (Fig. 299A) and breadth $rd\theta$, in the plane of the figure. The cross-section of the belt is then $rdrd\theta$ and its circumference is $2\pi a$ $2\pi r \sin\theta$, whence its volume is $2\pi r^2 dr \sin\theta d\theta$. By the last section, the magnetic field at any point of this belt is $H = ev\sin\theta/r$ and in all cases (**305**) magnetic energy per unit volume is $\mu H^2/8\pi$. Hence for the energy of the belt considered we have

$$\mu\frac{e^2v^2}{4r^2}\sin^3\theta d\theta.dr = -\mu\frac{e^2v^2}{4r^2}(1-\cos^2\theta)d\cos\theta.dr.$$

To find the total energy through all space we integrate first for all values of θ from 0 to π keeping r constant; and then for all values of r from b to infinity. Since $-\int_0^\pi(1-\cos^2\theta)d\cos\theta$ is equal to $\frac{4}{3}$, the first integration gives $\mu\frac{e^2v^2}{3r^2}dr$; the second gives $\mu\frac{e^2v^2}{3b}$.

Besides this magnetic energy the sphere possesses electrostatic energy of amount sensibly unchanged (for slow motions) from the value possessed when at rest. Hence, when in slow motion, the energy due to its charge is greater than when at rest by the amount $\mu\frac{e^2}{3b}v^2$. If the charge is on a material body of mass m we must include the energy of its motion $\frac{1}{2}mv^2$; the total amount is seen to be $\frac{1}{2}\left(m+\frac{2\mu e^2}{3b}\right)v^2$. The charge may be considered, therefore, as increasing the effective mass of the body by an amount proportional to the square of itself. The term $\frac{2\mu e^2}{3b}$ is called the *electromagnetic mass* of the charged body. When rapid motions are considered, this mass, like the magnetic field itself, is not accurately represented by the expression here deduced.

These equations are true whether electromagnetic or electrostatic units are employed; but if electrostatic units are used for expressing e, the value of μ for air must be taken as (**395**) $1/(9\times10^{20})$ C.G.S. units.

CHAPTER XXIX

GALVANOMETERS

356. Galvanometer.—A galvanometer, in the strict sense of the word, is an instrument which serves to measure the intensity of a current by the action which it exerts on a magnetic needle.

It consists essentially of a frame on which wire is coiled in a vertical plane, and of a magnetic needle suspended horizontally at the centre of the coil (Fig. 300). The plane of the wire is placed in the magnetic meridian, and therefore the needle is parallel to it in its position of equilibrium. The field due to the earth and that due to the current near the middle of the coil are thus at right angles.

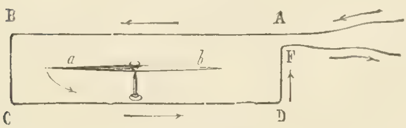


FIG. 300.

Let us suppose that the field due to the current is uniform in the region which is occupied by the needle, and let G be its strength for unit current; it will be GC for a current of strength C, and under the action of the two fields the needle will come to rest in a position which makes an angle α with its original direction. If H is the horizontal component of the magnetic field of the earth and M the moment of the needle, the angle α is given by the equation of equilibrium

$$MH \sin \alpha = MGC \cos \alpha,$$

from which

$$C = \frac{H}{G} \tan \alpha.$$

Thus, in the case in which the two fields are uniform and at right angles, the deflection is independent of the magnetic moment M of the needle, and therefore of its shape and magnetisation, and *the tangent of the deflection is proportional to the strength of the current.*

This condition that the field be uniform is of prime importance for the galvanometer. It dispenses with any empirical graduation.

It is easily realised if small needles and small angles of deflection are used. This is the case with all galvanometers of modern construction. The inconveniences of small deflections are amply compensated by the accuracy with which they can be measured by the method of reflection (**84**).

The scale reading gives directly tan 2α. If the deflection does not exceed 3 to 4 degrees, the error may usually be neglected which arises from taking $2 \tan \alpha = \tan 2\alpha$, and therefore in assuming that the current-strength for a fixed distance of the scale is proportional to the displacement $x-p$ of the image. For a deflection of 2° the error due to this assumption is less than 1 part in 800.

357. Absolute Measurement of Currents—Tangent Galvanometer.—The absolute measurement of current-strength depends on the measurement of G and H. We know how H is obtained (**243**). The value of the quantity G, which we will call the *galvanometer-constant*, is deduced from the dimensions of the coil of wire. This requires that the windings should be accurately circular, and wound with such regularity that the radius can be exactly measured. The radius should, moreover, be very large in comparison with the dimensions of the needle, if we are to assume that the field in which the needle hangs is uniform.

A much more uniform field is obtained by using two equal circular coils, placed parallel to each other at a distance equal to their common radius (**264**).

A current of unit strength going once round a circle of radius a, produces a magnetic field which, at any point on the axis of the circle, is directed along the axis, and at distance x from the centre is equal to (**262**)

$$\frac{2\pi a^2}{(a^2+x^2)^{3/2}}.$$

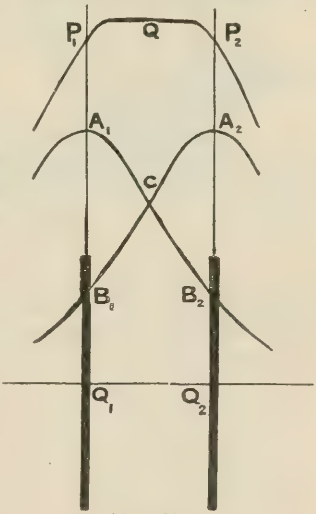


FIG. 301.

The values which this expression assumes for various values of x are shown graphically in Fig. 301, by the ordinates of the curve A_1CB_2. The value $x=0$ corresponds with the centre of the circle; the field here is greatest, and is represented by the maximum ordinate Q_1A_1. As x increases, the magnetic field, as shown by

the curve, decreases, at first slowly, then more and more rapidly until x becomes $= \frac{1}{2}a$, after which the rate of decrease of field again gets slower and the representative curve becomes less steep. At the point of greatest steepness, C, the curve changes from being concave towards the axis to being convex, and at this point there is no curvature either way. In other words, at a distance along the axis equal to half the radius of the circle, the magnetic field decreases for a little way at a uniform rate. Hence, if two equal circles or circular coils are placed so that their axes coincide, and at a distance apart equal to their common radius, the magnetic field on the axis at a point half-way between the circles is uniform, since the decrease of field corresponding to a small increase of distance from one circle is compensated by the equal increase of field due to decrease of distance from the other. The curve P_1QP_2, which represents the magnetic field due to two equal circular currents placed as described above, exhibits this result. The ordinates of this curve are the sums of the corresponding ordinates of the curves A_1CB_2 and A_2CB_1 representing the fields due to each of the two circles acting separately. For a distance on either side of the middle point equal to one-tenth of the common radius, the field decreases only by about one part in 9000, so that, with two circles, each of 25 centimetres radius, the magnetic field is almost exactly uniform for a distance of 2·5 centimetres on each side of the middle point.

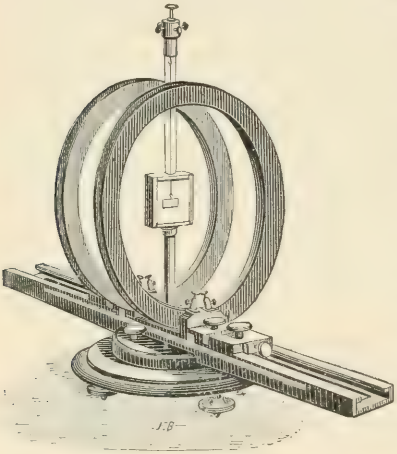


FIG. 302.

Fig. 302 represents a galvanometer constructed upon this principle, which is due to Helmholtz. With an instrument of this kind, having in all n turns of wire arranged in two equal coils of mean radius a, with a distance $2x = a$ between the mean planes of the coils, the magnetic field at the centre due to a current of unit strength in the wire is

$$G = \frac{2\pi n a^2}{(a^2 + x^2)^{3/2}} = 4{\cdot}496\frac{n}{a}.$$

If the circles are accurately set with their common axis perpendicular to the magnetic meridian, the field $= CG$ due to the current is at right angles to the earth's horizontal field, H, and the resultant field makes with the meridian an angle, α, given by the relation

$$\tan\alpha = \frac{CG}{H},$$

and the axis of a small delicately suspended magnet at the centre of the instrument sets itself in this direction. The strength of a current which causes a deflection, α, of the magnet is therefore

$$C = \frac{H}{G}\tan\alpha = \frac{Ha}{4{\cdot}496n}\tan\alpha = 0{\cdot}2224\,\frac{a}{n}\,H\tan\alpha.$$

For example, suppose there are 60 turns of wire, 30 on each circle, that the mean radius is 20 centimetres, and $H = 0{\cdot}1825$: then the current which deflects the magnet through 45° is

$$C = 0{\cdot}01353\ (\text{C.G.S.}) = 0{\cdot}1353 \text{ ampere}.$$

If the deflections are observed by the method of reflection, with a scale at a distance of 1 metre, a movement of the reflected ray through 1 centimetre indicates an angle α, such that $\tan 2\alpha = \frac{1}{100}$. In such a case we may take $\tan\alpha = \frac{1}{200}$, and the strength of the current is

$$\frac{{\cdot}1353}{200} = {\cdot}0006765 \text{ ampere}.$$

In the case of a single circle with the magnet hung at the centre, the value of x in the above formula $= 0$, and the field at the centre due to a unit current is given by the simpler formula

$$G = \frac{2\pi n}{a},$$

and the strength of a current which produces a deflection, α, becomes, in C.G.S. measure,

$$C = \frac{Ha}{2\pi n}\tan\alpha.$$

358. Sine Galvanometer.—Fig. 303 represents a sine galvanometer. The circle is set at first in the plane of the magnetic meridian, and, when a current passes, it is turned about a vertical diameter in the direction of the deflection, until it has again the

same position in regard to the needle. This condition is indicated when the pointer *cd* fixed to the needle has the same position as at first on the graduated circle A. Let α be the angle through which the instrument has been turned, read off on the horizontal circle, C, then since the couple due to the current, MGC, balances the action of the earth, $MH \sin \alpha$, we have

$$MGC = MH \sin \alpha,$$

whence

$$C = \frac{H}{G} \sin \alpha.$$

The current-strength is proportional to the sine of the angle through which the instrument is turned, whence the name of the apparatus. It is not necessary that the field due to the current should be uniform.

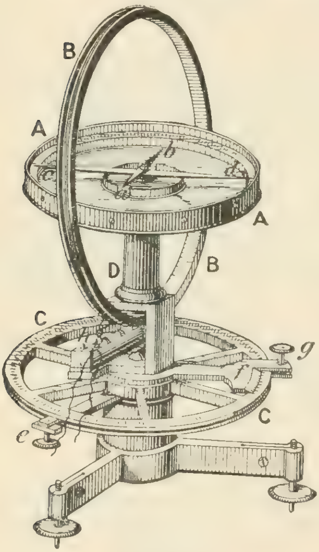


FIG. 303.

By this apparatus current-strengths greater than H/G cannot be measured.

359. Shape of the Coil.—In many galvanometers, no attempt is made to calculate the constant G directly; sensitiveness is the only character looked to. This is greater according as the value of $\tan \alpha$ is greater for the same current. The sensitiveness is thus measured by the ratio $\frac{G}{H}$, and therefore is greater as G increases and H is less.

The magnetic field at the centre due to unit current in a circle of radius a is $\frac{2\pi}{a}$; the length of wire being $2\pi a$, the field due to unit-length of the wire, or the specific effect, is $\frac{1}{a^2}$, and therefore varies inversely as the square of the radius. We are thus led to diminish as much as possible the radius of the windings and therefore the length of the needle.

The shape to be given to the coil, so as to get the greatest advantage from the wire used, is determined by the condition that the specific effect is to be the same at all parts of the outer surface;

for, if it were otherwise, there would be an advantage in transferring some of the wire from the position which it occupies to another where the specific effect is greater. This leads to the adoption for the meridian section of the coil of a figure approximately like that obtained by squeezing a circle along its vertical diameter (Fig. 304).

For a circle of radius r the magnetic field along the axis at a point O, distant p = OP from the circle (Fig. 304) is $\frac{2\pi r^2}{p^3}$ (262), and the field due to unit-length of wire is $\frac{r}{p^3}$. The condition just stated, that this quantity is to be constant $\left(=\frac{1}{N^2} \text{ say}\right)$ at all points on the surface of the coil, may be written

$$p^2 = N^2 \sin\theta.$$

For $\theta = 90°$, this evidently gives $p = N$.

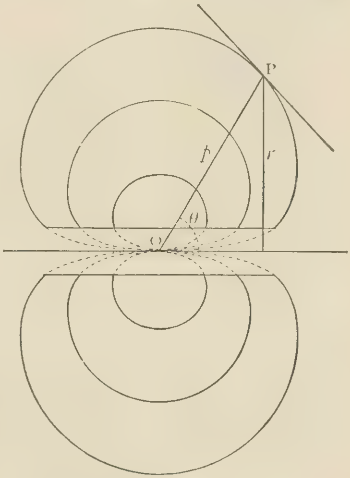


FIG. 304.

The figure gives successive curves for values of N, which increase in arithmetical progression; the dotted parts of the curve correspond to the central space left for the magnet, which is hung at O, with its axis perpendicular to the plane of the figure. In practice, the cross section of the coil has not the exact shape represented in the figure, but is a rectangle agreeing as nearly as may be with the theoretical shape. In any case, the magnet is at the centre of a long coil, so that the small part of the field in which it moves may be regarded as fairly uniform.

Considerations similar to those of (279) show that the action exerted by a coil of given shape is independent of the size of the wire if the density of the current is constant. But in the case of a given electromotive force, the current depends on the resistance of the wire, and the external action is a maximum when the resistance of the galvanometer-coil is equal to that of the rest of the circuit.

360. Compensating Magnet.—Two methods may be employed either separately or simultaneously to diminish the value of H.

The effect of the earth may be compensated by that of a magnet which produces in the place occupied by the needle a sensibly uniform field in the opposite direction to that of the earth. This magnet, which is called the *compensating* or *adjusting magnet*, is usually supported on a rod (Fig. 305), to which it is fixed by a clamping screw, whereby it can be set at any desired height; and by means of a tangent screw, the rod itself, along with the magnet, may be turned about its axis.

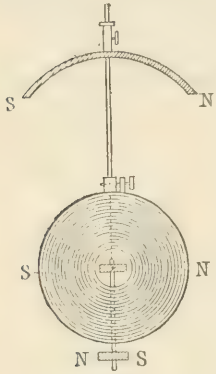


FIG. 305.

The direction of equilibrium of the needle is that of the resultant of the two fields. Thus if OA (Fig. 306) is the direction of the terrestrial field, OB that of the field of the magnet, OC will be the direction of the needle. It will be seen that when the two fields are nearly equal, the resultant makes an angle of nearly 90° with the common direction. In this case the direction of the resultant, and therefore the position of equilibrium of the needle, is powerfully influenced by variations of the earth's field.

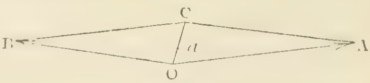


FIG. 306.

361. Astatic Needles.—The second method consists in using a system of astatic needles (**181**). The couple of the earth on the system may be reduced at will. On the other hand, if only one of the needles is placed inside the coil and the other outside, the couple due to the coil is slightly increased, as the couple which it exerts on both needles is in the same direction (Fig. 305).

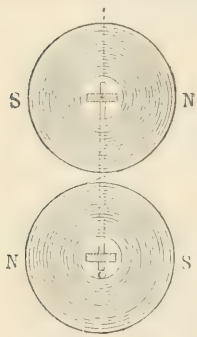


FIG. 307.

But a more symmetrical apparatus and greater deflective couple is obtained by using two coils, one over the other (Fig. 307), and placing one of the needles in the middle of each.

If the current passes in opposite directions in the two coils, the effects are obviously concordant.

362. Damping.—When a current corresponding to a permanent deflection α is started in a galvanometer, the needle receives an impulse which, neglecting passive resistances, would cause it to pass beyond its position of equilibrium by an angle α, so that the first throw would be equal to 2α. The needle then continues to oscillate until it has expended its energy against passive resistances.

The oscillations may at once be reduced to a very small amplitude if the current is broken after a time equal to one-third of the period of a single swing of the needle. From the laws of simple harmonic motion, the needle will in this time have traversed the arc $\frac{\alpha}{2}$, and will have acquired just sufficient velocity to bring it to α. The circuit is then closed again and the needle remains practically at rest.

An analogous method is used to bring it back to zero with no velocity; the circuit is broken during the first third of the period of swing, it is made again during the second, and is then permanently broken.

Even with these precautions the needle would oscillate for a long time before coming perfectly to rest if the oscillations were not damped. This may be effected either by increasing the resistance of the air, by light vanes attached to the magnet and oscillating with it, or by causing the motion of the magnet to generate induction currents in the conductors which surround it; these currents, according to Lenz's law (**299**), tend to oppose the motion. When the resistance of the circuit is small, the currents produced in the wire itself are sufficient to damp the motion; in galvanometers of high resistance the needle is sometimes surrounded by a mass of copper.

Whether the damping is due to the resistance of the air or to induced currents, experiment shows that the amplitude of oscillation decreases as the terms of a geometrical progression. We may infer from this that the retarding causes are at each instant proportional to the velocity of the needle. If α_0 α_1, α_2, . . . are the successive amplitudes, we have

$$\frac{\alpha_0}{\alpha_1} = \frac{\alpha_1}{\alpha_2} = \frac{\alpha_2}{\alpha_3} = \text{constant.}$$

Let the value of this constant be called e^λ, e being the base of the Napierian logarithms. The quantity λ, or the Napierian logarithm of the ratio of two consecutive amplitudes, is called the *logarithmic decrement* of the oscillations, and is taken as a measure of the damping. The duration, t, of the damped oscillation is a little greater than the duration, T, of the oscillation of the same system without damping.

By gradually increasing the damping, the oscillations may ultimately be suppressed, and an *aperiodic* or *dead-beat* motion obtained; the needle, when removed from its position of equilibrium, returns thither with a velocity which increases at first,

passes through a maximum, and finally becomes zero, when the needle attains its position of equilibrium.

363. Shunts.—The limits within which a galvanometer may be used may be greatly extended by means of *shunts*. This term is applied to a branch circuit connecting the two binding screws of a galvanometer, whereby a known fraction only of the current is allowed to pass through the galvanometer.

Let g be the resistance of the galvanometer, s that of the shunt, C the total current, and c the current in the galvanometer which produces the observed deflection; then, from the law of compound circuits (**119**),

$$C = c\frac{g+s}{s}.$$

The factor $\frac{g+s}{s} = m$, by which the galvanometer current must be multiplied to get the total current, is called the *multiplying power* of the shunt. Galvanometers are usually provided with three shunts, which have respectively the multiplying powers 10, 100, 1000, and therefore the resistances $\frac{1}{9}$, $\frac{1}{99}$, $\frac{1}{999}$, of that of the galvanometer respectively (Fig. 308).

The wire of the galvanometer being attached to the terminals G and G′, and those of the battery either to the same terminals or to others, P and P′, if one of the holes, A, B, C, is stopped with a plug, the corresponding shunt is introduced. When the plug is placed at O, the galvanometer is "short-circuited," and is thus protected against the passage of an accidental current.

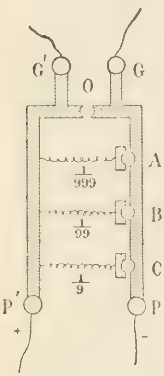


FIG. 308.

364. Differential Galvanometer.—If two identical wires are coiled close together, their effects are added together if currents in the same direction are passed through both; but their combined effect is the difference of their separate effects if the currents are in opposite directions, and therefore vanishes when the currents in opposite directions are equal.

A differential galvanometer should satisfy two conditions: the two circuits should exert equal effects on the needle, and should have the same resistance. The first condition can be tested by connecting the two coils in series, but so that the current traverses them in opposite directions; the second by connecting the coils in parallel, so that the currents again circulate in opposite directions; the needle should be stationary in both cases.

365. Thomson's Galvanometer.—The general plan of this is represented in Figs. 304 to 307. It has either one or two coils. The one-coil galvanometer is not, in general, astatic. The galvanometer with two coils is represented in Fig. 309. It has usually a considerable resistance—from 8000 to 10,000 ohms, or even more. The needles are about 8 millimetres in length, and usually consist of a series of small parallel bars (Fig. 310), thus giving a greater magnetic moment for the same weight.

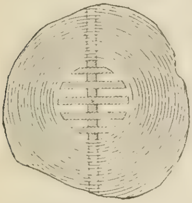


FIG. 310.

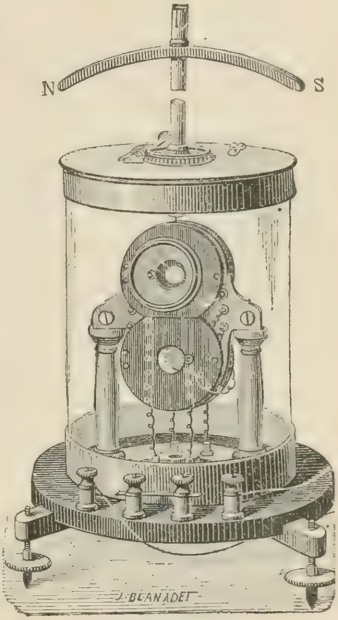


FIG. 309.

The small mirror is fixed to one of the sets of needles, or, better still, between the two coils, between which a sufficient space is left. The damping is effected by a thin plate of mica, that moves with the needle. The two coils can be connected in series or in parallel circuit, or the instrument can be used as a differential galvanometer.

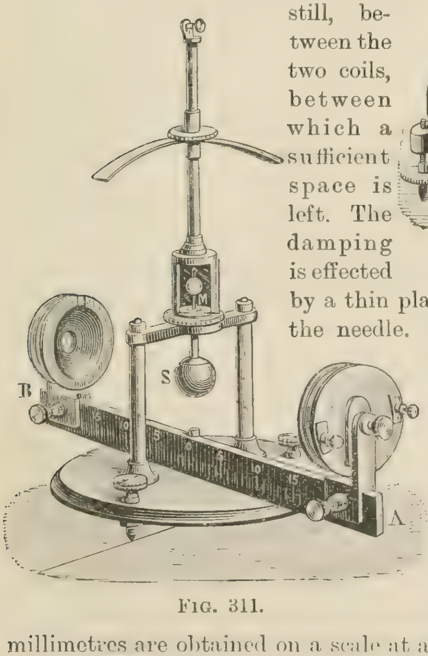


FIG. 311.

Thomson's galvanometers attain the highest degree of sensitiveness. With a current of a thousandth of a micro-ampere, deflections of 50 millimetres are obtained on a scale at a distance of a metre.

366. Wiedemann's Galvanometer.—Fig. 311 represents a galvanometer of the form introduced by Professor G. Wiedemann.

The needle is a horse-shoe magnet which oscillates in a cavity in a solid sphere of copper (Fig. 312). By this means a damping effect is obtained sufficient to make the instrument aperiodic. The coil is divided into two parts, which may be moved along a divided scale, and thus the sensitiveness may be varied at pleasure. There may be several sets of coils with wires of various thicknesses. The mirror is placed in a small chamber with a glass front, and can be turned in any direction without its being necessary to twist the rest of the instrument.

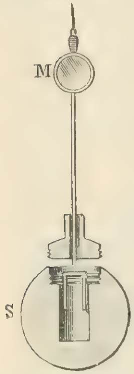


FIG. 312.

By means of an external magnet, the strength and direction of the field can be modified at pleasure, as in Thomson's galvanometer.

If the instrument is to be used as a ballistic galvanometer, the copper sphere is removed, to allow oscillation to take place freely.

367. Deprez and d'Arsonval's Galvanometer.—A rectangular coil (Fig. 313), movable about a vertical axis formed by two metal wires by which the current enters and leaves, is placed between the poles of a horse-shoe magnet, and a cylinder of soft iron, which becomes magnetised by influence, is supported inside the coil. When the current passes, the coil tends to set itself with its plane perpendicular to the field (**267**), and takes such a position that the deflecting couple due to the electromagnetic forces balances the torsion of the suspending wire. The damping is produced by induction currents due to the motion of the coil in the field, and as the field is very strong, the apparatus is nearly dead-beat when the circuit is closed by a low resistance. The damping effect is often increased by coiling the wire on a light silver frame.

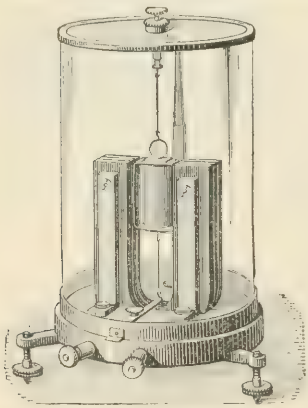


FIG. 313.

The deflections are read by means of a small mirror attached to the coil. If S is the total effective surface enclosed by the coil, and B the average strength of the induction, and T the coefficient of torsion of the wire, the equation of equilibrium is

$$SBC \cos a = Ta$$

from which is deduced

$$C = \frac{T\alpha}{SB \cos \alpha}.$$

For small deflections α is equal to $\sin \alpha$, and the current is sensibly proportional to the tangent of the deflection.

367.* String Galvanometer.—The use of the movement of a current-carrying conductor in a magnetic field as an indicator of the current was made first by Cumming in 1827, and again by Highton (in his Gold Leaf Telegraph) in 1846. In 1903 Einthoven devised his (so-called) String galvanometer, consisting of a very fine silvered-quartz fibre, or the finest procurable gold wire, stretched between the poles of a powerful electromagnet. The pole pieces are bored so that the fibre can be seen, and the lateral motion of it is observed through a microscope. With a field-strength of 20,000 units and a magnification of 600 times it is possible to observe the lateral motion of the fibre due to a current of 10^{-12} amperes. This galvanometer has the advantages that its suspended system is very nearly dead-beat and has a very short period—at least this is the case when the instrument is not pushed to its highest sensitiveness. The period can be shortened by increasing the tension of the fibre, and when it is short the instrument can be used in connection with the registration of a fluctuating current on a movable photographic plate.

A sensitive electrometer, of very small capacity, is obtained in a somewhat similar way by stretching a fine wire between two oppositely charged plates, its movement to or from the plates being observed.

368. Electrodynamometer.—In electrodynamometers the current is measured by the action which two conductors exert on each other. Other things being equal, the action is proportional to the product CC' of the two currents, or to the square of the current if the two conductors are each traversed by the same current. In the latter case the direction of the action is not changed when the direction of the current is reversed.

The apparatus usually consists of two coils, one fixed, the other movable, the movable coil being placed inside the fixed coil so that the centres coincide and the axes are at right angles to each other.

Under the influence of the current the axes tend to set parallel, and the moment of the couple which acts on the movable coil is equal to the product of its magnetic moment into the intensity of the field due to the fixed coil. For a current C, the moment of the movable coil formed of n turns of wire enclosing a total surface $S = n\pi r^2$ is equal to SC; the strength of the field is GC, G being the constant of the fixed coil. The moment of the couple is there-

fore SGC^2. In Weber's electrodynamometer the movable coil is supported by a bifilar suspension, which at the same time serves to convey the current. The angle of deflection α represents the equilibrium between the electrodynamic couple and that of the suspension, so that if T is the coefficient of torsion of the bifilar,

$$SGC^2 \cos \alpha = T \sin \alpha$$

and therefore

$$C^2 = \frac{T}{SG} \tan \alpha :$$

the square of the current is proportional to the tangent of the deflection. The action of the earth's field on the suspended coil can be eliminated by setting the suspended coil with its axis at right angles to the meridian and reversing the current in the fixed coil. The readings with the current direct and reversed correspond, if the earth's action is comparatively small, to the sum and difference of the effects of current and earth. The mean of the numerical values of these readings gives the approximate value of the effect due to the current alone, and may be inserted in the above equation.

369. Pellat's Absolute Electrodynamometer.—In Pellat's absolute electrodynamometer (Fig. 314) the movable coil is fixed at the end of a balance-beam, so that its axis is vertical, and it is placed in the uniform field due to a long regularly wound coil. If n_1 is the number of turns per centimetre of the fixed coil, we have

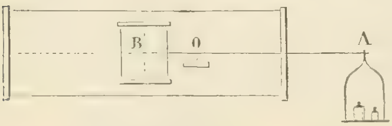


FIG. 314.

$$G = 4\pi n_1,$$

and the moment of the couple which acts on the beam is $4\pi n_1 SC^2$. This is counterpoised by the weight of a mass, p, placed at a distance, a, from the knife-edge; we have thus

$$pga = 4\pi n_1 SC^2,$$

from which we can deduce

$$C = A \sqrt{p},$$

where

$$A = \frac{1}{2\pi r}\sqrt{\frac{ag}{nn_1}},$$

a constant which is determined once for all by direct measurements.

370. Balance Electrodynamometers.—The strength of a current may also be deduced from the square root of the attraction or repulsion exerted between two parallel coils kept at a fixed

distance. In an apparatus of this kind, one of the coils is suspended to the beam of a balance, and the force between it and the fixed coil is weighed.

A very carefully made instrument on this general principle has been set up at the National Physical Laboratory, Bushy. It was designed by Professors Viriamu Jones, Ayrton, and Mather. In order to make it possible to measure the dimensions of the coils with great accuracy they are made by winding bare copper-wire in a single layer on accurately turned marble cylinders. There are two fixed cylinders set with their axes vertical, and each wound with two coils, separated by a short interval, and so connected that when they are in use the current circulates round them in opposite directions. Two movable coils hung from opposite ends of the beam of a delicate balance are suspended, one inside each of the fixed cylinders, so as to be coaxal with it and at such a height that its mean plane is half-way between the ends of the coils wound on the corresponding cylinder. By this arrangement, when a current is sent through all the coils, the two coils on each of the fixed cylinders exert a vertical force in the same direction on the coil suspended within it, and these forces, being in opposite directions on the two suspended coils, tend to turn the balance the same way. The strength of the current is deduced from the change of the weight needed to restore the equilibrium of the balance when a current is sent through the coils first in one direction and then in the other. The calculation is founded on the equation

$$f = nC^2(M_2 - M_1)$$

where f is the force on one suspended coil exerted by one of the coaxal fixed coils, M_1 and M_2 the coefficients of mutual induction between the fixed coil and the circles representing the two ends of the suspended coil, n the number of turns per unit length of the latter, and C the current to be measured. If everything were perfectly symmetrical, the observed change of weight would be given by eight times the above expression.

Electrodynamometers are not capable of such a high degree of sensitiveness as is readily obtainable with a galvanometer. But they have the advantage, as compared with galvanometers, that when two coils are traversed by the same current, they are independent of changes in the direction of the current. They are therefore of great use in the measurement of alternating currents.

Since the effect is proportional to the square of the current it is also proportional to the rate of production of heat in the circuit. A dynamometer is therefore a measurer of power, and practical forms are called *Watt-meters.*

CHAPTER XXX

ELECTROMAGNETIC MEASUREMENTS

371. Ordinary Measurements — Measurement of the Strength of a Current.—The measurements most frequently met with in practice are those of *current*, of *quantity*, of *electromotive force,* of *resistance,* of *capacity*, and of *inductance.*

The absolute measurement of the strength of a current is made directly by a tangent galvanometer, or by the absolute electrodynamometer. The tangent galvanometer gives (**357**)

$$C=\frac{H}{G}\tan\alpha.$$

According to this formula the strength of the current depends upon the constant G, which has to be calculated from the dimensions of the instrument, and H, which has to be determined by a separate experiment.

The electrodynamometer has the advantage of not depending (except in regard to a correcting term applicable to some forms of the instrument), upon H, the accurate determination of which is always a matter of some difficulty; on the contrary, when associated with a tangent galvanometer which is traversed by the same current, it furnishes one of the best methods of obtaining the value of the horizontal component.

Currents are most frequently measured by a calibrated galvanometer. For this purpose the deflection produced by a known current must be ascertained. The same current is passed simultaneously through a tangent galvanometer, and through the galvanometer to be calibrated, which may be shunted if necessary. Let α and α' be the simultaneous deflections observed on the two instruments; G' the field due to unit-current in the galvanometer under examination, and H' the horizontal component at the place where the needle hangs, we have

$$C=\frac{H}{G}\tan\alpha=\frac{H'}{G'}\tan\alpha';$$

an equation which gives the constant by which the indication of the latter galvanometer must be multiplied to obtain the value of the current either in C.G.S. units or in amperes.

With galvanometers of high sensitiveness, the calibration may be effected more directly. It is only needful to observe the deflection produced when the galvanometer is joined up in a circuit of very high resistance, with a cell of known electromotive force.

Suppose we are using a Daniell's cell (or, better, a secondary cell), that the total resistance of the circuit is 30,000 ohms, and that the galvanometer shunted to one-thousandth gives a deflection of 100 scale divisions.

The total current is $\frac{1 \cdot 07}{30{,}000}$ amperes, but only one-thousandth of this passes through the galvanometer. Hence one division corresponds to a current of

$$\frac{1 \cdot 07}{30{,}000 \times 1000 \times 100} = 3 \cdot 6 \times 10^{-10} \text{ amperes} = 3 \cdot 6 \times 10^{-11} \text{ C.G.S.}$$

The resistance of the coil is not needed, as it is negligible in comparison with the total resistance of the circuit. If the resistance of the galvanometer is g, its effective resistance when shunted to the mth is g/m, and is therefore usually a small part of the resistance of the circuit.

372. Measurement of a Quantity of Electricity—Ballistic Galvanometer.—When a discharge traverses a galvanometer, the needle is thrown from its position of equilibrium through a certain arc, and comes to rest after a series of oscillations. If the discharge is completed before the needle has had time to be appreciably displaced, the angle of throw measures the quantity of electricity which has traversed the galvanometer.

The mechanical problem is the same as that of a blow acting on a pendulum. Let M be the magnetic moment of the needle, K its moment of inertia in reference to the axis of rotation, and G the galvanometer constant: when the strength of the discharge current is C, the deflecting moment exerted on the needle is GMC, and in the infinitely short time dt this gives the needle an angular velocity, $\frac{GMCdt}{K}$. If the needle has not appreciably moved during the time occupied by the whole discharge, the final velocity is the sum of all the terms of the same kind, a sum which is evidently equal to $\frac{GMQ}{K}$, Q being the total quantity of electricity. Hence

if ω is the angular velocity imparted to the needle,

$$K\omega = GMQ \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Again, the initial energy of motion of the needle, namely, $\frac{1}{2}K\omega^2$, is expended, during the swing, in doing work against the horizontal component of the earth's magnetic field; if θ is the angle of deflection, this work is (**195**) $MH(1-\cos\theta)$, whence we get

$$\tfrac{1}{2}K\omega^2 = MH(1-\cos\theta) = 2MH\sin^2\tfrac{1}{2}\theta \quad . \quad . \quad (2)$$

If τ be the period of one complete vibration of the needle, we have

$$\tau = 2\pi\sqrt{\frac{K}{HM}}, \quad \text{or} \quad K = \frac{MH\tau^2}{4\pi^2} \quad . \quad . \quad . \quad (3)$$

Multiplying together equations (2) and (3), and taking the square root, we get

$$K\omega = \frac{MH\tau}{\pi}\sin\tfrac{1}{2}\theta,$$

and by substituting this value in (1), we obtain, to determine the quantity of electricity that has traversed the galvanometer,

$$Q = \frac{H\tau}{G\pi}\sin\tfrac{1}{2}\theta,$$

or, if θ is small enough to be identified with its sine,

$$Q = \tfrac{1}{2}\frac{H\tau}{G\pi}\theta = A\theta.$$

The constant A is found by determining the factor $\frac{H}{G}$ as above (**371**), and the time of oscillation of the needle,—or, more simply, by determining the angle of throw corresponding to the discharge of a known quantity of electricity, for instance, that obtained by turning a coil of a known surface (**310**) in the earth's field through an angle of 180°.

In the preceding calculation the arc θ is that through which the needle would be deflected if there were no damping. The angle really observed is a smaller angle θ_1. To get the necessary correction, the next deflection, θ_2, on the same side should be observed, then

$$\theta = \theta_1 + \frac{\theta_1 - \theta_2}{4}.$$

For between the two deflections, θ_1 and θ_2, the needle has made four quarter vibrations, and it may be assumed that the damping during the first quarter vibration, when the needle swings from

rest to θ_1, is equal to the mean damping during each of the four following quarter vibrations.

373. Alternating Currents.—If a series of discharges, alternately in opposite directions, and succeeding each other at very short intervals, are passed through the galvanometer, the deflection measures the algebraic sum of the electricity which passes, and, if this is zero, the needle remains stationary, or only undergoes accidental deflections. This is the case with a sinusoidal current and with the alternate currents obtained in a circuit by rapidly making and breaking an adjacent circuit (**313**). In this case the electrodynamometer may be employed; the deflection is proportional to the mean square of the current, that is to say, to the number which is the mean of the squares of successive current-strengths (**331**). It must not be forgotten that the electrodynamometer brings into the circuit not only its own resistance, but also its self-inductance. The electrometer gives a method free from this difficulty (**379**).

374. Measurement of Resistance.—A resistance is measured by comparing it with a known resistance by means of a current. It is impossible to compare two resistances without passing a current through them any more than we can compare two masses without submitting them to the action of a force,—that of gravity, for example.

The legal unit of resistance is the ohm, defined as being the resistance at 0° of a column of mercury of one square millimetre in cross section and 106·3 centimetres long. Copies are made either in mercury (Fig. 315) or in wire; the wire, frequently german-silver or platinum-silver, is wound on a bobbin and sunk in paraffin (Fig. 316). These copies have their marked value for a definite temperature. The variation of the resistance of mercury in a glass tube is given by the formula

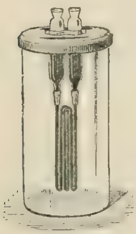


Fig. 315.

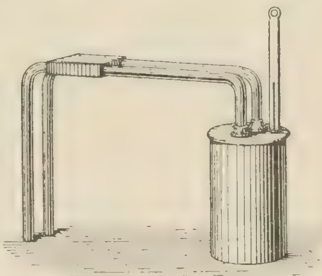


Fig. 316.

$$R = R_0(1 + 0{\cdot}0008726t + 0{\cdot}00000106t^2),$$

that of german-silver by the formula

$$R = R_0(1 + 0{\cdot}00044t).$$

A thermometer in the box containing the wire indicates the temperature.

375. Resistance Boxes.—It is necessary to have at our disposal a series of resistances, the values of which increase regularly. They consist ordinarily of coils placed in a box, and provided with plugs, by means of which any one or more of them may be inserted into the circuit at pleasure.

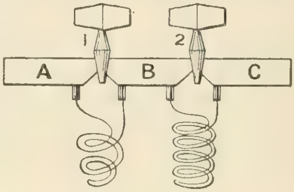


FIG. 317.

The wire is usually coiled after having been folded back on itself. In this way the effects of self-induction are diminished, but the capacity of the wire is increased by bringing close to each other parts of the wire which are at different potentials. It is best to form the coils of successive layers wound in opposite directions. The wire should be completely insulated.

The two ends of the same coil terminate in two contact pieces

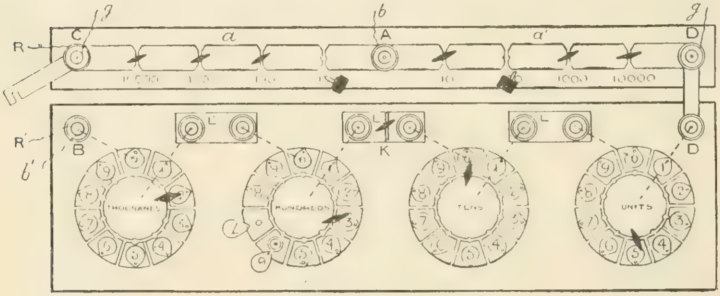


FIG. 318.

of copper, A and B (Fig. 317), with an interval between them which may be closed by a plug. When the plugs are in their places they connect the contact pieces into a virtually continuous rod of negligible resistance, but when any given plug is removed the corresponding coil is brought into the circuit.

The most convenient arrangement is that of "dial" boxes (Fig. 318). Each dial consists of nine equal coils connected with each other by ten copper blocks numbered from 0 to 9. There is no coil between the blocks 9 and 0, though generally there is one between 9 and the central block. In the centre is a circular block connected to the zero of the following dial by one of the copper strips, L, L′, L″. The plugs are placed between the central block and the blocks of the dial,

The first dial gives units, the others give tens, hundreds, &c. If the plugs are placed as shown in the figure, they bring in a resistance of 2305 ohms between the terminals B and D. K is a safety plug, which, when it is removed, cuts off all connection between the two plugs B and D. By means of special plugs provided with binding screws which can be inserted in holes in the blocks of the dial, conductors can be connected with any intermediate point of the dials. The rest of the figure is explained at the end of the next section.

376. Wheatstone's Bridge.—The method of comparison most usually adopted is that of *Wheatstone's Bridge*. The distribution of currents in the various branches of the network of conductors was fully discussed in (**120***); but, for the present application, it is sufficient to consider the following problem:—

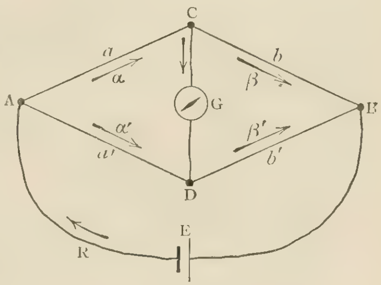


FIG. 319.

Two points, A and B, of a circuit (Fig. 319) are connected by two conductors. It is required to make a connection or *bridge*, CD, between the two branches, such that no current passes by this bridge.

For this it is necessary and sufficient that the two points C and D be at the same potential.

Now, if a a', b and b' are the resistances of the four branches AC, AD, CB, DB, V the potential at the point A, V' that of the point B, the fall of potential from A to C is

$$(V-V')\frac{a}{a+b},$$

and from A to D

$$(V-V')\frac{a'}{a'+b'}.$$

Consequently, for the potentials at C and D to be the same, we must have

$$\frac{a}{a+b}=\frac{a'}{a'+b'},$$

and therefore

$$\frac{a}{a'}=\frac{b}{b'}, \text{ or } ab'=a'b;$$

in other words, the resistances of any two adjacent sides of the

quadrilateral, ABCD, must be to each other in the same ratio as those of the other two adjacent sides.

It will be seen that the state of equilibrium is independent of the resistance of the branches which contain the battery and the galvanometer.

This arrangement furnishes a very simple means of comparing two resistances, b and b'; for when there is no current in the bridge CD, their ratio is that of the two resistances a and a', which can be adjusted at pleasure.

Suppose that the four sides of the quadrilateral are made up of metal bars having no appreciable resistance, but that any desired resistance can be inserted in any of them. Let b be the resistance to be measured, b' the standard resistance with which it is to be compared, and which can be adjusted so as to produce equilibrium.

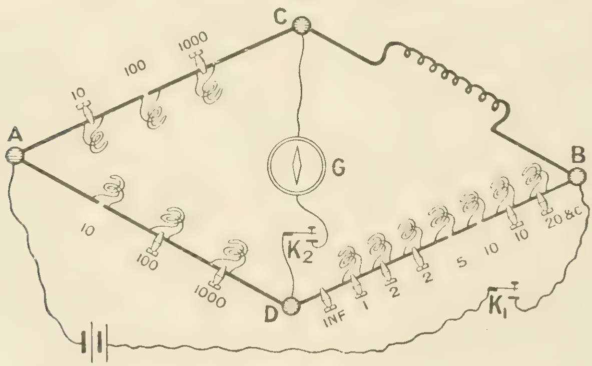


FIG. 320.

If we take $a=a'$, then $b=b'$; but the arms a and a' may be in any ratio, and the condition of equilibrium is that the resistances of the branches b and b' shall be in the same ratio. The arms a and a' are known as the *ratio arms*.

This arrangement of Wheatstone's is frequently compared to a balance, the resistances a and a' corresponding to the two arms of the beam; but, contrary to what occurs in the case of weights, equilibrium requires that the resistances to be compared shall be in the same ratio as the arms of the balance.

Each of the arms a and a' contains resistances of 10, 100, 1000 ohms. In Fig. 320 the ratio of the two arms is shown as that of 10 to 1, and since the resistance b' is equal to 15 ohms, the resistance measured is 150 ohms.

The relation proved above only holds good when the currents are steady; in order to eliminate the effects peculiar to the period

in which the currents are changing in value, the branch containing the galvanometer is closed by the key K_2, only after the arm containing the battery has been closed by depressing the key K_1.

The portion CAD of Fig. 318 represents the ratio arms of a dial resistance box. The coils in the arms a and a' are 10, 100, 1000, and 10,000 ohms. Hence we can get any decimal ratio between 10^{-3} and 10^3.

377. Wire Bridge.—Suppose that in the ordinary quadrilateral (Fig. 321), one of the summits, C, is replaced by a straight wire A′B′, along which the end, C, of the galvanometer branch may be moved. If l is the length of the wire A′B′, which should be perfectly cylindrical and homogeneous, x the distance A′C, and if we suppose resistances expressed in units of length of the wire A′B′, we have for the condition of equilibrium

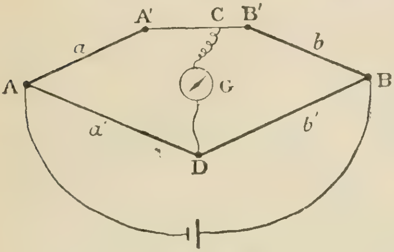


FIG. 321.

$$\frac{a'}{b'}=\frac{a+x}{b+l-x}.$$

The wire, which is usually of german-silver, forms one side of an elongated rectangle (Fig. 322), the other three sides of which are made up of broad copper bands of negligible resistance, at gaps in which the two resistances to be compared, a' and b', and two known resistances, a and b, are inserted. If these two latter are replaced by bars without resistance, we have simply

$$\frac{a'}{b'}=\frac{\text{A}'\text{C}}{\text{B}'\text{C}}.$$

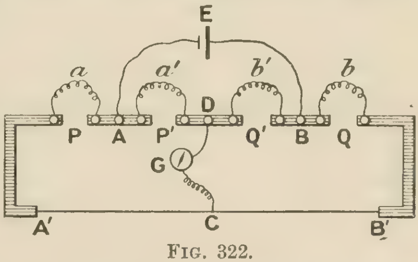


FIG. 322.

If, as is usually the case, the resistance between A and A′ and between B and B′ is not negligible in comparison with that of the wire A′B′, we may proceed as follows: having found a position C_1 of the sliding contact for which the galvanometer shows no deflection, interchange the resistances a' and b' to be compared, all else remaining as before, and find the new position C_2 of the slider

required to give no deflection. Put L for the length of wire of the same size and material as A′B′ which would have the same resistance as the total resistance from A through PA′B′Q to B, and let c_1 be the length of the same wire whose resistance would be equal to that from A through P and A′ to C_1, and c_2 the length whose resistance would equal that from A through P and A′ to C_2: then the first experiment gives

$$\frac{a'}{b'}=\frac{c_1}{L-c_1},$$

and the second gives

$$\frac{a'}{b'}=\frac{L-c_2}{c_2}.$$

Hence, adding numerations and denominators of the two equal fractions, we have

$$\frac{a'}{b'}=\frac{L+c_1-c_2}{L-c_1+c_2}=\frac{L-d}{L+d},$$

if we put $d=c_2-c_1$ for the distance between the two positions C_2 and C_1 of the slider, reckoned positive when the scale-reading for C_2 is greater than that for C_1 and negative in the opposite case.

The required ratio of a' to b' is thus expressed in terms of L, which can be determined once for all for the given apparatus, and of the *difference*, d, of two readings of the graduated scale A′B′. The result is thus independent of possible index-error of the slider C, or of possible want of coincidence between the ends of the scale A′B′ and those of the wire A′B′.

The value of L can be found in various ways: a simple method is to insert in the gaps P′ and Q′ two resistances, say p and q, whose ratio r is known independently. Then, proceeding as above, we get

$$\frac{p}{q}=r=\frac{L-d}{L+d},$$

whence

$$L=\frac{1+r}{1-r}d.$$

The apparatus is particularly convenient for comparing resistances which are almost identical with each other, as, for example, comparing a copy with a standard.

A good plan for this purpose is to insert the resistances to be compared, which may be represented by a and b, in the two end gaps P and Q, the two middle gaps P′ and Q′ being filled by any two

equal or nearly equal resistances a' and b'; then, after finding the balancing position of the sliding contact C, we interchange the resistances a and b, and again find the balancing position. Suppose that in the first case, with a at P and b at Q, the distance A′C is found to be m centimetres, and that, after interchanging a and b, a balance is obtained with $A'C = n$ centimetres, then if r is the resistance of 1 centimetre of the wire A′B′, we have

$$a - b = (m - n)\, r;$$

that is to say, the difference between the two resistances to be compared is equal to the resistance of the part of the graduated wire A′B′ over which the movable contact C is shifted. To prove this, we have only to remember that the total resistance AA′B′B is not altered by interchanging a and b, and that, since a' and b' remain unaltered, the second position of the point C must divide AA′B′B in the same ratio as the first.

378. Resistance of a Battery: Mance's Method.—Suppose the battery, of which the resistance is required, to be inserted in the branch b of the arrangement shown in Fig. 319, the battery shown in the figure between A and B being replaced by a simple make-and-break key. The galvanometer G will then be traversed by a current, but it is possible, by properly adjusting the resistance b' of the branch DB to make the current through the galvanometer independent of the resistance of the branch AB. The galvanometer will then show the same deflection whether the key in the last-mentioned branch is open or closed. When this is the case, the resistance of the battery is given by the expression (**120***, 3).

$$b = \frac{a}{a'} b'.$$

The galvanometer used for this experiment must be capable of carrying the whole current without being more than moderately deflected: a tangent-galvanometer would often be suitable. If a sensitive galvanometer is used it must be shunted by a low resistance and the coil turned so as to make its mean plane nearly coincident with the vertical plane containing the deflected needle; or a controlling magnet may be used to keep the deflection small.

379. Measurement of Electromotive Forces.—(1) **Electrometer Method.**—The electrometer may be used to compare the electromotive force of a battery with that of a standard. A galvanometer may also be used provided we join it up with a high resistance. If the resistance of the element can be neglected in

comparison with that of the rest of the circuit, the electromotive forces are, as when using the electrometer, proportional to the deflections.

The difference of potentials between two points, A and B, of a wire traversed by a current may also be measured in the same way. If C is the strength of the current, and R the resistance of the wire between the two points, we have $e = RC$; a measurement of e thus enables us to determine C when R is known. This method is frequently used for measuring strong currents.

It is specially applicable to the measurement of alternating currents. In using the electrometer, the needle being connected with one pair of quadrants (**83**), the two pairs of quadrants are put respectively in connection with the two ends A and B of the resistance R. The deflection is proportional to the square of the difference of potentials, and therefore does not change with the sign of this difference. If the alternate currents succeed each other at intervals which are very short in comparison with the time of oscillation, the needle takes up a fixed deflection proportional to the mean square of the current (**331**). If θ is the deflection, C_e the root of mean square current strength, and A the constant of the instrument, we have

$$\theta = A V_e^2 = AR^2 C_e^2,$$

and therefore

$$C_e = \frac{1}{R}\sqrt{\frac{\theta}{A}}.$$

380. (*b*) Method of Compensation.—A current is passed (Fig. 323) through a homogeneous wire so that the fall of potential from A to B is greater than the electromotive forces which are to be compared. The positive pole of the battery is connected through a galvanometer with the point A: the negative pole is connected with a sliding key, c, which is moved along the wire. The distance $x = \text{AC}$ is determined at which the key must be placed so that the galvanometer needle may be at zero. If C is the strength of the current in the wire AB, and k the resistance of one centimetre of this wire, then, since in order that there may be no current from A to c through the galvanometer, the difference of potentials between

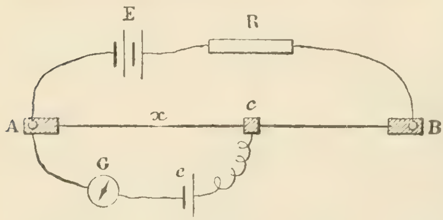


FIG. 323.

A and c must be equal to the electromotive force to be measured,

$$e = Ckx.$$

With the standard cell, we shall have, similarly,

$$e' = Ckx',$$

whence

$$\frac{e}{e'} = \frac{x}{x'}.$$

381. Closed Circuits.—We have thus obtained the electromotive force of a battery on open circuit. It is also a matter of interest to learn the electromotive force of the battery when at work, that is to say, the electromotive force as modified by the effects of polarisation. The two batteries to be compared, E and E′, are put in the same circuit (Fig. 324), so that the electromotive forces act in the same direction round the circuit; a bridge containing a galvanometer is interposed between any point A and a point B, which is found by trial so that the needle remains at zero. Let R and R' be the resistances of the two segments AEB and AE′B of the circuit: when there is no current in AB, the current is of the same strength in the two segments. Kirchhoff's second law (**118**) gives

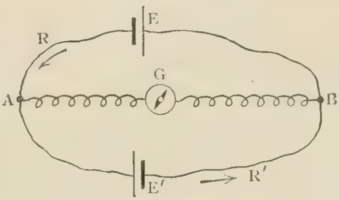


FIG. 324.

$$E = RC,$$
$$E' = R'C,$$

from which

$$\frac{E}{E'} = \frac{R}{R'}.$$

In order to avoid a measurement of the two resistances R and R', a known resistance is added to each of them, x for the first, x' for the second, so that there is still no current in AB; we have then

$$\frac{E}{E'} = \frac{R}{R'} = \frac{R+x}{R'+x'} = \frac{x}{x'}.$$

This latter device may only be employed when the polarisation is small, as this will change in value when the resistance of the circuit, and therefore the strength of the current, is altered.

382. Standards of Electromotive Force.—The standard elements most frequently used are those of Daniell, Latimer Clark, and Weston.

A *Daniell's* cell is not well adapted for a standard, as its electromotive force varies with the concentration of the liquids from 1·06 to 1·14 volt. Even with the same solutions there are variations of as much as 3 per cent. due to the condition of the metals. The variation with temperature depends also on the state of concentration of the liquids.

Latimer Clark's element consists of pure mercury as a negative metal covered with a paste obtained by mixing mercurous sulphate with a saturated solution of zinc sulphate. The positive metal is a zinc rod dipping into this paste of sulphate. Insulated wires, leading to the mercury and the zinc respectively, form the connections. This battery is not well adapted for continuous work, as it polarises easily, but it furnishes a very constant and trustworthy standard of electromotive force. Its electromotive force is 1·4324 volt at 15° C., and it diminishes by about 0·0011 volt for an increase of temperature of one degree.

The *Weston* or *cadmium* element is composed of cadmium, a paste of cadmium sulphate, a paste of mercurous sulphate, and mercury, and has an electromotive force of 1·0183 volt at 17° C. At any other temperature t between 10° and 30° C., the electromotive force is given by $E_t = E_{17} - 3{\cdot}4 \times 10^{-5}(t-17) - 0{\cdot}066 \times 10^{-5}(t-17)^2$.

Thermo-electric couples furnish stable standards, provided the temperatures of the junctions are accurately known. If the junctions are at 0° and 100° respectively, the electromotive force of an iron-copper couple is 0·001307 volt (**147**).

383. Measurement of Capacity.—In order to compare the capacities of two condensers, the condensers are charged by connecting the surfaces of each in succession with the poles of the same battery, and are then discharged through a ballistic galvanometer: the amounts of charge, and therefore the capacities, are as the angles of throw of the needle.

By charging the same condenser successively by means of two different batteries, we may compare in the same way the electromotive forces.

Two capacities may also be compared by an arrangement analogous to Wheatstone's bridge (Fig. 325). The two condensers, C and C', take the place of the branches b and b' of the usual quadrilateral, one surface of each being connected to earth, or to the back contact of the key K. The resistances a and a' are then adjusted until there is no deflection of the galvanometer on raising or lowering the key K.

The condition of equilibrium is that the two points B and B′ shall be at the same potential, that is to say, that at each instant

the charges Q and Q' shall be proportional to the capacities C and C': as these charges are proportional to the currents which produce them in the same time, and as these latter are inversely as the resistances, we have the proportion

$$\frac{C}{C'}=\frac{a'}{a}.$$

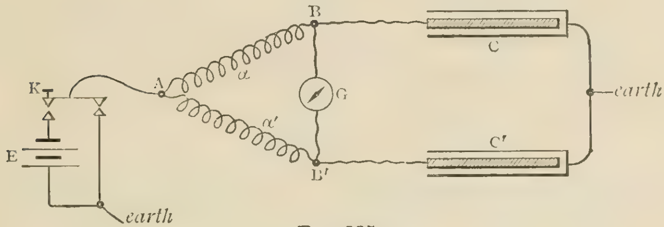


FIG. 325.

384. Mutual Inductance.—The two coils whose mutual inductance, M, is to be measured, are connected up in two separate circuits. One coil, with a battery, a key for making or breaking contact, and a tangent galvanometer, forms a primary circuit, and the other is connected with a ballistic galvanometer to form a secondary circuit. The primary circuit is closed, and the strength of the primary current, C, is read off on the tangent galvanometer as soon as the deflection has become steady. The primary circuit is then broken, and the throw, θ, of the ballistic galvanometer is observed, the reading being corrected for damping if needful. If r is the total resistance of the secondary circuit (including the galvanometer), the quantity of the secondary current is (**313**)

$$Q=MC/r;$$

but it is also given by the ballistic galvanometer (**372**) as

$$Q=\frac{H\tau}{G\pi}\sin\tfrac{1}{2}\theta.$$

Equating these two values, we get the coefficient of mutual induction—

$$M=\frac{rH\tau}{CG\pi}\sin\tfrac{1}{2}\theta.$$

The ratio H/G may be found as described in (**371**); or, if C' is the strength of the steady current which gives the small permanent

deflection θ' when passed through the ballistic galvanometer, we have

$$H/G = C'/\tan\theta',$$

and therefore

$$M = r\frac{C'\tau}{C\pi}\cdot\frac{\sin\frac{1}{2}\theta}{\tan\theta'}.$$

Lastly, if the same battery is used for giving the primary current C and the current C', and if R and R' are the total resistances respectively in circuit in the two cases, we may dispense with a measurement of the primary current and write

$$M = r\frac{R\tau}{R'\pi}\cdot\frac{\sin\frac{1}{2}\theta}{\tan\theta'}.$$

385. Self-Inductance.—The coil whose self-inductance is to be measured is connected as one arm of a Wheatstone's bridge arrangement, the other three arms being formed by resistances without appreciable inductance. If the resistances are so adjusted that, on completing the battery-circuit, there is no permanent deflection of the galvanometer, there is still a temporary deflection in one direction when the battery-circuit is completed and in the opposite direction when the battery-circuit is broken, resulting from the electromotive force of self-induction when the current begins or ceases in the coil under examination. The measurement consists in observing, on the one hand, the throw of the galvanometer needle due to this cause, and on the other hand, the steady deflection produced by a small disturbance of the balancing condition of the resistances when the battery is permanently connected.

Let L be the self-inductance of the coil to be measured and P its resistance, and let Q, R, S, and r be the resistances of the other arms of the bridge and of the galvanometer branch, as marked in Fig. 326. Further, let C be the current in the battery when connected and x the current through the coil of inductance L. The total quantity of the induced current is then (308)

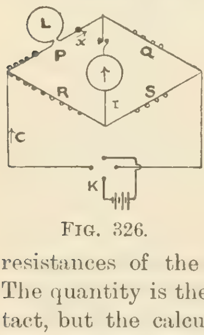


FIG. 326.

$$Lx \div \text{resistance of the circuit},$$

and of this a fraction depending on the relative resistances of the various branches traverses the galvanometer. The quantity is the same at making and breaking the battery contact, but the calculation is rather more obvious in the latter case. We shall accordingly suppose the battery to have been connected

long enough for the current to have become steady in every part of the circuit and then to be suddenly disconnected by opening the key K. The resistance of the circuit of which the given coil forms a part is then

$$P+R+\frac{(Q+S)\,r}{Q+S+r};$$

consequently the total quantity of the induced current which flows through the coil is

$$Lx\frac{Q+S+r}{(P+Q+R+S)\,r+(P+R)\,(Q+S)},$$

and of this the fraction $\dfrac{Q+S}{Q+S+r}$ passes through the galvanometer.

Putting q for this quantity, we have accordingly

$$q=Lx\frac{Q+S}{(P+Q+R+S)\,r+(P+R)\,(Q+S)}.$$

But **(113)** $\qquad x/C=(R+S)/(P+Q+R+S)$

since the resistances have been so adjusted that there is no current through the galvanometer when the battery is permanently connected, and therefore

$$q=LC\frac{(Q+S)\,(R+S)}{(P+Q+R+S)^2\,r+(P+Q+R+S)\,(P+R)\,(Q+S)}$$

$$=\frac{LCS}{(P+Q+R+S)\,r+(P+R)\,(Q+S)}$$

since $\qquad QR=PS.$

Further, if θ is the angle of throw of the galvanometer needle, the quantity q is given, as in the last section, by

$$q=\frac{H\tau}{G\pi}\sin\tfrac{1}{2}\theta,$$

and if a measuring galvanometer is included in the battery-circuit, so as to give the value of C, we thus get the self-inductance L expressed in terms of known quantities.

But the last measurement may be dispensed with by proceeding as follows. By applying the results proved in (**120***), we get for the current through the galvanometer, corresponding to a current C through the battery, the expression

$$C\frac{PS-QR}{(P+Q+R+S)\,r+(P+R)\,(Q+S)},$$

which vanishes when $PS = QR$, the condition usually fulfilled in the use of Wheatstone's bridge. Starting from this state of things, suppose the equilibrium to be disturbed by making, say, a small increase δP in the resistance P and an equal decrease in the resistance R, so that the sum $P + R$ is not altered. The galvanometer will now be traversed by a current, say c, given by

$$c = C\frac{(S+Q)\,\delta P}{(P+Q+R+S)\,r+(P+R)\,(Q+S)},$$

and if ϕ be the resulting steady deflection we have $c = \frac{H}{G}\tan\phi$. The denominator of this expression for c is identical with that of the expression for q; and on the assumption, which must be true to a very close approximation, that the current through the battery is of the same strength in both parts of the experiment, we get by dividing the value of q by that of c—

$$\frac{\tau \sin\frac{1}{2}\theta}{\pi \tan\phi} = L\frac{S}{(S+Q)\,\delta P},$$

or finally,

$$L = \frac{S+Q}{S}\,\delta P\,\frac{\tau \sin\frac{1}{2}\theta}{\pi \tan\phi}.$$

In practically carrying out the experiment, it is desirable to reverse the battery connection instead of simply breaking it, as the effects to be observed are thereby doubled, and moreover we thus avoid the change in the resistances of the conductors which is liable to result from a change in the magnitude of the currents traversing them. A make-and-break key must be included in the galvanometer circuit: this key must be closed during the observations for q, but must be open while connection with the battery-circuit is being made or reversed during the adjustment of resistances so as to satisfy the condition $PS = QR$, and also for the observation of the steady deflection due to the current denoted by c above, and must be closed only when the current in each branch has become steady. This may take a minute fraction of a second or several seconds, according to the time-constant of the coil to be examined.

CHAPTER XXXI

ELECTRICAL UNITS

386. Physical Magnitudes.—Every statement as to the numerical value of a physical quantity consists of *two* factors, one of which is a *number*, and the other the *unit* to which the number refers. Thus, in the statement that the strength of a current is 20 amperes, the factors are 20 and *one ampere:* the former answers to the question *how many?* the latter to the question how many *of what?* Often the answer to the second question may be taken as understood from the nature of the matter in hand, in which case it does not need to be expressed. Thus, if it is understood that, in speaking of the strength of a current, we always refer to the ampere as the unit, the number 20 would be a sufficient statement by itself; but in the absence of such an understanding, even if we knew that the statement before us related to the strength of a current, the mere number would leave it uncertain whether the current referred to was 20 amperes, or 20 C.G.S. units, or 20 units of any other kind.

The measurement of a physical quantity consists in ascertaining, by an appropriate experimental process, the relation which it bears to some other quantity of the same kind chosen as a standard of comparison or unit of value for the kind of magnitude in question. This unit constitutes the physical factor, expressed or understood, in the statement of the result of the measurement.

So long as we have to deal merely with magnitudes of one kind, the only points of fundamental importance to be considered in the choice of a unit are that it should be permanent, well-defined, and capable of being accurately compared with the quantities to be expressed by means of it. Thus, the yard, the metre, the mile, or the wave-length in a vacuum of the yellow light of incandescent sodium-vapour might, in principle, serve equally well as the unit of length: the choice of one rather than another is a matter of convenience.

But the main problem of physics does not consist in the comparison of quantities of the same kind, but rather in the discovery

and accurate investigation of relations between quantities of different kinds.

Hence it is important that, in choosing our units, we should take account not only of the conditions just mentioned, but should consider also the greater or less degree of clearness and simplicity with which they enable the mutual relations between different quantities to be expressed. The smaller the number of independent units any statement of the mutual relations of quantities contains, the simpler and clearer it is. Hence the last-mentioned condition is equivalent to saying that the best system of units is that which makes it possible to express the results of physical investigations by means of the smallest number of independent units.

An example will make these matters clearer. The relation between length and volume is such that the product of the three linear dimensions at right angles to each other of any rectangular space is equal to its volume. But whether the numerical value of a volume is equal to the product of the numerical values of the three linear dimensions depends on the units of volume and of length respectively. Let V denote any concrete volume, and v the unit of volume, then the numerical value is $\frac{V}{v}$. Similarly, if a, b, and c are the corresponding linear dimensions, and l is the unit of length, $\frac{a}{l}, \frac{b}{l}, \frac{c}{l}$ are their numerical values, and $\frac{abc}{l^3}$ is the numerical value of their product.

We cannot, in general, equate this with $\frac{V}{v}$, but require to employ a factor of proportionality, say ρ, depending on the values of v and l. We may then write

$$\frac{V}{v} = \rho \cdot \frac{abc}{l^3}.$$

Since V and abc are different expressions for the same concrete volume, they are necessarily equal, and therefore

$$\rho = \frac{l^3}{v}.$$

For example, if v is *one gallon* and l is *one foot*, $\rho = 6{\cdot}232$, and the statement contained in the first equation may be expressed in words as follows: *the number of gallons in a given volume is equal to 6·232 times the number obtained by multiplying together its length, breadth, and depth, each expressed in feet.* Here we adopt two in-

dependent units, for volume and length respectively. If, however, while keeping the gallon as unit of volume, we take as unit of length the edge of a cube measuring one gallon, we get $l = v^{\frac{1}{3}}$; or, again, keeping the foot as unit of length, if we take the cubic foot as unit of volume, we have $v = l^3$. In either case, $\rho = 1$, and our first equation takes the much simpler form

$$V = abc,$$

which is now numerically true.

It will be seen that the essential condition of this simplification is the avoidance of an unnecessary number of independent units.

387. Fundamental and Derived Units.—We have seen that a single independent or *fundamental* unit suffices for the measurement of both length and volume. We may take a certain length, l, as a fundamental unit, and derive from this l^3 as unit of volume. Or we may adopt a certain volume, v, as a fundamental unit, and derive from it $v^{\frac{1}{3}}$ as unit of length. Again, we may take a definite superficial area, s, as a fundamental unit, and derive from it $s^{\frac{1}{2}}$ as unit of length and $s^{\frac{3}{2}}$ as unit of volume.

Other units may be derived by referring simultaneously to two or more fundamental units. Thus if t denote the unit of time, the unit of velocity derived from this and the unit of length is the ratio l/t of the unit length to the unit time. Arithmetically, we can give no meaning to the ratio of two heterogeneous quantities like length and time, but this example shows that such a ratio may be capable of being interpreted physically as a quantity of a third kind. Reciprocally, we might adopt a definite velocity, c, as unit of velocity and derive from this the unit of length, as $l = ct$. Here the product of two heterogeneous quantities is interpretable as a quantity of a third kind.

We may extend similar considerations to the case of electrical quantities. For instance, it results from the discussion of electrical capacity in Chapter VI. that the capacity of an electric field may always be represented as the product of two factors, one of which is purely geometrical, while the other, the dielectric coefficient of the medium occupying the field, is purely physical. The geometrical part of the expression, in its turn, consists of the first power of the unit of length multiplied by a numerical factor. Consequently it is not needful to adopt an independent unit of electrostatic capacity; the unit may be derived by defining it as the capacity of a field occupied by the standard dielectric substance, air or a vacuum, and such that the geometrical factor applying to it is the unit of length.

388. Choice of Fundamental Units.—It will be seen from what has been said, that if we adopt as a principle that quantities are to be represented by means of as small a number of independent units as possible, the question still remains,—what are these units to be? Are we, for example, to take the unit of velocity as fundamental and derive from it the unit of length? or are we to treat the unit of length as fundamental and found on it the unit of velocity?

In order to answer such questions, we have to consider not only the qualities that have already been pointed out as essential to a satisfactory physical unit of any kind, but also the natural or logical sequence of ideas. This no doubt is, in part at least, relative to the historic order in which physical knowledge has been developed, and to the point of view from which we habitually approach the consideration of the various scientific conceptions. But taking, as we inevitably must do, these conceptions as we find them, the notion of length, or of the distance between two points, is simpler than that of velocity, and therefore logically precedes it.

The examination of the other quantitative notions of physics, and the discussion of them in the same sort of way as we have here discussed velocity and electrostatic capacity, have led physicists to the adoption of *three* fundamental units, namely, a unit of *length*, a unit of *mass*, or of quantity of matter, and a unit of *time*, in terms of which and of three other physical ideas—change of *temperature*, *dielectric coefficient*, and *magnetic permeability*—it is found that all the physical magnitudes that have been hitherto recognised can be expressed.

389. Absolute Units.—Units derived from the above-mentioned fundamental units are spoken of as *absolute* units, in contradistinction to the special units that might be adopted for each different kind of physical quantity; and magnitudes stated in terms of them are said to be expressed in absolute measure.

For example, we might adopt as unit of velocity the velocity acquired by a heavy body when it falls in a vacuum for a specified length of time at a specified place, and we might state the velocity of a railway train or of a rifle bullet in terms of this standard. But such a statement would not tell us how far the train or the bullet would move in a given time: it would only tell us the ratio between this distance and that corresponding to the velocity of the falling body. If, however, a velocity is stated in absolute units, that is, not merely by comparison with some other velocity, but in terms of the fundamental units of length and time, we can estimate completely its results.

390. Dimensions of Derived Units.—The formulæ which express the various derived units in terms of the fundamental units express at the same time the essential relations between the quantities represented, and show, in many cases, that magnitudes, between which no connection is obvious superficially, are essentially homogeneous. The exponent of the symbol of a fundamental unit in the formula of a derived unit is said to give the *dimension* of the latter with respect to the former. Thus the unit of volume, $v = l^3$, is of the *third* dimension with respect to the unit of length.

We proceed to give the dimensions of the most important derived units that we have to deal with in studying electricity and magnetism; but as the experimental measurement of electrical and magnetic magnitudes depends almost exclusively on the observation of mechanical effects, we give in the first place the dimensions of the chief quantities occurring in mechanics.

391. Mechanical Units.—*Velocity* (c).—We have already seen that the derived unit of velocity is the ratio of the unit of length to the unit of time. This is expressed by the equation

$$c = LT^{-1}.$$

Acceleration (a).—An acceleration is the ratio of an increment of velocity to the time in which it occurs, or the time-rate of increase of velocity. Hence

$$a = cT^{-1} = LT^{-2}.$$

Force (f).—A force is measured by the product of a mass acted on by the force into the resulting acceleration of this mass, or

$$f = Ma = LMT^{-2}.$$

The consideration of force as time-rate of change of momentum leads to the same expression.

Work or *Energy* (w).—The work done by a force is the product of the force into the displacement of its point of application projected on the line of action of the force, or fL. The energy of a moving body is the product of a mass into the square of a velocity, or Mc^2. Writing for f and c their values in terms of the fundamental units, both these expressions give

$$w = L^2MT^{-2}.$$

Turning moment or *Torque* (γ).—This is given by the product of a force into a length measured at right angles to the direction

of the force. If we do not distinguish between lengths measured in different directions, we have

$$\gamma = L^2MT^{-2},$$

which represents a moment as apparently homogeneous with work. We might write more appropriately

$$\gamma = L_xL_yMT^{-2},$$

the suffixes denoting that the two L's are to be measured at right angles to each other.

Power (p).—Power is used in mechanics for the *time-rate of doing work;* hence

$$p = fLT^{-1} = L^2MT^{-3}.$$

392. Electrical and Magnetic Units.—The measurement of electric and magnetic magnitudes depends on the observation of mechanical effects produced under accurately defined conditions. These effects are—(*a.*) *electrostatic*, involving, among other conditions, the *dielectric coefficient* of the medium in which they occur; (*b.*) *magnetic*, whether produced by, or exerted upon, ordinary magnets or conductors carrying currents. These involve the *magnetic permeability* of the medium.

We proceed to give dimensional formulæ for the most important magnitudes of each kind, that is, omitting all numerical factors, to give formulæ showing how each magnitude is related to the fundamental physical units. But, in the first place, it will be convenient to point out the relation between the dielectric coefficient and the magnetic permeability of a medium. This will enable us to pass, in the case of a given magnitude, from its value deduced from electrostatic effects to the value deducible from electromagnetic phenomena, or *vice versâ.*

With this object, we will examine the expressions obtained from the electrostatic and electromagnetic points of view respectively, for an electric charge or quantity of electricity.

According to Coulomb's law, two equal electric quantities, Q, concentrated at points whose distance apart is L in a medium of dielectric coefficient, K, exert upon each other a mechanical force, f, given by the equation

$$f = \frac{Q^2}{L^2K}.$$

Putting in the value of f from (391) and reducing,

$$Q = L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-1}K^{\frac{1}{2}}.$$

This formula gives the dimensions of Q on the electrostatic system.

To get a corresponding expression founded on electromagnetic action, we may start from the relation between electric current and electric quantity, namely,

$$Q = CT.$$

We may proceed further by considering that, if a current of strength C flows round each of two plane areas, $A(=L^2)$, placed at right angles to each other at a distance great as compared with their linear dimensions, in a medium of permeability μ, each current exerts a turning moment on the other, the value of which is proportional to the product of the magnetic moment of one current into the field due to the other (**255**, **256**). Hence the dimensional equation

$$\gamma = \frac{CL^2 . CL^2}{L^3}\mu,$$

or

$$C = L^{\frac{1}{2}} M^{\frac{1}{2}} T^{-1} \mu^{-\frac{1}{2}}.$$

Consequently,

$$Q = L^{\frac{1}{2}} M^{\frac{1}{2}} \mu^{-\frac{1}{2}}.$$

Equating this with the electrostatic expression already found, we get

$$K^{\frac{1}{2}} \mu^{\frac{1}{2}} = L^{-1} T,$$

or

$$K = L^{-2} T^2 \mu^{-1}\,;\ \mu = L^{-2} T^2 K^{-1}.$$

By similarly equating the electrostatic and electromagnetic expressions for any other electric magnitude we should arrive at the same result. This may be stated in words as follows: the product, $K\mu$, of dielectric coefficient into magnetic permeability for any medium is homogeneous with the reciprocal of the square of a velocity characteristic of the medium.

393. Electrostatic Units.—Starting from the expression for a quantity of electricity deduced from electrostatic action we proceed to deduce comparable expressions for the other most important electrical magnitudes.

Electric charge:—

$$Q = L^{\frac{3}{2}} M^{\frac{1}{2}} T^{-1} K^{\frac{1}{2}}.$$

Surface density (**16**).—This is the ratio Q/L^2; hence

$$\sigma = L^{-\frac{1}{2}} M^{\frac{1}{2}} T^{-1} K^{\frac{1}{2}}.$$

Electric displacement or *Electric induction* (Maxwell) (**44**).—Since $D = \sigma$ they are of the same dimensions.

Electric force at a point in an electric field (**20**).—The dimensions are obtained by dividing surface-density by K; or, conversely, displacement = electric force × dielectric coefficient:

$$f_e = L^{-\frac{1}{2}} M^{\frac{1}{2}} T^{-1} K^{-\frac{1}{2}}.$$

Electric potential (**32**).—The product of electric force into length is difference of potentials:

$$V = L^{\frac{1}{2}} M^{\frac{1}{2}} T^{-1} K^{-\frac{1}{2}}.$$

Capacity.—The capacity of an electric field (**51**) is the ratio of its charge to the difference of potentials of its boundaries; or, again, it is the ratio of electrostatic energy to the square of difference of potentials:

$$S = LK.$$

Current is time-rate of electric displacement through an area:

$$\begin{aligned} C &= \sigma L^2 T^{-1} = Q T^{-1} \\ &= L^{\frac{3}{2}} M^{\frac{1}{2}} T^{-2} K^{\frac{1}{2}}. \end{aligned}$$

Resistance is ratio of difference of potentials to current:

$$R = L^{-1} T K^{-1}.$$

Magnetic moment.—A current multiplied by an area round which it flows, and by μ, the permeability of the medium, gives the moment of the equivalent magnetic shell (**255**):

$$m = CL^2 \mu = L^{\frac{3}{2}} M^{\frac{1}{2}} K^{-\frac{1}{2}}.$$

Intensity of magnetisation—Magnetic induction.—The dimensions are the same, namely, magnetic moment divided by L^3:

$$A = L^{-\frac{3}{2}} M^{\frac{1}{2}} K^{-\frac{1}{2}}.$$

Strength of magnetic field.—The dimensions are couple-moment divided by magnetic moment (**179**):

$$H = L^{\frac{1}{2}} M^{\frac{1}{2}} T^{-2} K^{\frac{1}{2}}.$$

Inductance (*Mutual* or *Self*).—This is the ratio of energy to square of current (**304**):

$$\frac{L^2 M T^{-2}}{L^3 M T^{-4} K} = L^{-1} T^2 K^{-1}.$$

394. Electromagnetic Units.—We have already (**392**) obtained the dimensions of *quantity of electricity* and *electric current*:

$$Q = L^{\frac{1}{2}} M^{\frac{1}{2}} \mu^{-\frac{1}{2}}; \qquad C = L^{\frac{1}{2}} M^{\frac{1}{2}} T^{-1} \mu^{-\frac{1}{2}}.$$

From these we may obtain the dimensions of the remaining magnitudes. For convenience of comparison we will take the same order as before.

Surface density and electric induction—

$$\sigma = L^{-\frac{3}{2}} M^{\frac{1}{2}} \mu^{-\frac{1}{2}}.$$

Electric force— $\dfrac{\sigma}{K} = L^{\frac{1}{2}} M^{\frac{1}{2}} T^{-2} \mu^{\frac{1}{2}}.$

Electromotive force and difference of potential—

$$L^{\frac{3}{2}} M^{\frac{1}{2}} T^{-2} \mu^{\frac{1}{2}}.$$

Capacity— $L^{-1} T^{2} \mu^{-1}$

Resistance— $L T^{-1} \mu.$

Magnetic moment— $L^{\frac{5}{2}} M^{\frac{1}{2}} T^{-1} \mu^{\frac{1}{2}}.$

Magnetic induction— $L^{-\frac{1}{2}} M^{\frac{1}{2}} T^{-1} \mu^{\frac{1}{2}}.$

Magnetic field— $L^{-\frac{1}{2}} M^{\frac{1}{2}} T^{-1} \mu^{-\frac{1}{2}}.$

Inductance— $L\mu.$

395. Remarks on the Two Systems of Units.—Frequently in statements as to the dimensions of electric and magnetic quantities, the factors K and μ are not expressed, that is, it is tacitly assumed that each of them has the value unity, or is of *no* dimensions with respect to the fundamental physical units. If we put $K = 1$ in the above formulæ, we get the expressions usually given for the dimensions of the various units on the "electrostatic system;" while, by making $\mu = 1$, we get the usual corresponding expressions on the "electromagnetic system." The result is that the same magnitude appears as having different dimensions on the two systems: for example, an electric charge appears as having the dimensions $L^{\frac{3}{2}} M^{\frac{1}{2}} T^{-1}$ on the electrostatic system, and the dimensions $L^{\frac{1}{2}} M^{\frac{1}{2}}$ on the electromagnetic system; resistance, as having the dimensions $L^{-1} T$ on the electrostatic system, and LT^{-1} on the electromagnetic system.

The assumption underlying the formulæ given above, in which K and μ appear, is that each magnitude is of definite invariable dimensions, but that these dimensions cannot be completely and explicitly assigned, inasmuch as all that we can state as to the dimensions of K and μ is that their product is of the same dimensions as the inverse square of a velocity, or, more concisely, that

$$K\mu v^{2} = \text{a pure number}.$$

In the account which we have given (Chapter XIX.) of magnetic induction we regarded it as of essentially the same nature as a magnetic field. If this be so it follows that μ is in reality a mere number, and the dimensions of K are those of the square of the reciprocal of a velocity, or $T^2 L^{-2}$.

Many physicists, however, prefer to regard induction and field as being of essentially different physical character, induction being related to intensity of field in much the same way as strain is related to stress in an elastic body; or, more generally, as effect to cause. The identity in the numerical values of the two quantities in air or other non-magnetisable medium is regarded as being a consequence of the choice of units only; and the relation between them, which will be true whatever units are adopted, is written $B = \mu_0 H$, where μ_0 is called the absolute permeability of the air. The statements which we have made above embrace both these views, a suitable interpretation being given to μ.

396. Choice of Fundamental Units—C.G.S. System.—The quantities to be adopted as the fundamental units of *length, mass*, and *time*, may evidently be chosen arbitrarily according to considerations of convenience. The units now almost universally adopted for scientific purposes, not only in electricity, but in all departments of physics, are those agreed upon by a Congress of Electricians that met in Paris in 1881, namely—

As unit of length—one *Centimetre* (very nearly one thousand-millionth part of a quadrant of the earth measured from the equator to the pole);

As unit of mass—one *Gramme* (very nearly the mass of one cubic centimetre of water at the temperature of its maximum density);

As unit of time—one *Second* (one eighty-six thousand four hundredth part of a mean solar day).

It was agreed that units founded upon these quantities should be called Centimetre-Gramme-Second absolute units, or, more shortly, C.G.S. units.

Such units are sometimes much smaller, in other cases much larger, than the quantities which have to be expressed by means of them. To avoid the consequent necessity of using very large numerical multipliers or divisors, multiplication by one million ($=10^6$) is expressed by the prefix *mega-*, and division by one million by the prefix *micro-*.

It has been found convenient also to give special names to some of the most important derived units of the C.G.S. system. Thus

the unit of force, namely, the force which, when acting on a gramme, causes its velocity to change at the rate of one centimetre per second in a second, is called a *dyne*. The weight of a gramme is equal to about 981 such units. The weight of one milligramme is therefore almost equal to one dyne, and the weight of one kilogramme almost equal to one *mega-dyne*.

The C.G.S. unit of work or energy is the work done or energy expended when a force of one dyne acts through a distance of one centimetre, and is called an *erg*. One *kilogramme-metre* is about 981×10^5 ergs, or, roughly, 100 *megergs*.

The unit of heat, or the quantity of heat needed to raise the temperature of one gramme of water by one degree centigrade at a specified part of the scale, may be called a *water-gramme-degree*, or simply a gramme-degree. It is equivalent to about $4{\cdot}18 \times 10^7$ ergs, or nearly 42 megergs.

397. Practical Units.—The C.G.S. units of electrical magnitudes are in many cases so much smaller, or so much larger, than the quantities usually dealt with in practice, that subsidiary units, derived from them by multiplying or dividing by powers of 10, are employed for practical purposes. For example, on the electromagnetic system, the C.G.S. unit of resistance is equal to the resistance of only about one twenty-thousandth of a millimetre of copper wire of one millimetre diameter; the C.G.S. unit of electromotive force is about one hundred-millionth of that of a Daniell's cell; on the other hand, the C.G.S. unit of capacity is more than two thousand times the capacity of an electric field filled with air, with spherical boundaries one centimetre apart, and with a radius equal to that of the earth.

Consequently, units of a more convenient size for practical purposes were agreed upon by the Electrical Congress, already mentioned as having met in Paris in 1881, in the case of five of the magnitudes that are most frequently referred to, namely, *resistance*, *electromotive force*, *strength of current*, *quantity of electricity*, and *capacity*.

The practical unit of resistance is called the *ohm*, and is defined as 10^9 C.G.S. electromagnetic units. It is, almost as nearly as can be measured, equal to the resistance at 0° C. of a column of mercury of one square millimetre section, and 106·3 centimetres long.

The practical unit of electromotive force is called the *volt*, and is defined as 10^8 C.G.S. electromagnetic units. It is about 6 per cent. less than the electromotive force of a Daniell's cell. Almost as nearly as can be measured, the electromotive force of a Latimer Clark's cell at 15° is 1·432 volt.

The practical unit of current-strength is called the *ampere*. It is the strength of the current which an electromotive force of one volt can maintain in a circuit of resistance one ohm: its value in C.G.S. electromagnetic units is therefore $10^8/10^9 = 1/10$. Almost as nearly as can be measured, a current of one ampere, passed through an aqueous solution of nitrate of silver, deposits silver at the rate of 0·0011183 gramme per second, or 4·0259 grammes per hour.

The practical unit of quantity is called the *coulomb*. It is the quantity which, in one second, passes every complete transverse section of a conductor traversed by a steady current of one ampere: hence it may also be called an *ampere-second*, and is equal to 0·1 of a C.G.S. electromagnetic unit.

The practical unit of capacity is called the *farad*. It is the capacity of an electric field which has a charge of one coulomb when the difference of potentials between its boundaries is one volt. In C.G.S. electromagnetic measure a farad is therefore equal to the ratio of $10^{-1} : 10^8$, or $10^{-9} : 1$. Even the farad, though only one thousand-millionth of a C.G.S. unit of capacity, is too large for ordinary purposes, and consequently capacities are very commonly stated in terms of the millionth part of a farad, called a *micro-farad*, equal to 10^{-15} C.G.S. electromagnetic unit of capacity.

In addition to the above, a practical unit of inductance has been adopted and is called the *henry*. The self-inductance of a circuit is one henry if an opposing electromotive force of one volt results from a variation of the strength of current in the circuit at the rate of one ampere per second. The mutual inductance of two circuits is one henry if an electromotive force of one volt in one of them results from a variation of the strength of current in the other at the rate of one ampere per second. The value of the henry in C.G.S. measure is consequently 10^9 centimetres.

The electric energy expended when a coulomb of electricity falls in potential by one volt is called a *joule*. Its value in C.G.S. units is $10^{-1} \times 10^8 = 10^7$ ergs.

The rate at which energy is expended or power exerted when a current of one ampere flows from a given point of a circuit to a point where the potential is lower by one volt is called a *watt*; it represents 10^7 ergs per second.

A gramme-degree of heat, being the equivalent of $4{\cdot}18 \times 10^7$ ergs, is equal to 4·18 joules; conversely, 1 joule is equal to 0·239 gramme-degree. One horse-power is equal to 746 watts; one kilowatt is 1·34 horse-power.

398. Recapitulation.—We may recapitulate here in tabular form the results of the foregoing discussion as to the dimensions of electrical and magnetic quantities.

Table of Dimensions.

	Dimensions in Terms of—	
	L, M, T, and K.	L, M, T, and μ.
Dielectric coefficient . .	K	$L^{-2}T^{2}\mu^{-1}$
Magnetic permeability . .	$L^{-2}T^{2}K^{-1}$	μ
Quantity of electricity . .	$L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-1}K^{\frac{1}{2}}$	$L^{\frac{1}{2}}M^{\frac{1}{2}}\mu^{-\frac{1}{2}}$
Magnetic charge (pole) . .	$L^{\frac{1}{2}}M^{\frac{1}{2}}K^{-\frac{1}{2}}$	$L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-1}\mu^{\frac{1}{2}}$
Electric potential, Electromotive force . .	$L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-1}K^{-\frac{1}{2}}$	$L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-2}\mu^{\frac{1}{2}}$
Magnetic potential . .	$L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-2}K^{\frac{1}{2}}$	$L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-1}\mu^{-\frac{1}{2}}$
Electric force (strength of electric field) . .	$L^{-\frac{1}{2}}M^{\frac{1}{2}}T^{-1}K^{-\frac{1}{2}}$	$L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-2}\mu^{\frac{1}{2}}$
Strength of magnetic field .	$L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-2}K^{\frac{1}{2}}$	$L^{-\frac{1}{2}}M^{\frac{1}{2}}T^{-1}\mu^{-\frac{1}{2}}$
Electric displacement (surface density) . .	$L^{-\frac{1}{2}}M^{\frac{1}{2}}T^{-1}K^{\frac{1}{2}}$	$L^{-\frac{3}{2}}M^{\frac{1}{2}}\mu^{-\frac{1}{2}}$
Magnetic induction . .	$L^{-\frac{3}{2}}M^{\frac{1}{2}}K^{-\frac{1}{2}}$	$L^{-\frac{1}{2}}M^{\frac{1}{2}}T^{-1}\mu^{\frac{1}{2}}$
Electric current . . .	$L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-2}K^{\frac{1}{2}}$	$L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-1}\mu^{-\frac{1}{2}}$
Conductivity . . .	$LT^{-1}K$	$L^{-1}T\mu^{-1}$
Resistance	$L^{-1}TK^{-1}$	$LT^{-1}\mu$
Capacity	LK	$L^{-1}T^{2}\mu^{-1}$
Inductance	$L^{-1}T^{2}K^{-1}$	$L\mu$
Magnetic moment . .	$L^{\frac{3}{2}}M^{\frac{1}{2}}K^{-\frac{1}{2}}$	$LM^{\frac{1}{2}}T^{-1}\mu^{\frac{1}{2}}$
Strength of magnetic shell .	$L^{-\frac{1}{2}}M^{\frac{1}{2}}K^{-\frac{1}{2}}$	$L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-1}\mu^{\frac{1}{2}}$

It may be noted that the pairs of magnitudes that are bracketed together in this table are reciprocally analogous, the dimensions of one in terms of K being the same as those of the other in terms of μ. In every case, the ratio of the dimensions in terms of K to the dimensions in terms of μ is a power of $LT^{-1}K^{\frac{1}{2}}\mu^{\frac{1}{2}}$.

399. Comparison of Practical and C.G.S. Units.—It may be useful also to collect in tabular form the values of the practical

units giving the corresponding electrostatic and electromagnetic values side by side.

	Name of Practical Unit.	Value in C.G.S. Measure.	
		Electro-magnetic.	Electro-static.
Resistance	Ohm	10^9	$10^9 \div v^2$
Electromotive force . .	Volt	10^8	$10^8 \div v$
Strength of current . .	Ampere	10^{-1}	$10^{-1} v$
Quantity of electricity .	Coulomb	10^{-1}	$10^{-1} v$
Capacity	Farad	10^{-9}	$10^{-9} v^2$
Inductance	Henry	10^9	$10^9 \div v^2$
Energy	Joule	10^7	
Power	Watt	10^7	

The symbol v in this table stands for the velocity $K^{-\frac{1}{2}}\mu^{-\frac{1}{2}}$ with which electric or magnetic disturbances are propagated in the medium to which K and μ refer. This velocity, in the case of air, is very nearly $3 \times 10^{10} \frac{\text{cms.}}{\text{sec.}}$

CHAPTER XXXII

DETERMINATION OF THE OHM

400. Standards.—When once the practical units are defined, the next step is to embody the definitions in actual material standards. This can be practically carried out for the standards of resistance, of electromotive force, and of capacity. In other cases, as, for example, in the case of electric currents, it is not practicable to produce and preserve an actual unit as a standard of reference; we have to depend on measuring instruments whose indications bear a known relation to the theoretically defined units. Similarly, in mechanics we have actual concrete standards of length and mass, but no concrete unit of time or of velocity. If, however, we possess a standard of one kind, and appropriate measuring instruments, the others may be easily deduced from it. As the standard of resistance is that which is most trustworthy, both as to accuracy and permanence, the efforts of many physicists in various countries have been directed to its determination.

The problem to be solved was this: What is the length, at a specified temperature, of a column of a given uniform cross-section of a chemically well-defined material, the resistance of which is equal to 10^9 electromagnetic C.G.S. units, that is to say, to one ohm? In practice the problem has been to find the length of a column of pure mercury, of one square-millimetre cross-section, which has, at 0° C., a resistance of one ohm.

The resistance of a conductor in the electromagnetic system is a quantity of the same nature as a velocity (**395**), that is to say, the quotient of a length by a time. Hence, it follows that the measurement of the absolute value of a resistance amounts ultimately to the measurement of a length and of a time. If other quantities occur in the calculation of the experiments, they can only enter as ratios of magnitudes of the same kind.

The formulas of Ohm and of Joule each furnish a method of measurement. But Joule's formula contains the coefficient J, representing the mechanical equivalent of heat, and in the present state of our knowledge this is not known with sufficient accuracy to form the basis for the determination of the ohm.

Ohm's formula bases the measurement of resistance on a comparison of the absolute value of a current with that of an electromotive force. The galvanometer directly gives the former. In reference to the second, it is to be observed that the only electromotive forces which can be directly measured are those due to electromagnetic induction. The method may be founded on the observation either of an instantaneous or of a continuous effect. We shall explain the principles of four of the simplest and at the same time most trustworthy methods that have been employed hitherto.

401. Rotating Coil—Weber's Method.—A coil of known area, S, movable about a vertical diameter, is connected in circuit with a ballistic galvanometer and set exactly at right angles to the magnetic meridian; it is then rapidly turned through 180°, and the throw of the galvanometer is observed. If R is the total resistance of the circuit, we have on the one hand, for the quantity of electricity which traverses the circuit (**310**),

$$Q=\frac{2SH}{R}\mu;$$

and on the other (**372**),

$$Q=\frac{1}{2}\frac{H}{G}\frac{\tau}{\pi}\theta.$$

From which

$$R=4SG\mu\frac{\pi}{\tau}\frac{1}{\theta}.$$

It will be observed that the result is independent of the value of the earth's field. Besides the area of the coil and the deflection of the needle, the only quantities to determine are the time of oscillation of the needle and the constant of the galvanometer coil. Of these quantities, G represents a magnetic field due to a unit current; its dimensions are therefore H/C, or (**398**) L^{-1}; the dimensions of the area S are L^2, while π and θ are purely numerical. The whole expression is therefore of the dimensions $LT^{-1}\mu$, or if, in accordance with the electromagnetic system, we take $\mu=1$, the result of the experiment gives R expressed as a velocity.

402. Mutual Induction of Two Coils—Kirchhoff's Method.—Let M be the mutual inductance of two fixed coils one of which is joined up with a ballistic galvanometer forming a circuit of total resistance R, while the other is in circuit with a battery which gives a steady current of strength C. When the current is reversed, we have (**313**)

$$Q=\frac{2MC}{R},$$

and on the other hand,

$$Q=\frac{1}{2}\frac{H\tau}{G\pi}\theta.$$

The strength of the current is given by the deflection α of a tangent galvanometer, or

$$C=\frac{H}{G'}\tan\alpha.$$

From these equations we obtain

$$R=4M\frac{G}{G'}\,\frac{\tan\alpha}{\theta}\,\frac{\pi}{\tau}.$$

The result is again independent of H. The ratio $\frac{G}{G'}$ of the constants of the galvanometers, ballistic and tangent, is obtained by passing the same current simultaneously through both (**371**). The physical magnitudes in terms of which the fixed value is expressed are M and τ, of which the ratio is of the dimensions $LT^{-1}\mu$.

403. Constant Current—Lorenz's Method.—A metal disc rotates uniformly about its geometrical axis: if a is the radius of the disc, T the time of a single revolution, H the component of the field parallel to the axis, the electromotive force acting between centre and circumference is (**317**)

$$e=\frac{\pi a^2 H}{T}=\frac{AH}{T},$$

A being put for πa^2, the area of the disc.

Instead of using the terrestrial field, let the axis of rotation be placed at right angles to the magnetic meridian, and let it coincide with the axis of a concentric circular coil traversed by a current C. If M is the flux of magnetic induction which passes through the disc from face to face, in consequence of a current of unit strength in the coil, we have

$$CM=AH,$$

and therefore

$$e=\frac{CM}{T}.$$

Suppose, lastly, that the centre and circumference of the disc are respectively connected with two points separated by resistance R on the circuit of the current C which traverses the coil; then the difference of potentials between these points is CR, and if R is so chosen that

$$CR=e,$$

we can, by making the connections the right way about, cause this difference of potentials to balance the electromotive force due to the rotation of the disc, a condition which is easily recognised by a galvanometer connected in circuit with the disc remaining undeflected. Equating the two values of e, we get the remarkably simple expression

$$R=\frac{M}{T}.$$

404. Lippmann's Method.—This may be regarded as, to a great extent, a modification of Lorenz's method. A coil C (Fig. 327) rotates uniformly about its vertical diameter within the uniform field (**266**) of a long, horizontal, hollow coil, MN, carrying a constant current, which at the same time circulates in the conductor AB, the resistance of which is to be determined. The revolving coil is connected, through the capillary electrometer E, with the points A and A_1, but the circuit is only closed at the moment when the plane of the coil is parallel to the field, in other words, when the electromotive force due to the rotation passes through its maximum. The speed of rotation and the resistance R between the points A and A_1 are so adjusted that the induced electromotive force is compensated by the difference of potential between the points A and A_1. As there is no current in the coil, self-induction does not come into account.

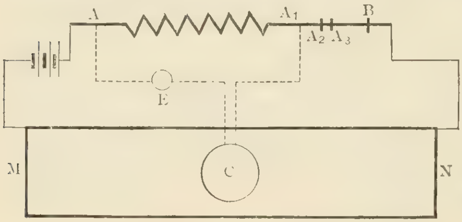


FIG. 327.

If A is the total area of the coil, T the time of a revolution, and n_1 the number of turns per centimetre of the fixed coil, the maximum electromotive force is

$$\frac{2\pi}{T}A\ 4\pi n_1 C=e,$$

and if R is the resistance between the two points A and A_1 we have,

$$R=\frac{e}{C}=\frac{8\pi^2 n_1 A}{T}.$$

As in the preceding method, the current cancels out. This method has the advantage, as compared with that of Lorenz, that

greater electromotive forces can be used, so that those which may arise at the contact of sliding pieces may be neglected in comparison.

405. Material Standard of Resistance.—Many experiments by the above and other methods agree in indicating that the resistance expressed in absolute electromagnetic units, as $10^9 \frac{\text{cm.}}{\text{second}}$, or *one ohm*, is very nearly indeed the same as the resistance at 0° of a column of mercury 1 square millimetre in cross-section and 106·3 centimetres long. Practically, however, the only way of producing a column of mercury for use as a standard of resistance and estimating its cross-section is to enclose the mercury in a glass tube, and to assume that the cross-section of the column is the same as that of the tube which contains it. Reciprocally, the most accurate method of measuring the cross-section of a glass tube is based upon a determination of the mass of mercury that fills a measured length of it, and upon the determination of the density of mercury. Hence a statement of the value of the ohm, which, like that given above, involves the cross-section of a column of mercury, is not expressed in terms of the quantities which are directly measured. From this point of view it is more direct to specify the length and mass of the mercury column, which has a resistance one ohm, than to specify its length and cross-section. This is the method adopted by the Board of Trade, as well as by the Governments of Germany and the United States in the official definition of the ohm for legal and commercial purposes. In an Order in Council of 1894, the ohm is stated to be "represented by the resistance offered to an unvarying electric current by a column of mercury at the temperature of melting ice, 14·4521 grammes in mass, of a constant cross-sectional area, and of a length of 106·3 centimetres."

From the definition of specific resistance (**115**) it follows that two conductors of the same material have the same resistance if their masses are proportional to the squares of their lengths; hence the above statement as to the magnitude of the ohm is very nearly equivalent to the following—the resistance at 0° of 12·79 grammes of mercury in the form of a column of uniform cross-section 100 centimetres long.

406. Measurement of the Velocity 'v'.—We have seen (**392**) that the product $K\mu$ of the absolute inductive capacity into the magnetic permeability of a medium, say air, in which electrostatic and electromagnetic forces can be observed, has the dimensions of the reciprocal of the square of a velocity. Since the properties denoted by K and μ respectively are definite for a given medium

under given circumstances, the product $K\mu$ must be expressible in terms of a definite velocity characteristic of the medium. We will denote this velocity by v, and will investigate its numerical value for a vacuum or air, which is from this point of view nearly the same thing.

Suppose a given quantity of electricity, Q, to be expressed, on the one hand, as s electrostatic units, and, on the other hand, as m electromagnetic units. In accordance with (**398**), we may write

$$Q = s[L^{\frac{3}{2}} M^{\frac{1}{2}} T^{-1} K^{\frac{1}{2}}] = m[L^{\frac{1}{2}} M^{\frac{1}{2}} \mu^{-\frac{1}{2}}],$$

or

$$(K\mu)^{-\frac{1}{2}} = \frac{s}{m}[LT^{-1}] = v,$$

where s/m is a purely numerical factor and $[LT^{-1}]$ is the unit of velocity, or, in concrete value, one centimetre per second: v is therefore a velocity of s/m centimetres per second. It is clear that the absolute magnitudes of the quantities expressed by unity on the electrostatic and electromagnetic systems respectively must be in the inverse ratio of the numerical values of the same quantity when expressed in terms of these two units, or

$$\frac{[Q_m]}{[Q_s]} = \frac{s}{m},$$

where $[Q_m]$ denotes the quantity of electricity expressed as unity on the electromagnetic system, and $[Q_s]$ the quantity expressed as unity on the electrostatic system. Consequently, it is often said that the velocity v is equal to the ratio of the electromagnetic to the electrostatic unit of electric quantity.

It will be seen by reference to the table of dimensions of electric magnitudes (**398**) that the product $K\mu$, and therefore the velocity v, might be expressed in terms of the electrostatic and electromagnetic values of any one magnitude. Thus, if the symbol of any magnitude, with the suffix m or s attached to it, denotes its numerical value in electromagnetic or electrostatic measure respectively, we have

$$\frac{Q_s}{Q_m} = \frac{C_s}{C_m} = \frac{E_m}{E_s} = \sqrt{\frac{R_m}{R_s}} = \sqrt{\frac{S_s}{S_m}} = \sqrt{\frac{L_m}{L_s}}.$$

Practically, the only magnitudes which admit of direct electrostatic measurement are electric charge or quantity, difference of potentials or electromotive force, and capacity. There are consequently three distinct methods that may be employed for the measurement of v.

407. Measurement by Means of a Quantity of Electricity.—The experiment consists in charging a condenser, and measuring by means of Coulomb's balance a known fraction of the total charge, and therefore the charge itself; and, on the other hand, in measuring this charge by the ballistic galvanometer. The quotient of the number furnished by Coulomb's balance, by the number given by the ballistic galvanometer, gives the desired value. This experiment was first made by Weber and Kohlrausch.

408. Measurement by Means of an Electromotive Force.—A steady current is passed through a wire. Let C be the strength of the current and R the resistance between the two ends A and B of the wire, both in C.G.S. electromagnetic units. The electromagnetic difference of potential between the two points is $C_m R_m$. An absolute electrometer (**86**) connected with the points A and B gives the same difference in electrostatic units. The quotient of the former number by the latter is the value of v.

409. Measurement by Means of a Capacity.—A condenser is formed of two parallel plates separated by a layer of air; one of them is so large that it may be considered as infinite, and the other is surrounded by a guard ring (**86**). If A is the area of the movable plate, and e the distance between the two plates, the electrostatic capacity is given by the formula (**53**)

$$S_s = \frac{A}{4\pi e}.$$

The condenser is charged to a difference of potentials $V - V'$, and is then discharged through a ballistic galvanometer (**372**). We have

$$Q_m = \frac{1}{2}\frac{H\tau}{G\pi}\theta = S_m(V - V')_m.$$

Let us assume that the difference of potentials $V - V'$ is that between two points A and B of a wire of resistance R carrying a current C; this gives

$$(V - V')_m = R_m C_m = R_m \frac{H_m}{G'} \tan \alpha,$$

and consequently

$$S_m = \frac{1}{2R_m}\frac{G'}{G}\frac{\tau}{\pi}\frac{\theta}{\tan \alpha}.$$

The ratio of S_s to S_m is equal to v^2.

410. Method of Electrical Oscillations.—Lodge and Glazebrook determined v by photographing an oscillating spark on a moving sensitive plate. The time-period of the oscillations was

obtained by measuring the distance between corresponding points in the successive images of a single spark, and determining the velocity of the plate. The circuit consisted of a coil of self-inductance L_m, and an air condenser of capacity S_m in electromagnetic and S_s in electrostatic measure. The time-period for such a circuit is given (**339**) by the formula

$$T = 2\pi\sqrt{L_m S_m} = 2\pi\sqrt{\frac{L_m S_s}{v^2}}.$$

S_s was obtained from the dimensions of the condenser, L_m by Maxwell's method (**385**); hence v could be calculated. The mean value, after numerous corrections had been made, was $3{\cdot}009 \times 10^{10}$ cms. per second.

411. Numerical Value of v.—The values obtained for v by these various methods are all very nearly 3×10^{10} C.G.S. Accordingly the constant v is a velocity of 3×10^{10} centimetres per second. This velocity is exactly that of light, and thus the reasoning of (**350**) is confirmed.

Maxwell made a noteworthy remark in reference to this. He assumes, and the hypothesis has been confirmed by Professor Rowland's experiments (**355**), that an infinitely long straight wire having a charge, Q (measured in electrostatic units), per unit length, moving in its own direction with a velocity c, would be equal to a current of strength Qc in electrostatic, and of strength $\frac{Qc}{v}$ in electromagnetic measure.

Two such wires placed parallel at the distance d, and moving in the direction of their length with the same velocity, would act like two parallel currents in the same direction. The attraction exerted by either on unit length of the other is (**268**)

$$\frac{2Q^2c^2}{v^2} \cdot \frac{1}{d}.$$

At the same time, in consequence of their charges, each will repel unit length of the other with a force which a simple calculation shows is equal to

$$\frac{2Q^2}{d}.$$

The attraction is accordingly equal to the repulsion, and the two wires are without action on each other, if

$$c = v;$$

hence the velocity v is that with which each wire must move in the direction of its length in order that the two wires may have no action on each other.

CHAPTER XXXIII

CONSTANT-CURRENT MACHINES

412. Constant-Current Machines—Generator; Motor.—We shall take as type of these machines Faraday's disc (**275, 317**). A metal disc (Fig. 328) driven by any kind of motor is made to rotate about an axis in a uniform field. Two contacts or *brushes*, one pressing on the circumference and the other on the axle, connect the disc with an external circuit. A current is produced, and if the rotation is uniform, the electromotive force of the circuit is perfectly constant. If H is the component of the field parallel to the axis, A the area of the disc, n the number of turns in a second, and R the resistance of the circuit, the current is (**317**)

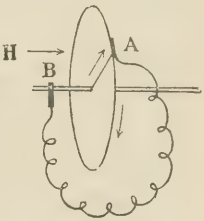


FIG. 328.

$$C = \frac{nA\mu H}{R}. \qquad . \qquad . \qquad . \qquad . \qquad (1)$$

The machine is reversible: that is to say, if driven by a motor it will produce a current; if connected with a battery it will produce work. For the same direction of the current the rotations are in opposite directions in the two cases, but in either of them the product $CA\mu H$ represents the work corresponding to one turn. In the first case, the machine consumes mechanical energy and produces electric energy, in this case it is said to act as a *generator;* in the second case it consumes electric and produces mechanical energy, and acts as a *motor*.

413. Magneto- and Dynamo-Electric Machines.—If the field in which the disc rotates is produced by fixed magnets, or by electromagnets excited by an extraneous source, so as to be independent of the working of the machine, this is said to be a *magneto-electric* machine. For a given speed, whether on open or closed circuit, it acts like a battery of constant electromotive force. Formula (1) shows, moreover, that this electromotive force is proportional to the speed.

But we may suppose the field in which the disc rotates to be maintained by the current which traverses the disc. It is sufficient for this purpose for the disc to be in circuit with the wire of the coil, or of the electromagnet, which produces the field, in other words, with the wire of the *field-magnet*. The magnetic induction is, in that case, a function of the strength C of the current. The machine is then a *dynamo-electric* machine, or, as it is usually called, a *dynamo*. The whole of the current may traverse the coil of the field-magnet, in which case the machine is said to be *series-wound*; or a portion of the current may be shunted, giving a *shunt-wound* dynamo; or, lastly, we may have a combination of the two, in which case the machine is a *compound dynamo*. Whatever be the arrangement, provided the volume of the copper and the density of the current are the same, the same field is always produced (**279**).

414. Dynamo without Soft Iron.—Two cases are to be distinguished according as the field-magnet of the dynamo does or does not contain soft iron.

If the field-magnet is a coil without a soft iron core, the magnetic induction is numerically equal to the field, and is therefore proportional to the strength of the current, and if P is a constant we may put

$$AH = PC.$$

Combining this with equation (1) we obtain the relation

$$n = \frac{R}{P} \qquad \qquad (2)$$

between the constant P, the resistance R, and the speed; the condition here expressed depends only on the construction of the machine, and must be satisfied if it is to act as a generator.

For the mechanical energy taken in per second from the motor is nPC^2, the energy expended in the same time in generating heat is RC^2. Assuming that there is no other work than generation of heat, we have

$$nPC^2 = RC^2,$$

that is to say, equation (2).

The condition which this equation expresses is that in every revolution the machine expends exactly the amount of energy put into it.

For a speed $n_1 < n$, the machine expends more than it receives. In this case it cannot maintain a current, and if a current is started it dies down again. For any velocity $n_2 > n$ the machine

takes in, on the contrary, more energy than it expends as heat, and, if the resistance of the circuit remained constant, the strength of a current would increase until the wires fused and the machine was destroyed. Lastly, for the value n, the strength of current will increase until the work done during each turn is equal to that furnished by the motor, and will therefore only depend on the power of the motor. Such a machine could not be utilised in practice.

415. Dynamo with Soft Iron—Characteristic.—If the field-magnet has an iron core, the magnetic induction is still a function of the current, but a more complicated one.

It increases with the magnetisation, at first almost proportionately to the strength of current, then less rapidly, and finally tends towards a limit when we get near saturation. If $\phi(C)$ is the magnetic flux cut during one turn of the disc when the current-strength is C, the function $\phi(C)$ represents the electromotive force for one turn per second; for a speed of n turns per second the electromotive force will be $n\phi(C)$. The product $C\phi(C)$ represents the positive or negative work of the electromagnetic forces for one turn of the disc for any speed.

All the properties of the machine are known if the function $\phi(C)$ is known. The curve obtained by taking C as abscissa and $\phi(C)$ as ordinate is called the *characteristic.*

416. Machine used as Motor.—Let us suppose the machine joined up with a battery of constant electromotive force E, and that its axle is acted on by a break, which keeps the work done per turn at the constant value W for any speed. As the electric measure of the work done by the machine for each turn is $C\phi(C)$ for a current of strength C, we must have $C\phi(C)=W$. Let R be the resistance of the circuit. If the battery cannot produce the current C in the resistance R, the machine will not turn; if it can produce a current of greater strength, the current will, nevertheless, still be equal to C; the machine will turn with a greater or less speed so as always to use up the energy furnished to it in excess of C^2R per unit of time. For a velocity of n' turns per second we shall have

$$EC=C^2R+n'C\phi(C),$$

or putting $E'=n'\phi(C)$,

$$C=\frac{E-E'}{R}.$$

The efficiency is

$$u=E'/E,$$

and the work done by the machine in each second

$$E'C = E' \cdot \frac{E - E'}{R}.$$

This work, being expressed by the product of two factors the sum of which is constant, is a maximum when the two factors are equal, that is to say, when

$$2E' = E,$$

in other words, when the current-strength C is half that which the battery would produce in a circuit of resistance R if the machine were at rest. The corresponding efficiency is equal to 0·50. This is a result at which we have previously arrived (**136**).

417. Machine acting as Generator.—Suppose the machine is driven by a prime-mover, and the current is employed to do work such as the generation of heat, the production of chemical decomposition, or driving a motor. The problem is to determine the strength of current corresponding to a given speed, or conversely.

Let us suppose a simple circuit of the total resistance R, and the machine rotating with a speed of n turns per second: Ohm's law gives

$$C = \frac{n\phi(C)}{R},$$

from which follows

$$\frac{\phi(C)}{C} = \frac{R}{n}.$$

If OM is the characteristic (Fig. 329), and the abscissa OP represents C, the ordinate MP represents $\phi(C)$, and $\frac{\phi(C)}{C} = \frac{\text{MP}}{\text{OP}}$ is the tangent of the angle MOP. Hence to solve the problem, we need only take $\text{OA} = 1$, draw a perpendicular at A, and on this perpendicular take a length $\text{AB} = \frac{R}{n}$, and join the point B with the origin; the point where the right line OB cuts the curve determines the position of M, and therefore $\text{OP} = C$, the strength of the current.

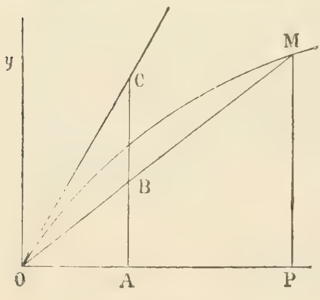


FIG. 329.

It will be seen that the current will be constant if n is made to increase proportionately to R.

For a given speed, the current diminishes as R increases, the point M continually approaching the origin. This goes on until

R/n becomes equal to AC/OA, OC being the tangent to the curve at the origin. If the resistance has a still greater value the machine ceases to work. The reason of this has been already given in (**414**); it is that the machine expends more energy than it receives.

If the induction is proportional to the current, as is the case when the field-magnet has no soft iron, the characteristic becomes simply a straight line passing through the origin; from which it follows that for a given speed there can be equilibrium only when the resistance of the circuit has a definite value, whatever the strength of the current.

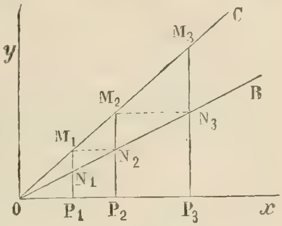


FIG. 330.

For let OC (Fig. 330) be the straight characteristic, and OB the line making an angle with the axis whose tangent is equal to $\frac{R}{n}$, the ratio which must exist between the electromotive force and the current, in order that the latter may be stable. If, at a given moment, the strength of the current were OP_1, the machine would give rise to an electromotive force, M_1P_1, greater than N_1P_1, necessary to maintain the existing current. This electromotive force would therefore produce a stronger current, OP_2, which in turn would develop an electromotive force M_2P_2, and so on; the current would thus continue to increase without limit. The result would be reversed if the line BO lay farther from Ox than CO.

If the electromagnet has any residual magnetism, or if it has a second wire traversed by a current independent of the machine, $\phi(C)$ is no longer zero when $C=0$; in this case the characteristic does not pass through the origin, but is displaced parallel to itself by a quantity OO_1 (Fig. 331), so that for $C=0$, we have $\phi(0)=\text{OH}$. The current is obtained by the same construction as before, but the machine always starts whatever the resistance.

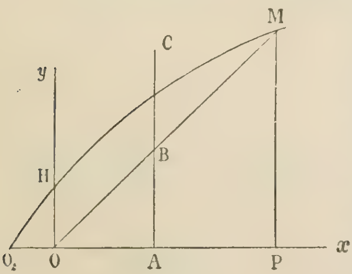


FIG. 331.

418. Let us suppose that the circuit having resistance R contains a constant electromotive force E', independent of the current, that of a battery or of a voltameter, for instance. The electro-

motive force E' must be taken positive if it acts in the same direction as the machine, and negative if it is in the opposite direction. Let us consider the latter case; the equation

$$C=\frac{n\phi(C)-E'}{R} \quad . \quad . \quad . \quad . \quad (3)$$

gives

$$C\frac{R}{n}=\phi(C)-\frac{E'}{n}.$$

It will be seen (Fig. 332) how the construction must be modified: the line, the tangent of whose inclination is $\frac{R}{n}$, instead of passing through the origin must be drawn through a point o′ of the axis of y, at a distance $oo'=\frac{E'}{n}$ upwards or downwards, according as E' is negative or positive.

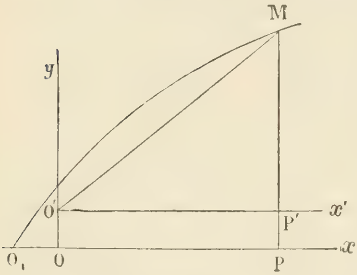


FIG. 332.

419. Let us suppose, lastly, that the machine has to drive a motor. Let $\phi(C)$ and $\phi'(C)$ be the characteristic functions for the two machines, n and n' their speeds. Both are traversed by the same current; and the current-strength, the efficiency, and the work given out by the motor are expressed by the equations

$$C=\frac{n\phi(C)-n'\phi'(C)}{R} \quad . \quad . \quad . \quad . \quad (4)$$

$$u=\frac{n'\phi'(C)}{n\phi(C)} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$W=n'C\phi'(C) \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The problem may be stated in several ways. Most frequently the prime mover drives the generator at a fixed speed, being able to supply energy to it up to a definite rate. The motor has to do an amount of work per revolution which may or may not be defined. In the former case the current is determined by the condition

$$W=C\phi'(C)$$

and the speed n' is given by equation (6). In the second case n' and C are arbitrary, and they are determined either by the

condition of having a definite efficiency or by that of obtaining a maximum rate of work.

It is clear that, if the motor is prevented from rotating, the current has its maximum strength, but the work done and the efficiency are both nothing. If the motor were able to move without mechanical resistance, its velocity would increase until its electromotive force became equal to that of the generator; the current would then be nothing, and the efficiency unity, but only because both factors of the ratio would vanish (comp. **136**). Between these two extremes there may be all possible efficiencies from 0 to 1, and the conditions for a maximum rate of work may be satisfied. Although the value of W', when C is replaced by its value derived from equation (4), is represented by a product of two factors the sum of which is $n\phi(C)$, the maximum does not occur when the two factors are equal, for this sum is not constant, C being a function of n', and the condition of the maximum cannot be determined so long as $\phi(C)$ is not given.

If the two machines are identical, the formulæ reduce to

$$\left.\begin{aligned} C &= \frac{(n-n')\phi(C)}{R} \\ u &= \frac{n'}{n} \\ W' &= n'C\phi(C) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (7)$$

The first shows that, for a given current, the difference of speeds is proportional to the resistance of the circuit; the second that the efficiency is equal to the ratio of speeds; the third that the work transmitted is proportional to the speed of the motor. It is not correct to say, as is often done, that the maximum is when $n' = \frac{n}{2}$, and corresponds to the efficiency 0·50.

420. Transmission of Energy.—Let us suppose that n being given, we wish to obtain from the motor an efficiency u. Equation (5) gives n', equation (4) gives C, and substituting the values in (6) we obtain the equation

$$W' = n^2u(1-u)\frac{[\phi(C)]^2}{R},$$

which shows that we may transmit the same quantity of energy with the same efficiency and the same speed of the generator and motor, provided that we make $[\phi(C)]^2/R = \text{constant}$; that is to say, provided the characteristic function of the two machines is varied in proportion to the square root of the total resistance of the cir-

cuit. Suppose, for instance, that the resistance is multiplied by 25; if we make the electromotive force per turn in each machine five times as great as before, the strength of the current given by formula (4) will be one-fifth: the two machines will do the same work as before for the same number of turns, and the energy used up in overcoming resistance will remain as it was.

In order to make the electromotive force of the two machines five times as great, their construction and dimensions may be modified. But we may also, at each end of the circuit, couple on the same axle five machines connected in series and each of them identical with the original one.

421. The results obtained in this chapter apply to machines with a strictly constant current, like a Faraday's disc, moving in a uniform field, or one that is symmetrical about the axis. Unfortunately, machines of this type only give feeble currents. The very powerful currents obtained from ordinary machines result from much more complicated conditions which we shall investigate in the next chapter. They give continuous currents in this sense that their direction is constant, but the strength is not uniform. Moreover, the electromotive force for a given current is no longer proportional to the speed, and if $\phi(C)$ represents the electromotive force for a speed of one turn per second, the electromotive force for a speed of n turns per second is not exactly represented by $n\phi(C)$.

CHAPTER XXXIV

CONTINUOUS-CURRENT MACHINES

422. Gramme Machine—Essential Parts.—The principal part of the machine is a coil formed of a series of loops or sections, B, all coiled in the same direction, and completely encircling a soft iron ring, A (Fig. 333). Each loop is in connection with the next by means of a conducting *commutator-strip*, L, mounted on an insulating cylinder concentric with the ring, and rigidly connected with it, called the *collector*. The two ends of each loop or section of the coil are fixed to two consecutive strips, and each strip is soldered to the termination of the preceding section, and to the beginning of that which follows. The successive sections thus form a continuous coil constituting a closed circuit. If any two diametrically opposite strips were connected with the poles of a battery, the current would divide equally between the two halves of the ring, and the soft iron core would be magnetised so as to have one pole where the current enters, and the other where it leaves. Each section of the coil may be made up of one or of several turns of wire.

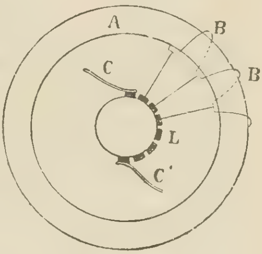


FIG. 333.

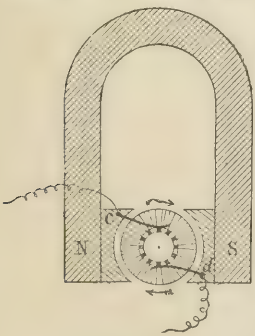


FIG. 334.

The ring is placed between the poles of a horse-shoe magnet or of an electromagnet, and is made to rotate uniformly (Fig. 334). Two brushes, *c d*, rest on the collector at the ends of a diameter, which is nearly at right angles to the line of the poles of the magnet, and collect the currents in the ring; these are the poles of the machine.

Without the soft iron ring the field between the branches of the magnet, or rather between the *pole-pieces* of soft iron which are

attached to it, would be nearly uniform; the ring, placed in this field, acquires by influence poles at the two ends of a diameter which are of the opposite kind to those of the adjacent pole-pieces; it thus absorbs the lines of force proceeding from the pole-pieces, and conducts them through its mass, so that only a very small number of lines pass through the empty space inside the ring (Fig. 335). It follows from this that the coils moving in the field cut the magnetic flux only by the part of each which is on the outside of the ring. As far as the explanation of the action of the machine is concerned, it is immaterial whether the iron core is supposed stationary, and the whole set of coils supposed to slide round it; or whether the core itself turns carrying the coils with it; in the latter case the poles are displaced within the ring, but they remain fixed in space. In this case, which is that met with in practice, it is important, in order to prevent currents from being induced in the mass of the ring, or so-called Foucault currents (**326**), that it should be formed of a bundle of iron wires, or thin plates placed close together; the wires or plates are insulated from each other, and their planes are perpendicular to the axis of the ring, in other words, to the plane of the windings, and therefore to the direction in which the electromagnetic induction acts. The ring is usually in the form of a hollow cylinder.

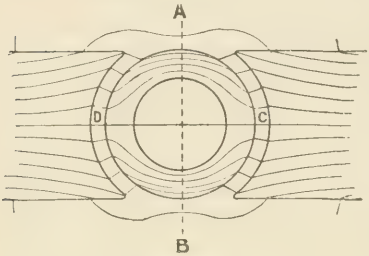


Fig. 335.

423. Action of the Machine.—Let us confine our attention in the first place to a single section of the coil while the machine makes one complete revolution. In order to account for the effects of induction produced in this section, we may consider either the variations of the magnetic flux through the area enclosed by it, or the flux cut at each instant by the various elements of the contour (**303**). According to the first point of view, the plane of the coil at A and at B (Fig. 335) is perpendicular to the magnetic flux, but in passing by ACB from the first position to the second the coil turns through 180°. If N_a be the total flux through the iron ring, counting both halves, the change of flux in the coil is simply equal to N_a. An equal variation, but of opposite sign, takes place in the coil in the second part of the revolution as it passes from B to A through D.

According to the second mode of looking at the matter, it is

easy to see that no flux is cut except by that portion of a winding, which covers the ring on the outside. The electromotive force of induction is nothing at A, increases from A to C, then decreases from C to B. At B it passes through zero, changes sign, and, in the second half rotation, from B to A through D, it again passes through the same series of values, but with the signs reversed.

The direction of the electromotive force is thus opposite on the two sides of a vertical diameter perpendicular to the line of the poles. In the conditions for which the figure is drawn, the lines of the field go from left to right, and the motion of the coil is like that of the hands of a watch; the current then goes from front to back of the figure in the outside part of the winding during the motion BDA, and from back to front during the motion ACB. The electromotive force varies during a complete revolution in the way represented by a curve like ABA (Fig. 336), more or less resembling a curve of sines.

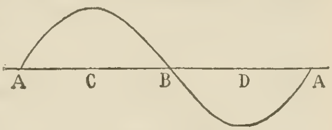


FIG. 336.

Let us now consider the condition at a given moment of the whole system of moving coils: all those on the left of the diameter AB are the seat of electromotive forces in the same direction, acting, for instance, in consequence of the mode of winding, so as to make the current traverse the coil from B to A; and since all the sections communicate with each other, these electromotive forces are added together, and tend to produce a current from B to A. All the sections of the coil on the right of the diameter AB are also the seat of electromotive forces in the same direction; relatively to each section of the coil, these electromotive forces act in the opposite direction to the former, but as the face of each section, which was previously turned upwards, is now turned downwards, the electromotive forces still tend to make the current go from B to A, and the total electromotive force in this half of the coil is manifestly equal to that in the first half.

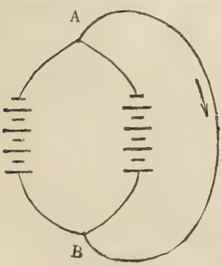


FIG. 337.

Since the two brushes connect the points A and B with the outer circuit, this will be traversed by a continuous current from A towards B, equal to the sum of the currents which traverse the two halves of the ring. The case is the same as that of a battery, the elements of which are arranged in two equal and parallel series (Fig. 337); to make the analogy complete, it must be assumed that the electromotive force of the elements

decreases symmetrically in each series from the middle where it is a maximum to the ends where it is zero.

424. Variations of the Current.—The current is continuously in the same direction, but is not quite uniform. One brush is alternately in contact with a single commutator-strip and with two.

In the latter case, the corresponding section of the coil is short-circuited, and its contribution to the resultant electromotive force is withdrawn for the time being. Hence arise periodical variations both of electromotive force and of resistance, and therefore of the strength of the current. The curve which represents the current as a function of the time, instead of being a straight line parallel to the axis of the time, is indented, the length of each indentation corresponding to the time necessary for one commutator-strip to take the place of the preceding one.

The indentations are the smaller and the current the more uniform the greater the number of subdivisions of the commutator.

425. Displacement of the Brushes—Angle of Lead.—If the machine turns uniformly on open circuit, the facts are in exact agreement with the preceding discussion. A difference of potential is established between the two poles formed by the brushes, and this difference is a maximum when the line of the brushes is in the plane of symmetry of the field-magnets. Moreover, for a constant strength of the field, the difference of potential is proportional to the speed of rotation of the ring.

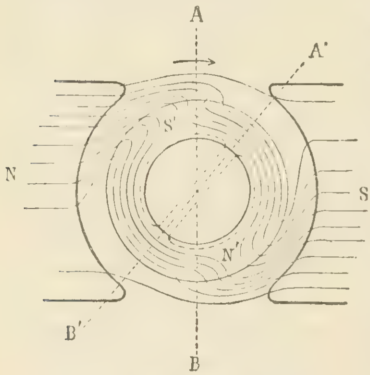


FIG. 338.

If the circuit is closed other phenomena are produced; strong sparks pass between the brushes and the commutator-strips which they leave. These sparks would soon destroy the commutator, and, in order to prevent them, experiment shows that the line of the brushes must be displaced in the *direction of the motion*. A position, A′B′ (Fig. 338), varying with the strength of the current, is thus found, for which the sparks are totally suppressed, or at all events, reduced to a minimum. The angle which the line A′B′ of the brushes makes with the line of symmetry AB is called the *angle of lead*.

The machine is clearly reversible. If it works as a motor, for a current of the same strength and in the same direction, the ring

will turn in the opposite direction, but the position of the brushes for which sparking ceases is still, as before, the position A′B′; accordingly, the diameter of contact must be turned through the same angle, but in *the direction opposite to the motion.*

Let us consider the last section of the coil to the left of the line A′B′ at the moment at which the commutator-strip connected with the leading end is in contact with the brush; it is traversed by the current $\frac{1}{2}C$; the next moment it is short-circuited, the brush being in contact simultaneously with the strips connected with both ends, and the current therefore ceases; then, as soon as the leading strip is set free, the section enters the second half of the rotation and is again traversed by the current $\frac{1}{2}C$, but in the opposite direction.

The self-induction of the coil opposes so much resistance to the sudden starting of the current, that the greater part of the flow passes as a spark across the air-space between the commutator-strip and the brush which it has just left. It is the same case as that of the experiment of Fig. 287, except that the quantities of electricity that come into account in the present case are usually much greater.

These discharges and their destructive effects would be avoided if the section, while on short circuit, were submitted to an inductive action capable of gradually developing in it beforehand a current equal to that which it will receive when the commutator-strip quits the brush. This is done by bringing the coil into a part of the field which can produce this induction. It is evident from this that the angle of lead must increase with the strength of the current.

426. Reaction of the Armature.—The necessary displacement of the brushes complicates the action of the machine. The principal effect is to modify the shape of the field and the magnitude of the electromotive force.

The field results from the magnetisation produced both by the current in the coils of the field-magnets and by that in the ring. In the case represented in the figure and with the direction of winding assumed, the current in the ring would tend to produce a south pole at A, and a north pole at B, that is to say, a magnetisation perpendicular to that due to the field-magnets. If the line of the brushes remained in the plane of symmetry, the field would still be symmetrical, but if the line of brushes is brought to A′B′ (Fig. 338), there results a distortion of the field readily observable by means of iron-filings, an idea of which is given by the figure.

It is not accurate to say that the effective electromotive force is

the sum of the electromotive forces which would be given by each of the magnetisations separately, but it is certain that the second acts in the opposite direction to the first. Moreover it is evident that owing to the displacement of the brushes, the poles developed at A′ and B′ must exert an influence on the pole-pieces, supposing them not independently magnetised, tending to turn the ring in the direction of the actual motion. The effect of this action is to diminish the work against electromagnetic forces, and as the electromotive force is equal to this work for each unit of current, the effective electromotive force is diminished by so much.

When the machine acts as a motor, the same causes have the effect of increasing the electromotive force.

427. Losses of Energy.—Whenever a section of the coil is short-circuited by a brush, it loses the intrinsic energy (**304**) which it possessed, and which must be restored to it a moment after, without contributing to the action of the machine. The strength of the current being $\frac{1}{2}C$, if l is the self-inductance of the section, the loss of energy in each turn is $\frac{1}{8}lC^2$. As the effect occurs twice during each revolution for each section, if $2m$ is the number of sections, and n the number of revolutions per second, the loss for each revolution is $2ml\frac{C^2}{4}$, and in a second $2mnl\frac{C^2}{4}$, or $nL\frac{C^2}{4}$, L being the self-inductance of the whole ring. The loss is the same as would be caused by an increase of the resistance of the ring equal to nL.

Each portion of the ring during one revolution goes through a complete magnetic cycle, which represents a definite amount of work (**212**). Lastly, the precautions taken to diminish the Foucault currents (**326**, **422**) do not destroy them completely. These losses are usually greater than that which arises from the resistance of the ring, and it is to them chiefly that the heating of the machine is due.

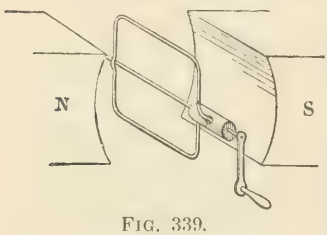


FIG. 339.

428. Siemens' Machine.—This only differs from Gramme's machine in the winding of the armature. A rectangular conducting frame (Fig. 339) turning uniformly between the poles of an electromagnet, may be regarded as an elementary constituent of such a machine. The two sides of the rectangle parallel to the axis are at each moment the seat of equal and opposite electromotive forces, but these

evidently act so as to cause a current to circulate in the same direction round the rectangle. The electromotive force is a maximum as the rectangle passes the plane parallel to the field, it is zero and changes sign when the rectangle is perpendicular to this plane.

The ring of the Gramme machine is replaced by a drum of soft iron on the outside of which wire is wound forming a series of rectangular sections analogous to the preceding, and placed in equidistant planes passing through the axis. Each of these sections is formed of several windings, and all are connected by means of a Gramme commutator, the end of one section and the beginning of the next terminating in the same segment of the commutator. Thus, to describe the simplest mode of winding, the wire from the strip No. 1 goes directly to the circumference of the drum, follows the corresponding generating line of the cylinder, crosses the second base diametrically, returns along the opposite generating line, and after being carried round eight times in this way, is brought to the commutator-strip No. 2, and so on for the other sections. Only half the whole number of strips are used for windings which completely cover the drum; the winding is continued in the same manner, using the second half of the commutator, and forming a second outer layer of wire, so that sections connected to strips which are diametrically opposite each other are exactly superposed.

Two brushes placed in a plane nearly perpendicular to the field receive a continuous current, but rather less uniform than that of the Gramme machine, the number of sections being generally less.

Thus the essential difference between the two machines is that in the Gramme ring the wire is coiled on the ring both outside and inside, and that on a Siemens' drum it is only on the outside; the drum might therefore be a massive cylinder.

The two systems give pretty much the same results. Each has its inconveniences and its advantages. With Siemens' mode of winding the inactive portion of the armature wire is shorter; the magnetisation of the core is less, and therefore the reaction of the armature is also less. But on the base of the drum there are portions of the wire in close contact with each other which are at very different potentials, and this makes the insulation more precarious. Besides, the sections are more difficult to fix upon the drum, and one cannot be uncoiled without uncoiling all, while in the ring the sections are completely separate, and one can be easily replaced without interfering with the rest.

429. Designing a Machine.—The properties of a dynamo-

machine like that of Gramme, considered as a whole, may be regarded as resulting from the relative motion of two closed circuits, each of which is the seat of a special kind of flux: the magnetic circuit, the seat of the flux of magnetic induction, and the electric circuit, the seat of the electric current: these two fluxes being connected by the equation $4\pi nC = \frac{1}{\mu}\frac{l}{S}N$ (**284**).

The magnetic circuit is made up of the field-magnets, the soft iron core of the ring or of the drum, frequently called the *armature*, and by the space between the pole-pieces and the armature, a space partly filled by air, partly by copper, which is sometimes called the *interferricum*.

The flux of magnetic induction is not the same in all parts of the circuit; part escapes at the ends of the pole-pieces, and a further portion at the entry into the armature. It is the aim in construction to render these losses as small as possible; they may be taken as being a constant fraction of the total flux for a given machine.

This fraction can be readily determined experimentally after the machine is made, by coiling round each part one or two turns of wire connected with a ballistic galvanometer, and reversing the inducing current (**315**).

Let N_a be the total flux of induction through the armature, divided equally between the two halves in the case of a ring-armature, N_b the flux in the interferricum, N_m that in the field-magnets: representing by p and q factors greater than unity, which may be determined in the manner just mentioned, we may write

$$N_b = pN_a$$
$$N_m = qN_b = pqN_a.$$

It is evident that if N_a is known we know also the electromotive force. At every revolution there is a change of flux through each turn of wire equal to N_a; if there are $2m$ turns of wire, and if n is the number of revolutions per second, we have

$$E = 2mnN_a.$$

Let R with a corresponding suffix represent the factor $\frac{1}{\mu}\frac{l}{S}$ for each part of the circuit; in other words, the magnetic reluctance of this part (**283**). The value of R varies with the magnetic induction; the value of μ, which is required for the calculation, is deduced in each case from the curves of magnetisation. If we apply formula (2) of (**284**), remembering that, in the armature,

there are as many turns of wire traversed in one direction as in the opposite direction, and that therefore the corresponding term in the expression for the magnetomotive force will vanish, we obtain

$$N_a(\mathrm{R}_a + p\mathrm{R}_b + pq\mathrm{R}_m) = 4\pi nC,$$

n being the number of turns of wire on the field-magnet, and C the strength of the current which traverses them.

Assuming a series of values for N_a, the corresponding values of C may be calculated, and a curve may be constructed taking C as abscissa and N_a as ordinate. In this way a kind of *characteristic* curve is obtained for the machine, which may be called its magnetic characteristic. This is not the same as the characteristic already spoken of (**415**), and is independent of the speed.

In actual machines the aim is to get the greatest possible value of N_a in the armature—values, for instance, amounting to as much as 15,000 to 20,000 units per square centimetre (**213**). In the field-magnets, on the contrary, the aim is not to exceed semi-saturation. The section of the field-magnets relatively to that of the armature is calculated from this consideration, and from the values taken for the coefficients p and q.

CHAPTER XXXV

ALTERNATE-CURRENT MACHINES

430. General Properties.—The type of all machines generating alternating currents is the coil rotating about a diameter in a uniform field (Fig. 340). The current is sinusoidal and has a retardation of phase ϕ relatively to the electromotive force defined by the equation

$$\tan 2\pi\phi = \frac{2\pi L}{RT};$$

the maximum value of the current is

$$C_o = \frac{E_o \cos 2\pi\phi}{R},$$

E_o being the maximum value of the electromotive force (**329**, equations (5) and (6)).

The work is equal to the product of the root-mean-square electromotive force into the root-mean-square of current, multiplied by the cosine of the difference of phase (**332**),

$$P = \frac{E_o C_o}{2} \cos 2\pi\phi.$$

FIG. 340.

It may be added, further, that if, in Fig. 341, OR represents the maximum value E_o of the electromotive force, and the angle ROC the lag $2\pi\phi$, OC represents the maximum of the effective electromotive force, and OI, which is at right angles to OC, represents the maximum electromotive force of self-induction (**330**).

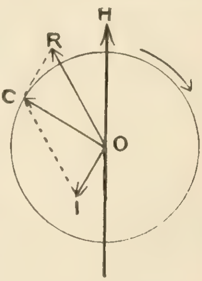


FIG. 341.

The root-mean-square of current is $\frac{\text{OC}}{R\sqrt{2}}$, while the work per second, or power, is $\frac{\text{OC}}{R\sqrt{2}} \times \frac{\text{OR}}{\sqrt{2}} \cos \text{ROC}$, or $\frac{\text{OC} \cdot \text{OC}}{2R}$. The phase of the electromotive force of self-induction being always 90° behind the current, the cosine of the difference of phase vanishes, and the

corresponding work is nothing. Hence self-induction produces no loss of energy; its only effect is to limit the power of the machine.

431. Coupled Machines. — Let us imagine two machines coupled together, two coils, for instance, mounted on the same axis, and making an angle of α with each other, the wires being joined so as to form a single circuit. Adopting the same mode of representation, the two lines OL and OF (Fig. 342) making the angle α, representing the given electromotive forces, the maximum value of the resultant electromotive force is represented by the diagonal OR of the parallelogram, and the construction is completed, as before. As the self-inductances are simply added together, the difference of phase relatively to the effective electromotive force or the current, that is to say, the angle ROC is independent of α.

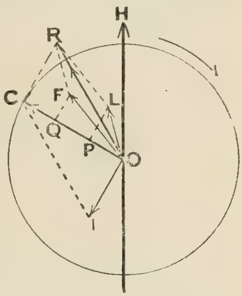


FIG. 342.

The work expended per second in the form of heat in the circuit is $\frac{\text{OC}}{R\sqrt{2}} \cdot \frac{\text{OC}}{\sqrt{2}}$, or $\frac{\text{OC}}{2R} \cdot \text{OC}$; the contribution of the first machine is $\frac{\text{OC}}{2R}\,\text{OP}$ and that of the second $\frac{\text{OC}}{2R}\,\text{OQ}$. These quantities of work are respectively proportional to the lines

OC, OP, OQ,

and it is apparent from the figure that the first is in all cases the algebraical sum of the other two.

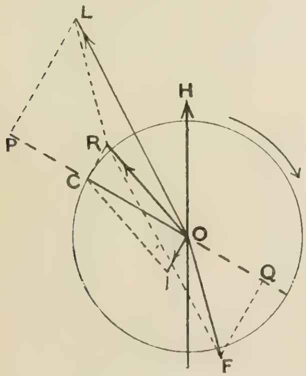


FIG. 343.

If the two projections fall on the same side of the point O, the amounts of work represented by them are of the same sign, and the two machines act as generators absorbing work. If they fall on opposite sides (Fig. 343), the first machine absorbs work and acts as a generator, the second produces work and acts as a motor. The work that must be done upon the combined system, and is dissipated in the form of heat, is the difference between the amounts of work taken in by the first machine and given out by the second.

432. Coupled Alternators continued. — Instead of two machines mounted on the same axle, let us consider two machines each driven by a special motor and not connected with each other otherwise than electrically. The motors are supposed to be regulated so as to drive the machines at very nearly the same speed. If the machines were independent, the smallest difference of speed would make them pass successively through every possible difference of phase: there would be no stable condition, unless they were able to act on each other in such a manner as to maintain their synchronism and their difference of phase.

Let us take the simplest case, that of two identical machines: the conclusion arrived at will be none the less generally applicable. Since the machines are supposed identical the line OL equals the line OF (Fig. 342), and these lines make equal angles with OR on opposite sides of it. It is obvious therefore that the angle LOC is necessarily greater than the angle FOC; that is to say, that the *leading* machine always makes, with the effective intensity, a greater angle than the *following* machine. It follows from this that the latter always does more work than the former, and consequently requires a greater torque to maintain it in steady motion. Unless this torque be supplied it will slacken speed; this will cause the phase-difference a between the two machines to increase, and this increase will continue until the difference of phase is 180°, that is, until the machines are in opposition; in which case the current is zero when the two machines are *identical*, and no electrical work is done by either. In fact, two such machines running in phase with one another are in unstable equilibrium; and when by any accidental circumstance (such, for example, as a slight increase of friction) one of them begins to lag ever so little behind the other, the forces called into play are such as to increase this lag.

Two alternate-current machines placed *in series* in the same circuit cannot, therefore, be made to work in the same *direction;* they can, however, if joined *in parallel.* For in the closed circuit containing the two machines, at two points of which the external circuit is connected, the two machines will act in opposition; if the external circuit has a relatively high resistance, which is always the case in practice, its influence on the *régime* may be neglected, and therefore the machines work under conditions of stable equilibrium. The qualities which best suit coupled working are a small self-inductance and resistance, for when one machine begins to lag it is most quickly pulled into step again when the current which the leading machine sends through it in consequence is as large as

possible; the reason being that this current acts as a motor current accelerating the lagging machine.

When one machine is used as a generator and another as motor, the two machines are connected electrically, but are mechanically independent. Equilibrium then corresponds to that difference of phase for which the work transferred to the motor is equal to the work done by it. The efficiency is equal to

$$\frac{\mathrm{OQ}}{\mathrm{OP}}$$

(Fig. 343), and the coefficient of loss in the circuit to

$$\frac{\mathrm{OC}}{\mathrm{OP}}.$$

433. Alternating Motors.—Most alternators can be used either as generators or as motors. In the latter case the field-magnets are excited by an independent continuous current machine, and an alternating current is sent through the armature coils. The armature can move indifferently in either direction. Nevertheless, as the number of dead points is considerable, the load can only be put on after the armature has been made to rotate by hand or otherwise with something like the velocity which it ought to have. The reactions finally determine the synchronism of the motor and of the alternating generator that drives it.

There are, however, several motors constructed on a different principle, in which the alternating current of the generator is sent into the field-magnets, and the armature, forming a closed circuit of itself, has no connection with other parts.

As is well known, two equal simple harmonic vibrations of the same period, at right angles to each other, differing in phase by a quarter of a vibration (or by 90°), compound to a uniform rotation in a period equal to that of either component (**335**). As magnetic fields are compounded geometrically, the two alternating electro-magnets give a field of constant strength, the direction of which revolves uniformly, and they consequently drag the armature round in the same direction with an amount of lag that depends on the work done by the armature. As in this arrangement brushes are dispensed with, there is no sparking.

434. Transformers.—Instead of directly using the alternating current, it may be used to excite a current of the same kind in a neighbouring circuit. The primary current corresponding to certain numbers, E and C, of volts and amperes, the secondary current would correspond to other numbers E' and C'. If the transformation is effected without loss of energy, we have

$$EC = E'C',$$

and by a suitable arrangement of apparatus the two factors of the product may be modified at will.

Such apparatus are called *transformers*. They play an extremely important part in the transmission of electrical energy to a distance. An example will illustrate the practical aspect of the question.

Suppose that it is desired to transmit the power corresponding to 500 amperes at an electromotive force of 100 volts, *i.e.*, 50,000 watts, to a distance of 1000 metres with a loss of 10 per cent.

For direct transmission, the conductor, 2000 metres in length, must have a resistance of only 0·02 ohm; this would imply a section of 16 square centimetres; the weight of the conductor would be 28,000 kilogrammes, and its price about £3360.

If the same amount of energy were transmitted by a current of 50 amperes with a potential difference of 1000 volts, and then transformed at the farther station into a secondary current of 500 amperes at 100 volts, the resistance of the connecting wire might be 2 ohms, its section 0·16 square centimetre, its weight 280 kilogrammes, and its price about £34.

435. Various Types of Transformers.—In any transformer there are three circuits to be considered: the primary circuit, the secondary circuit, and the magnetic circuit. It is most advantageous to use a closed magnetic circuit. The types which appear to give the best results are represented in Figs. 344 and 345. The arrangement of the first is like that of a Gramme ring; the primary circuit AB and the secondary *ab* are coiled simultaneously on a soft iron core, which usually consists of a coil of wire. In the second the two electrical circuits AB and *ab* are wound together, so as to form the core, and the iron wire is coiled on the outside. In the former case the *core*, and in the latter the *envelope*, is of soft iron; in the former the magnetisation is longitudinal, and in the second transverse (**288**). Both systems appear to give equally good results.

436. Theory of the Transformer.—Let us suppose that at the two terminals of the primary circuit AB a difference of potential is established which varies harmonically. If, as in (**333**), we neglect variations of μ and the effects of hysteresis, the two electrical circuits and the magnetic one will each be the seat of a flux which varies harmonically in the same period, but with different phases.

We will represent by the same letters, but with suffixes 1 and 2, corresponding quantities connected with the two circuits, primary and secondary; thus, n_1 and n_2 will be the number of turns

of wire, R_1 and R_2 the resistances, and c_1 and c_2 the simultaneous current-strengths.

The maximum strengths of current C_1 and C_2, which are directly deduced from the root-mean-square currents, may be regarded as data furnished by experiment.

The variation of the flux of induction is the same for each turn of the primary and of the secondary circuit; if we represent by N the maximum value of the flux of induction, the maximum values of the electromotive forces developed in the two circuits are (**328**)

$$E_1 = n_1 \frac{2\pi}{T} N, \text{ and } E_2 = n_2 \frac{2\pi}{T} N;$$

they are therefore in the same ratio as the numbers of the turns of

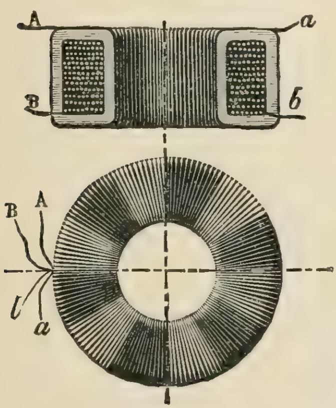


FIG. 344.

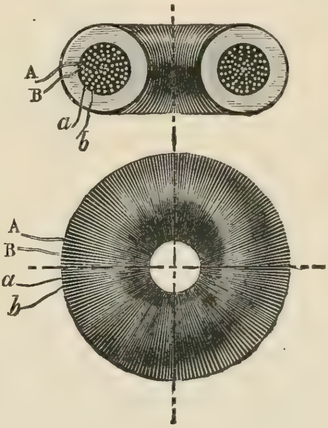


FIG. 345.

wire, and both differ in phase by 90° as compared with N. If we further assume that the secondary circuit is without self-induction, there is no difference of phase between the electromotive force and the current which it produces, and we have simply

$$E_2 = C_2 R_2;$$

E_2, E_1, and N are thus known.

Moreover, if R is the magnetic reluctance (**283**)

$$\mathrm{R}N \sin 2\pi \frac{t}{T} = 4\pi(n_1 c_1 + n_2 c_2);$$

this equation, in which $c_2 = C_2 \cos 2\pi \frac{t}{T}$, enables us to calculate $n_1 c_1$.

We have
$$n_1 c_1 = n_1 C_1 \sin 2\pi\left(\frac{t}{T} + \phi\right)$$
where $2\pi\phi$ is the acute angle of a right-angled triangle OAB (Fig. 346), the hypothenuse OB of which is $n_1 C_1$ and the opposite side AB is $n_2 C_2$. It remains to find the intensity and the phase of the electromotive force acting at the ends of the primary wire. If the values at a given moment are denoted by small letters, we have

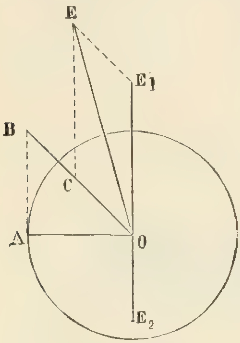


FIG. 346.

$$e = R_1 c_1 + e_1.$$

With the mode of construction already used, it will be seen that the electromotive force required is the projection of the diagonal E of a parallelogram, the sides of which are $C_1 R_1$ and E_1. To construct this parallelogram, let us take two rectangular axes, and draw the triangle mentioned above. The direction OA corresponds to the magnetic induction N, the direction OB to the current C_1; let us represent the electromotive forces of induction by the lines OE_1 and OE_2 perpendicular to OA; on OB take a length OC, equal to $C_1 R_1$, and complete the parallelogram; the diagonal OE is the maximum value of the difference of potential between the ends A and B of the primary wire, and the angle $2\pi(\phi + \psi)$ which it makes with OA is the angle of *lead* relatively to the magnetisation.

In ordinary transformers the angle $2\pi(\phi+\psi)$ is very near 90°; hence the angle $2\pi\psi$ is the complement of $2\pi\phi$. Observing that $\sin 2\pi\phi = \frac{n_2 C_2}{n_1 C_1}$, we have approximately
$$E = E_1 + R_1 C_1 \cos 2\pi\psi = C_2\left(\frac{n_1}{n_2} R_2 + \frac{n_2}{n_1} R_1\right).$$

From this equation we obtain
$$C_2 = \frac{\frac{n_2}{n_1} E}{R_2 + \left(\frac{n_2}{n_1}\right)^2 R_1},$$
or, observing that the factor $\left(\frac{n_2}{n_1}\right)^2$ is very small,
$$C_2 = \frac{\frac{n_2}{n_1} E}{R_2};$$

an expression which shows that the secondary current is very nearly that which would be produced in the secondary circuit by an electromotive force which was a fraction $\frac{n_2}{n_1}$ of that acting at the ends of the primary wire.

The resistance R_2 is made up of two parts: the resistance of the external circuit, and that of the wire ab; as this does not exceed a few hundredths of an ohm, it follows that the difference of potentials at the terminals of the secondary circuit is sensibly

$$\frac{n_2}{n_1}E.$$

The work per second done in the secondary circuit is

$$W_2 = \tfrac{1}{2}E_2C_2 = \tfrac{1}{2}R_2C_2^2;$$

the energy supplied at the terminals of the primary circuit is

$$W_1 = \tfrac{1}{2}EC_1 \cos 2\pi\psi = \tfrac{1}{2}E\frac{n_2}{n_1}C_2;$$

the efficiency is therefore

$$u = \frac{W_2}{W_1} = \frac{E_2}{\frac{n_2}{n_1}E} = \frac{R_2}{\frac{n_2}{n_1}\left(\frac{n_2}{n_1}R_1 + \frac{n_1}{n_2}R_2\right)}$$

$$= \frac{1}{1 + \left(\frac{n_2}{n_1}\right)^2\frac{R_1}{R_2}}.$$

This differs but little from unity. Experiment shows that the practical efficiency is greatest when the transformers work at full load. The magnetisation thus oscillates within narrow limits, and the losses due to hysteresis are small.

437. Ruhmkorff's Coil.—Ruhmkorff's coil is a true transformer, in which the aim is, by means of a strong current generated by a source of small electromotive force, to obtain, in the secondary wire, a considerable electromotive force capable of giving long sparks and of charging Leyden batteries,—capable, in short, of reproducing all the effects which are ordinarily obtained with electrostatic machines.

The apparatus represented in Fig. 347 consists of a cylindrical core of soft iron formed of a bundle of parallel wires; on this core is wound the primary coil formed of stout insulated wire, and outside this is coiled the secondary wire, consisting of a great number of turns of fine wire connected with the two terminals A

and B. This secondary wire is very carefully insulated so as to avoid internal discharges. Instead of coiling it in successive layers parallel to the axis, it is wound in separate sections perpendicular to the axis, and these are separated from each other by insulating partitions. In this way the potential increases from one end of the wire to the other without there being anywhere too great a difference between two adjacent layers.

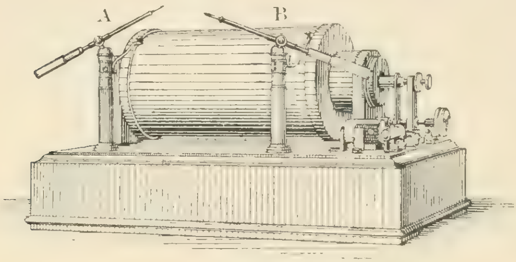


FIG. 347.

The primary current is given by a battery, and the induction results from the alternate making and breaking of the circuit. This arrangement, which would not be suitable in the case of a closed magnetic circuit in consequence of the retention of magnetisation (**207**), may be used in the case of a short cylinder, and it is even preferable to the use of a current varying harmonically, inasmuch as the variations of the current are more sudden.

438. Make and Break.—The make and break is often effected by means of a piece of soft iron, O, placed below the end M of the soft iron core (Fig. 348). The hammer, OD, and its anvil, with the corresponding pieces, *e*, *a*, H, form part of the primary circuit. When the current passes, the core is magnetised, the hammer is raised, and the circuit broken; the hammer in falling again closes the circuit, and so forth.

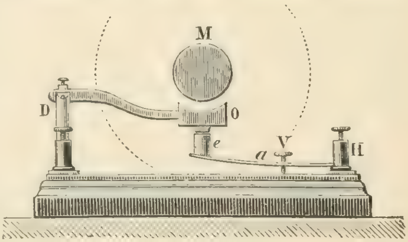


FIG. 348.

The break is made less sudden by the spark which passes between the hammer and the anvil. In order to make the break more rapid, Foucault used a platinum point vibrating vertically up and down and dipping into mercury. The spark is considerably reduced by covering the mercury with a layer of alcohol. The vibrating motion is kept up by a separate battery and electromagnetic contact breaker.

439. Wehnelt Interrupter.—If the ends of the primary of a coil, giving, say, a 6-inch spark or more, are connected to a large

battery through an electrolytic cell consisting of weak acid in which the electrodes are a lead plate (kathode) and a platinum-wire (anode), the latter being encased in a glass tube with the exception of a few millimetres at the end, a torrent of dense sparks passes at the secondary spark gap. The length of spark obtainable with this interrupter is not greater than with others, but the density of the spark is much increased, especially if the sparks taken are of less than the maximum length, in which case the discharge tends to assume the form of a flame. The action of this interrupter is not fully understood, though it depends on the interruptions brought about by the gas set free at the electrodes. It will not work at all unless there is a certain amount of inductance in the circuit—that of the primary is sufficient. If the secondary is short-circuited, the interrupter may cease to act owing to the diminution of the effective induction thereby brought about.

440. Fizeau's Condenser.—The power of the coil, when an ordinary spring or a mercury break is employed, is increased by connecting the two sides of the break with the two coatings of a condenser. The effect is a maximum when the condenser has a certain definite size.

When the circuit is broken the greater part of the energy of the primary current, instead of being expended in producing a spark at the contact-breaker, charges the condenser; this is then discharged in the form of an oscillating current through the primary wire.

At the first oscillation the current is in the opposite direction to the battery current; the effect is almost the same as if the battery current had been reversed instead of being broken, so that the variation of the magnetic flux through the secondary wire is virtually doubled.

It appears, from recent experiments made by Lord Rayleigh, that when the current is interrupted with sufficient rapidity by other means the presence of a condenser may be harmful rather than beneficial. Thus, with a particular coil, when the primary circuit was cut by a rifle bullet, good sparks were obtained about 6 cm. long when no condenser was used; while with a condenser, and using either the bullet or a mercury-under-oil break, only feeble brush discharges were obtained. Much depends upon the strength of the primary current, since its power of forming an arc at the break increases with its strength. Rayleigh found that when the primary current was weakened by interposing one or two ohms' resistance in the circuit, the usual condenser was a disadvantage even with a spring break.

If an adjustable air condenser be connected in parallel with the secondary spark gap, the spark length is decreased with increase of its capacity. It is probable, therefore, that the capacity of the wires of the secondary coil itself has a considerable influence upon the length of spark that can be obtained.

441. Direct and Inverse Currents.—If the two poles of the apparatus are connected by a conductor, this is traversed by equal currents alternately in opposite directions, which neutralise each other as regards action on the galvanometer.

If the wire is cut and the ends are separated, the current continues to pass, forming loud sparks, which sometimes attain a great length. The large coils made by Ruhmkorff, which are about 60 centimetres in length, and have 120 kilometres of secondary wire, are capable of giving sparks 45 centimetres long, and much larger coils than these are now made.

These sparks are always somewhat slender. They become denser and less frequent if the ends of the secondary wire are connected with the coatings of a condenser. In this way the capacity of the wire is increased. With large coils a single Leyden jar would be incapable of withstanding the great difference of potentials between the two poles, and accordingly two or more jars are used, arranged in cascade.

When the two ends of the wires attached to the terminals of the secondary coil are drawn gradually apart, it may be observed that in general the direct current passes more readily than the inverse one, and when the distance is sufficiently great the former alone passes. The coil gives then an intermittent current always in the same direction, which can be used to charge a Leyden battery.

When the poles are connected by a tube containing a rarefied gas, a Geissler's tube, for instance, the phenomena described in (**94**) are obtained. The investigation of these luminous phenomena by means of a rotating mirror shows, moreover, that provided the resistance of the tube is small the discharge is not a continuous phenomenon, but is due to a series of oscillations alternately in opposite directions.

This is also the case when the ends of the coil are separated so far that no spark can pass: by measuring the difference of potentials at the ends of the secondary wire, after successive very short intervals of time, it is found that, at each break of the primary current, the secondary wire is the seat of isochronous oscillations, which are rapidly damped out, as shown in Fig. 286 (**339**).

442. Yield of the Coil.—The yield, as in the case of electrostatic machines, may be measured by the time necessary to charge

a battery of Leyden jars to a given potential. The two coatings of the insulated battery are connected on the one hand with the arms of a spark-micrometer, with its knobs at a distance, D, and on the other with the two terminals of the coil, a break, $d<D$, being interposed in one of the connecting wires, so as only to allow the direct current to pass, and to prevent the battery from being discharged at each alternation.

At each spark of the coil the battery receives the same quantity of electricity; after a certain time it is discharged through the micrometer. This phenomenon is periodic, and one spark of the battery corresponds to a constant number of sparks of the coil. Experiment shows that the quantity of electricity corresponding to one spark of the coil gradually diminishes when the rapidity of the breaks is increased beyond a certain limit.

MISCELLANEOUS

CHAPTER XXXVI

MISCELLANEOUS TECHNICAL APPLICATIONS

443. Incandescent Lighting.—Electric energy is used for lighting purposes chiefly in two ways, namely, in *incandescent* lamps and in *arc lights.*

The former method consists in raising a conducting thread to so high a temperature that it becomes luminous. Platinum was first tried for this purpose, but it rapidly disintegrates. The best substance is a thin filament of carbon placed in a vacuum so as to prevent its being burnt. In the Edison-Swan lamp, for instance (Fig. 349), the carbon is a filament of bamboo calcined at a high temperature, and contained in a glass bulb which is exhausted by means of a mercury pump; it is fixed at the ends to two platinum wires fused into the glass, which are the electrodes.

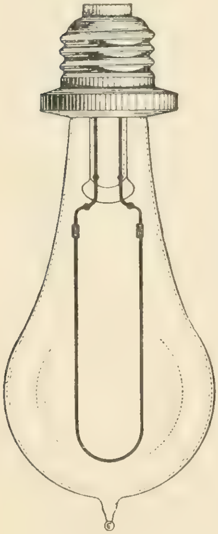


FIG. 349.

Before the bulb is finally sealed the filament is kept incandescent for some time in an atmosphere of a hydrocarbon. Carbon resulting from the decomposition of the hydrocarbon by the great heat is deposited on the filament, which thereby becomes more compact and regular, and acquires a steel grey lustre. This process is known as *treating.*

The rarefaction is pushed to one or two hundredths of a millimetre of mercury. At a lower pressure the filament rapidly disintegrates, and the bulb becomes covered with a layer of carbon; at a greater pressure, 2 or 3 millimetres, for example, the bulb becomes heated.

The illuminating power[1] rapidly increases with the strength of the current; but the life of the lamp is thereby shortened.

The conditions of maximum economy are those in which the gain in luminosity and the loss in the life of the lamp, each estimated by its money value, exactly compensate each other.

Equilibrium of temperature is attained when the rate at which heat is lost by radiation is equal to the rate at which it is generated in the carbon filament, the conditions of this generation being expressed by Joule's law. The ratio of the light emitted to the energy expended is a function of the temperature, and does not depend on the shape of the carbon filaments, provided they have the same emissive power. A treated lamp has less emissive power than an untreated one, and in consequence it requires a less expenditure of power to raise it to a given temperature. Experimentally it is found that both, at the same temperature, give out about the same amount of light, consequently the treated lamp has the greater luminous efficiency.

By measuring the quantity of heat imparted by the lamp to a calorimeter with opaque sides, which absorbs the whole of the radiation, and then to one which transmits the light, it is found that the light is about 5 per cent. of the total energy; in the case of an ordinary candle the ratio is scarcely 3 in 1000.

An Ediswan lamp of 16 candle-power, when made for a difference of potential of 100 volts, works with a current of 0·6 ampere, and such a lamp usually lasts about 1000 hours.

The energy consumed in this case per second is $100 \times 0{\cdot}6 = 60$

[1] The English official standard of illuminating power is a spermaceti *candle*, $\frac{7}{8}$-inch in diameter, six to the pound, burning 120 grains per hour. The French standard is the Carcel lamp, burning 42 grammes of oil in an hour: this is equivalent to about 8·9 English candles. The unit of light at the Physikalisch-Technische Reichsanstalt (Charlottenberg) is the Hefner lamp burning amyl-acetate and adjusted to give a flame 4 cms. in height at normal atmospheric pressure (76 cms.) in an atmosphere containing 8·8 litres of water vapour per cubic metre. The unit at the National Physical Laboratory (Teddington) is that given by the ten-candle power Harcourt pentane-lamp burning a mixture of pentane vapour and air so as to give a flame $2\frac{1}{2}$ inches in height at normal pressure in an atmosphere containing 8 litres of water vapour per cubic metre. By a memorandum, April 1, 1909, issued by the Bureau of Standards (America), the pentane-candle is accepted as the standard, and the following equivalents are stated: 1 pentane-candle = 1 bougie-decimale = 1 American candle = 1·11 Hefner unit = 0·104 Carcel unit It is expected that the pentane-candle will be accepted as an International unit.

joules. But 60 joules per second, or 60 watts, is nearly one-twelfth of a horse-power; hence 1 horse-power radiated from such lamps would amount to about 200 candle-power. The resistance of the lamp when hot is $\frac{100}{0\cdot 6} = 166\frac{2}{3}$ ohms.

The lamps are generally arranged in parallel circuit. Suppose there are n lamps of the same kind: let r be the resistance of one lamp, and c the current through each. On the other hand, let E be the electromotive force of the generator and R the resistance of the circuit, exclusive of the lamps; the resistance ρ of the whole of the n lamps in parallel is $\frac{r}{n}$, and we have

$$E = (R + \rho)\, nc = (nR + r)\, c\,;$$

the useful work is nrc^2, and the efficiency $\frac{r}{nR + r}$.

Suppose it is desired to work lamps of the preceding type by means of a battery. Let us assume that the only appreciable external resistance is that of the battery; in order to obtain the maximum rate of doing work (**135**), we must take $nR = r = 166\frac{2}{3}$ ohms. For Bunsen's elements the preceding formula gives

$$1\cdot 8x = 333\tfrac{1}{3} \times 0\cdot 6 = 200,$$

x being the number of elements. From this we deduce $x = 111\cdot 1$. Hence there must be 111 elements in series, whatever be the number of lamps to be maintained. The resistance alone must vary with the number of lamps, and will be equal to

$$\frac{166\frac{2}{3}}{111n} = \frac{1\cdot 5}{n} \text{ ohms.}$$

The use of Bunsen's elements is only mentioned by way of example. In practice the lamps are always fed by dynamos or accumulators. Take the case of an installation of 1000 lamps. With high-resistance Ediswan lamps arranged in parallel circuit, the dynamo must give 600 amperes with an electromotive force of 100 volts.

In recent years several new kinds of lamps have been devised. In the *Nernst* lamp the light is produced by the incandescence of a rod consisting of rare earths. The rod at

ordinary temperatures is practically a non-conductor; it therefore requires to be first heated by means of a current in an auxiliary conductor. The conductivity of the rod increases fast with the temperature and, when sufficient current is passing through it, the auxiliary circuit is automatically cut out by an electromagnetic break. The light given is very white; the efficiency is about 1·5 watt per candle-power.

In the Osram lamp the filament is composed principally of the metal osmium. The temperature of fusion of this metal is exceedingly high, and hence the filament can be raised to a high state of incandescence. The efficiency is about 1·25 watt per candle-power. It requires to be delicately handled as the filament is very brittle when cold and very soft when hot.

The tantalum lamp has a filament of that metal. Its efficiency is 1·7 watt per candle-power. The lamp has great stability.

In the Arons' mercury lamp the mercury is contained in a highly evacuated bulb. The two terminals are both inserted near the base of the bulb. A narrow central tube embraces one of these and serves to insulate the mercury inside it from the surrounding mercury in contact with the other terminal. If the bulb be shaken a little the two portions of mercury come into momentary contact and permit the arc to strike. When the envelope is of fused silica instead of glass very powerful currents may be sent through. Such a lamp serves as a very efficient source of ultra-violet radiations extending to a wave-length of 2540 tenth-metres.

In the Cooper-Hewitt mercury lamp the mercury is contained in a tube $1\frac{1}{2}$ foot long. The electrodes are at the two ends and consist of an iron anode and a mercury cathode. The arc is struck by inclining the tube so that the mercury runs down it. A current of $3\frac{1}{2}$ amperes at 150 volts is suitable, and the energy expended is about ·45 watt per candle. The light is very deficient as regards red rays.

Such a lamp can be employed as an *electric valve*, since, when the mercury is the anode and the iron electrode the cathode, a current passes with great difficulty. When an alternating electromotive force is applied the current flows only during half the time, and solely in one direction, provided that the maximum electromotive force lies within a certain range.

444. Voltaic Arc.—Davy having attached two rods of carbon to the poles of a battery of 2000 elements, found that, on gently drawing them apart after being in contact, a flame formed between the two points, to which he gave the name of the *voltaic arc*.

The phenomenon is so brilliant that it can only be looked at through a dark glass; a still more convenient way of observing it is to project the image of the arc and the two carbons on a screen by means of a lens. It is then seen that the arc has far less brilliance than the points of the carbons themselves (Fig. 350); that it consists of two distinct parts—one the arc, properly so called, which is blue, and the other, which has the ordinary appearance of a flame, and is reddish; that the positive carbon is brighter than the negative, and that a greater length of it is luminous, indicating that its temperature is higher; that the positive carbon is hollowed out in the form of a crater, from which the brightest light proceeds, while the negative one is sharpened to a point; finally, that the positive wears away more rapidly than the negative carbon. In a vacuum the effects are the same, apart from the action of air on the carbons, and it is clearly seen that matter is carried from the positive to the negative carbon.

FIG. 350.

The temperature of the arc is very high, and difficultly fusible substances, such as platinum, melt easily in it. Hence a permanent arc can only be produced with rods of carbon. For illuminating purposes it is best to have the positive carbon, which is the brightest, above, as the light is thereby better diffused. If the heat is to be used for melting or volatilising a substance, the lower carbon is made the positive one, and its extremity is hollowed out.

The light emitted is very rich in highly refrangible rays, and appears bluish as compared with sunlight. In the spectroscope the carbons give a continuous spectrum extending far towards the violet; the arc itself gives a fluted spectrum showing the rays of carbon, and those of any metals which may be present in the carbons.

Calorimetrical experiments made in a transparent and in an opaque vessel show that the fraction of energy converted into luminous rays is about one-tenth of the total energy expended in the arc.

The arc is acted upon by a magnet like ordinary movable conductors (**274**).

In air the arc produces ozone, and in an atmosphere of hydrogen it gives acetylene.

445. Electromotive Force of the Arc.—Measured either by an electrometer or by a high-resistance galvanometer, the difference of potential between the two carbons is found to be never lower than 30 volts. It varies from 30 to 70 volts. Experiment shows that this difference or fall of potential, which of course takes place in the direction of the current, consists of two parts. One of these is fixed, and is independent of the strength of the current and of the distance of the carbons, acting therefore like a true electromotive force; the other varies with the strength of the current and the distance of the carbons. According to experiments by Mrs. Ayrton, the connection between the observed difference of potentials, E, the length of the arc, L, and the strength of the current may be expressed by an equation of the form

$$E_C = p + qL,$$

when the strength of the current is constant; and by an equation of the form

$$E_L = s + \frac{t}{C},$$

when the length of the arc is constant; p, q, s, and t being numerical quantities determined by experiment. These expressions are included in the following more general equation

$$E = a + bL + \frac{e + fL}{C},$$

when a, b, e, f are again constants.

When the length of the arc was measured in millimetres, the current in amperes, and difference of potentials in volts, the values obtained with solid carbons agreed with the formula

$$E = 38{\cdot}9 + 2{\cdot}07L + \frac{11{\cdot}7 + 10{\cdot}5\,L}{C}.$$

This formula indicates an electromotive force at the carbons of about 52·6 volts as being required to maintain an arc of 4 mm. with a current of 10 amperes.

It appears from the above formula that, for an arc of constant length, an increase of current is accompanied by a decrease of the potential-difference between the poles, as though the arc had a "negative resistance." The explanation appears to be that in the electric arc the current is not conveyed by a conductor of fixed

dimensions like a metal wire, but by a mass of carbon-vapour, and perhaps intensely heated air, the effective volume of which increases with the strength of the current. That some process of adjustment, of the conducting medium of the arc to the strength of the current, does take place is shown by the fact observed by Mr. Duddell that, when the current is suddenly increased, the potential-difference also increases at the first instant (about $\frac{1}{5000}$ second.

When the distance between the poles is maintained invariable, but the current is increased—the increase being so gradual that for each value the shape of the carbons may become that proper to the particular current—the size of the crater increases; and when it is on the point of overlapping the sides of the carbon, the arc begins to hiss. The increase in current is meanwhile accompanied by a decrease in potential difference; but at the hissing point there is a sudden diminution amounting to about 10 volts, and past this point no further diminution occurs. In the open arc, whether silent or hissing, the outer envelope of the vaporous portion is always bright green. With the hissing arc the light issuing from the crater is also green or greenish-blue. This similarity in colour suggested to Mrs. Ayrton that just as the green in the outer envelope is probably due to the combustion of carbon in air, so in the hissing arc the internal green might have a like origin, for it only appears when air can get to the crater. This supposition is supported by the fact that on first striking an arc, hissing always occurs; for the carbon points are surrounded by air until an atmosphere of carbon vapour is produced. Proof was obtained by using a hollow positive carbon; if air or oxygen was blown down this, hissing (with accompanying fall in potential difference) was always produced, unless the electric current was so feeble that the arc was blown out. These phenomena did not occur when nitrogen or carbon dioxide was blown down; if the draught was very strong, however, the arc was blown aside and an *increase* of potential difference was produced, as is usual when the arc is lengthened. The way in which the hiss is produced is probably that air gets to the white hot carbon and sudden combustion takes place; this produces an atmosphere of carbon dioxide which prevents further combustion until it is swept aside; rapid repetition of these two phases is the physical origin of the hiss that is heard.

If one of the carbons of the arc be connected with one surface of a condenser of capacity S and the other with one end of a coil of self-inductance L, the other end of which is joined to the second surface of the condenser, the arc is thrown into a condition of

regular oscillation the period of which, when the resistance of the condenser circuit is sufficiently small, is given by (**338**)

$$T = 2\pi \sqrt{LS},$$

as can be verified by the pitch of the resulting musical note.

446. Work Expended in the Arc.—The difference of potentials multiplied by the strength of the current gives the rate of expenditure of energy in the arc. For the rate of expenditure in watts, Mrs. Ayrton's experiments give

$$W = 11{\cdot}7 + 10{\cdot}5\,L + (38{\cdot}9 + 2{\cdot}07\,L)\,C,$$

or 526 watts for an arc such as that referred to in (**445**).

A brilliant arc of about 900 candle-power requires a current of 15 amperes and a potential difference of 50 volts at the carbons—that is to say, 750 watts, or almost exactly one horse-power. The illuminating power increases much more rapidly than the energy expended: and therefore large lamps are relatively more economical than small ones.

447. Use of Alternating Currents.—With alternating currents the light is as steady as with continuous ones, although it gives rise to a peculiar humming sound, the pitch of which depends on the number of reversals of the current per second. The carbons wear away at equal rates, and both become pointed. By means of a kind of phenakistoscope their condition may be examined, and even photographed, at various phases of the period. The arc disappears when the current is zero, but it is spontaneously renewed in the incandescent gas. The brightness of the two carbons varies periodically, and passes through maxima and minima; further, each of the two carbons becomes in turn the more luminous when it is positive. These variations can be easily observed, even when they are reproduced a hundred times in a second.

If the difference of potential is continuously observed, it is found that it does not vary harmonically in the same way as the strength of the current. Its absolute value remains almost constant, but the sign changes very rapidly the moment the current is reversed, a result agreeing well with the existence of an inverse electromotive force.

448. Carbons.—Davy used wood charcoal. Foucault replaced this by the carbon deposited in the interior of gas retorts and known as gas-graphite; it is harder, is a better conductor, and does not waste so rapidly. Carbons prepared artificially are used now; they are purer, more homogeneous, and more regular in shape. They are prepared from a paste of powdered coke, lampblack, and a very thick syrup of gum and sugar. The whole is well mixed, pressed, passed through a draw-plate, dried, and hardened at a high temperature. The carbons are baked re-

peatedly, being each time dipped in boiling syrup. What are called cored carbons are now frequently used; the centre of these carbons is of a different composition from the rest. They take a better shape when in use, and give a more regular arc. It is very important that the carbons should be pure; in particular, they ought to be free from silica.

449. Electrical Furnace.—Another application of the heating effects produced by electricity is to the fusion of minerals and metals by the voltaic arc. The temperatures which can be obtained by combustion are necessarily limited by dissociation, but this does not apply to the voltaic arc. The question of temperature, however, is not the only one that comes into account; in many cases the reducing properties of the negative electrode appear to play an important part.

Sir William Siemens showed that electric energy might be economically employed to melt iron and steel. A crucible of graphite about 20 centimetres in diameter is used, surrounded by charcoal powder. A carbon rod, like those used for electric lighting, passes through the bottom of the crucible and forms the positive electrode. The negative electrode is kept at a proper distance above it by a kind of regulator. With a current of 36 amperes and a motive power of 4 horse-power, a kilogramme of steel is melted in fifteen minutes. It may be taken that the heating and fusion of a kilogramme of steel requires 450 kilogramme-degrees of heat, or $450 \times 4180 = 1,881,000$ joules of work. Now, 4 horse-power in fifteen minutes give $4 \times 745 \times 15 \times 60 = 2,682,000$ joules, so that the efficiency is about 70 per cent.

450. Electrical Welding.—Of all known modes of heating, the electric arc is that by which the greatest quantity of heat can be concentrated in a given point. This fact is made use of industrially for fusing together two pieces of the same metal—iron, for example—without using a second metal as solder. In order to join two pieces of sheet-iron, they are placed one over the other with their edges together; they are then connected with the negative pole of a battery of 80 to 90 volts, and a carbon rod connected with the positive pole is moved along the edge; the arc which passes between the carbon and the iron melts the iron and solders the two pieces firmly together.

Another entirely different method, for joining two rods end to end, consists in pressing the ends, suitably prepared, against each other, and then passing a current which fuses the parts in contact. The current, which only needs to act a short time, should have great strength with low electromotive force. It is obtained by

means of a transformer. The apparatus is arranged, for instance, so that a mean current of 20 amperes and 600 volts in the primary gives 1 volt and 12,000 amperes in the secondary circuit.

451. Applications of Electrolysis.—Electrolysis is now largely employed in producing a thin deposit of silver, gold, or nickel upon other metals with the object either of improving the appearance or of preventing surface corrosion. It is also used for taking casts of any object in copper, as in the reproduction of plates engraved on wood or metal. The great purity of the copper deposited electrolytically provides a very effective way of purifying commercial copper. Electrolytic copper is now the chief source of the metal used in practice on account of its high conductivity.

Electrolysis is also largely employed in the extraction of metals from their ores. The whole of the aluminium of commerce is now obtained in this way. In 1900 no less than 5000 tons were produced by plant working at 25,000 horse-power.

452. Electric Telegraphy.—The applications which form the subject of the preceding sections utilise the energy of the electric current, and the question of greatest importance is that of efficiency. Electric telegraphy belongs to that category of applications which only require a current giving a very small amount of useful work—in most cases only as much as suffices to call into action, by a sort of detent, some other mechanical energy, such as that of a weight or of a spring. The characteristic properties on which this class of applications depends are the facility with which the time and place at which electrical effects occur can be controlled, and the rapidity with which an electrical action at one point is followed by a resulting effect at a distant point.

453. Electrical Communication between Two Points.—Practice has shown that in order to put two stations A and B into electrical connection only one wire is needed, provided that at the station A that pole of the battery which is not joined up with the line wire, and at the station B the end of the wire itself, are connected with large copper plates buried in the ground. The action is as if the earth itself played the part of a *return* wire. This arrangement has a twofold advantage; in the first place, it is more economical, as the wire is the most costly part of the installation, and in the second place, there is only half the resistance in the circuit, since the resistance of a good earth-connection is almost inappreciable.

The same wire can be used for sending in either direction. Each station must have its battery P, its transmitter T, and its receiver R (Fig. 351): S and S are the copper earth-plates. The

receiver may be placed either at R or at R′. In the second case the two receivers always work at the same time, which enables the signals sent to be verified at the sending station.

454. Duplex Method.—It may even be arranged that the two stations communicate simultaneously by the same wire. One of the arrangements for this purpose is that represented in Fig. 352. The receiver is placed on the bridge of a Wheatstone's balance, the two sides a and a' of which are equal, while the two other sides are the line and a resistance ρ put to earth. If the two resistances are equal, it is readily seen that a current from A does not pass through the receiver at A as the bridge is in equilibrium, but that it acts on the receiver at B, and conversely. Three cases are possible. In the first case a single key is depressed, which is the case just described. Secondly, the two keys are depressed simul-

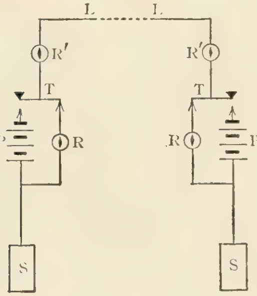


FIG. 351.

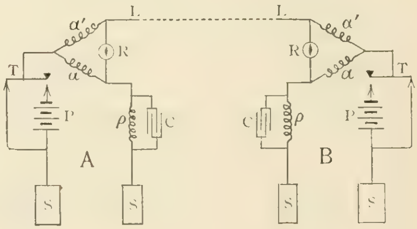


FIG. 352.

taneously; the two receivers then give the same signal under the influence of the same current if the two batteries have their opposite poles to the line, and under the influence of a local current if they are arranged in the same direction—that is, so that the same poles are joined to the line, their electromotive forces acting then in opposition to each other. Lastly, one of the keys is depressed, while the other is free and is not connected either with the battery or the earth: the current then gives the desired signal at the receiving station, but it acts also on the local receiver, though only for so short a time that no great inconvenience is caused; this difficulty may, moreover, be obviated by a somewhat more complicated arrangement, which we need not describe.

By means of condensers C, joined to the resistance ρ, the capacities of the real line and the artificial line may be balanced in the same way as their resistances are—a procedure which is found to be necessary.

455. Lines with Condensers.—It is often desirable to isolate the line wire completely by interposing condensers C at the ends (Fig. 353). When the key is depressed the local battery charges the first coating of the condenser, and the potential of the second becoming raised or lowered to a corresponding degree, sends to the line a flux of the same kind as that which the battery would have given. This flux charges the first coating of the second condenser, which by the same action causes a flux, always of the same sign, through the instrument of the receiving station. The advantage of this arrangement is that it protects the line against the action of earth-currents, which sometimes interfere with the working.

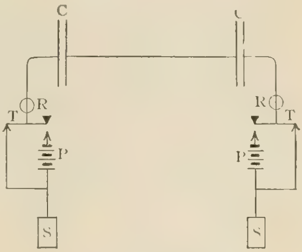


FIG. 353.

456. Air Lines.—The wire ordinarily used for air lines is galvanised iron 4 millimetres in diameter. Its resistance is 10 ohms, and its electrostatic capacity 0·01 microfarad per kilometre. The use of copper had been given up, owing to its small tenacity and its high price; but it is coming into use again in the form of a special kind of bronze, of which wires can be made which are almost as strong as iron, and have almost as high a conductivity as pure copper.

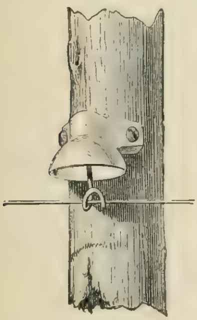


FIG. 354.

As the wire is surrounded by a medium which is absolutely devoid of conductivity, it is sufficient if it is insulated at the points of support. Porcelain is generally used for insulators (Fig. 354). There is a certain amount of leakage at the supports, which varies with the state of the atmosphere: an estimate of its magnitude at any time can be obtained by observing the extent to which the apparent resistance of the line is decreased.

457. Submarine and Subterranean Wires.—In lines laid in the ground or under water, where the surrounding medium is a conductor, an insulating envelope is needed. The conductor or *core* consists of a strand of copper wires CC (Fig. 355), surrounded by several layers of gutta percha, then by a layer of jute, and lastly by a protecting

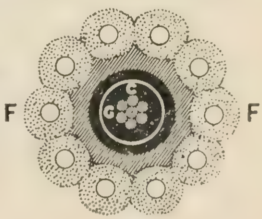


FIG. 355.

sheath of steel wire covered with tarred hemp. The whole thus forms a condenser having a considerable capacity. The cable from Ireland to Newfoundland has a capacity per kilometre of 0·22 microfarad, with a resistance of 1·62 ohm.

458. Rapidity of Signalling.—When one end of the cable is connected with the pole of a battery at constant potential, the current is not immediately set up through the whole extent of the line. It does not move in the wire like a cannon-ball, but rather like a mass of water issuing from a reservoir to fill a channel communicating with lateral basins, which have to be filled at the same time as the channel itself. The *wave front* advances with increasing inclination from the vertical, and between the first arrival of the water and the final establishment of the ultimate level an interval elapses which is longer the more distant the section considered is from the origin. Thus, on the Transatlantic cables from Ireland to Newfoundland, there is no trace of a current in Newfoundland until 0·2 of a second after connection has been made with the battery in Ireland; at the end of 0·4 second the current has only 0·07 of its final strength; it has one-half at the end of a second, and only after 3 seconds can it be considered that the full strength is attained. The growth of strength follows much the same law as the growth of strength of a solution by diffusion in a tube which, after having been filled with water, has one end placed in a vessel containing a solution of any salt. The apparent velocity, which it has sometimes been attempted to measure, on the assumption that the propagation is uniform, by determining the time necessary for producing a definite effect at a certain distance, depends on the nature, the form, and the position of the wire, and also on the sensitiveness of the apparatus employed for observing the effect.

459. Lord Kelvin's Theory.—Lord Kelvin gave the mathe-

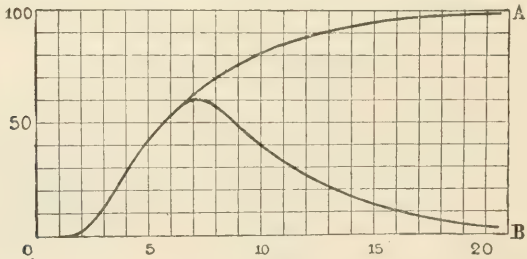


FIG. 356.

matical theory of these phenomena, and showed that if, for a given wire, and at a given distance from the origin, a curve A (Fig. 356)

is constructed, taking as abscissæ the intervals of time that have elapsed since connecting the end of the wire with the battery, and for ordinates the corresponding strengths of the current at the given distance, the final strength being taken as unity, this same curve may be used for all cables and all distances, provided the interval represented by unit-length along the axis of time is suitably chosen. Let α be this interval: if ρ is the resistance in ohms and γ the capacity in microfarads of unit-length of the cable, the value of α for a point M at a distance l from the origin O is given by the formula

$$\alpha = 233 \times 10^{-10} \gamma \rho l^2.$$

Let R be the resistance of the cable, and Γ its capacity, from O to M: we have

$$R = l\rho, \quad \Gamma = l\gamma, \quad \text{and } \gamma\rho l^2 = \Gamma R;$$

consequently

$$\alpha = 233 \times 10^{-10} \Gamma R.$$

It is to be observed that the value of α is independent of the unit of length taken. The product ΓR represents a time (**339**); it may be regarded as the constant characteristic of the cable.

The formula shows further that the value of α increases proportionally to the square of the distance l.

For the Transatlantic cable already mentioned we have, taking the kilometre as unit of length, $\gamma = 0{\cdot}22$ farad, $\rho = 1{\cdot}62$ ohm, $l = 5000$; from this we deduce

$$\alpha = 0{\cdot}2 \text{ seconds.}$$

In ordinary iron telegraph wires 4 millimetres in diameter $\gamma = 0{\cdot}01$ farad at most; $\rho = 10$ ohm. For the line from Paris to Bordeaux, which is nearly 600 kilometres long, we have

$$\alpha = 0{\cdot}0008 \text{ second,}$$

that is, 0·001 as a round number—or a time 200 times as small as with the submarine cable.

The examination of the curve of *arrival* shows that the current may be regarded as having attained its normal value after a time equal to 20α: if it is broken at the origin after the lapse of this time, it will still require 20α to vanish. If then two successive currents are to attain their maximum value, and produce signals which do not encroach on each other, they must not follow each other at smaller intervals than 40α.

The equations from which these results are derived are equations (1) and (2) in (**341**), but with the omission of the term containing

the self-inductance. The complete theory would require the retention of this term, and also an allowance for the leakage which always occurs through the surrounding dielectric (gutta percha, &c.), which is never a perfect insulator.

460. Use of Alternate Contacts.—By means of the arrival-curve the effect produced by a rapid succession of currents either in the same or in opposite directions may be estimated. Suppose, for instance, that the current is broken at the sending end of the wire, after having been made for a time $n\alpha$; in order to find the effect at the receiving station, it is only needful to draw a curve identical with the first, but shifted parallel to itself through a distance equal to $n\alpha$, then to construct a fresh curve, B, the ordinates of which are the differences of those of the two former curves. Suppose now that, after a fresh interval $n'\alpha$, a negative current of the same strength is started at the sending end. The original curve must be displaced parallel to itself by the amount $(n+n')\alpha$, and the differences obtained by subtracting its ordinates from those of B taken for a new curve, and so on. In this way dentated curves more or less regular are obtained, the ordinates of which are sometimes positive and sometimes negative, but in which the effect of each positive or negative contact is well marked, although the actual current at any instant depends on the previous contacts—as many as thirty or forty, for example, producing an effect if they follow each other at intervals of α.

461. Bell's Telephone.—In Bell's telephone (Fig. 357) a thin plate of iron, M, is fixed in a frame immediately in front of a bar-magnet, A, the end of which is surrounded by a coil, B, of fine wire. Any motion of this thin plate modifies the magnetisation of the bar, and causes a variation of the magnetic flux through the coil. The resulting induced currents pass into the coil of a second apparatus identical with the first, and cause corresponding changes in the magnetisation of the second magnet, and motions in the iron plate which are identical with those of the first, except as regards strength.

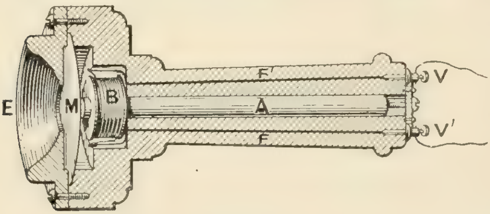


Fig. 357.

The system is perfectly symmetrical, and therefore reversible. The two apparatus, the transmitter and the receiver, are in reality two identical alternating current machines, the former acting as

generator and the latter as motor; the period continually varies from one moment to another, but at each instant is identical for the two. Experiments made by Lord Rayleigh on a telephone of this type show that a current of $4 \cdot 4 \times 10^{-8}$ amperes alternating with a frequency of 640 times per second, was quite able to produce an audible sound when passed through it. For lower frequencies than this the requisite current is greater.

462. Microphone.—The currents which work the telephone are extremely feeble, not exceeding a few hundred-thousandths of an ampere. The action has been greatly improved by using the current of a battery instead of induced currents. This is effected by means of Hughes's microphone.

A pointed rod of carbon A (Fig. 358) is in contact with two pieces of carbon C and C′, fixed to an upright piece of wood. The carbon forms part of a circuit made up of the battery V, the line wire, and the telephone receiver. With a permanent current the telephone does not speak, but any movement of the carbon changes the resistance of the circuit, and, modifying the strength of the current, entails a displacement of the plate of the telephone. Experiment shows that, when a person speaks before the instrument, the vibrations communicated to it are transmitted to the plate of the telephone so as to reproduce the spoken words with greater or less distinctness.

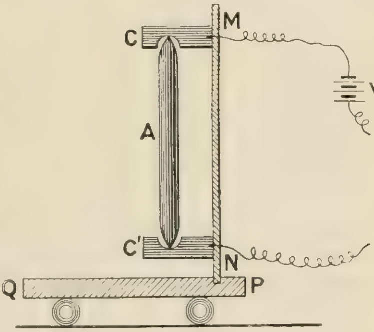


Fig. 358.

Theoretical considerations show that the effect of a periodical variation in the resistance is equivalent to a periodic variation of the electromotive force, the resistance remaining constant. They also show that if in a complex sound the ratios of the amplitudes and the differences of phase of the constituents are to be retained, and therefore the clearness and quality of the original sound, the variation of the telephonic currents must be a very small fraction of their mean strength.

The form of the original microphone has been variously modified. It is desirable to increase the number of contacts, so as to avoid scratching noises. The Ader microphone consists of ten carbon rods, laid in two sets of five each on three cross-pieces, also of carbon, fixed to the same piece of wood. Good results are also

obtained by using small fragments or filaments of carbon. One of the best microphones consists essentially of a thin plate of carbon resting on a packing of the filaments of incandescent lamps.

463. Transmission by the Telephone.—In the telephone, which is an apparatus producing very rapidly alternating currents, a considerable influence of the effects of induction and capacity is to be expected. Applying the theory which explains so well the phenomena of transmission in cables, we arrive at this result, that in the case of a vibratory motion the electrical effect would be propagated in the wire at a uniform rate, with a velocity dependent on the duration of the period; moreover, the amplitude of the disturbance would fall off as it proceeded along the cable, and this decrease would take place at different rates for different periods. Now it is known that the complex motions corresponding to articulate speech may be regarded as arising at each instant from the superposition of a certain number of harmonic vibrations of different periods. If the corresponding electrical effects were propagated with different velocities, they would undergo a kind of dispersion in the wire, and this, taken in conjunction with the different rates of damping, would lead to a change in the wave-form as it passed along the conductor; so that at the arrival station the quality, and the articulation which depends upon it, would be profoundly modified. Experiments show that this is not the case. This is due to the very rapidity of the vibrations bringing into play the effects of self-induction; this does not affect the relatively very slow vibrations employed in telegraphic transmission, and acts in the contrary direction to the capacity (**336, 337**). This question has been very fully worked out from the theoretical standpoint by Oliver Heaviside, who has, moreover, shown that a certain amount of leakage between the direct and return conductors is beneficial in preserving the character of the wave; for although its introduction necessarily weakens the electrical disturbance received, yet it can be so adjusted that waves of all frequencies are attenuated in the same proportion.

464. Wireless Telegraphy.—Since electric waves may travel independently of conducting circuits, signals may be sent without any cable, and they may be picked up by any apparatus capable of responding to the influence of the waves which pass through it. Coherers (**347**) and magnetic detectors (**348**) are now employed with considerable success for this purpose.

465. Electrical Thermometry.—The increase of resistance which metals undergo with increase of temperature serves as the

basis of a method for the detection and measurement of changes of temperature.

The metal platinum is found to be the best for the purpose as it is permanent and has a high melting point, besides having the essential requisite for sensitiveness, viz. a high coefficient of change of resistance with temperature. To construct a platinum thermometer fine platinum wire is wound round a mica strip, and fairly stout leads of copper or silver are fused to its ends. The whole is inserted in a glass or porcelain tube closed at the lower end (Fig. 359); and the leads pass out at the upper end through a wooden cap, and are fastened to binding-screws fixed therein. When in use the thermometer forms one arm of a Wheatstone's bridge. The ratio arms are taken equal to one another; in order to eliminate the resistance of the leads a blind pair identical with them but connected together at the lower ends is inserted in the tube of the thermometer, so as to be exposed to exactly the same temperature as the thermometer leads, and are connected so as to form part of the adjustable arm. The resistance of this arm is adjusted till balance is obtained. Owing to the equality of the ratio arms the blind leads automatically compensate for the resistance of the true leads. The remainder of the resistance in the adjustable arm is equal to the resistance of the platinum wire at the temperature of the measurement.

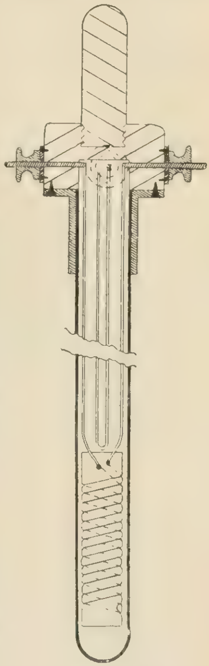


FIG. 359.

To deduce the temperature from a resistance measurement it is necessary to first standardise the thermometer by measuring its resistance at two known temperatures, *e.g.* the usual standard temperatures 0° and 100° C. The temperature corresponding to any intermediate resistance may then be calculated by proportional parts; or, in symbols

$$\frac{t}{100} = \frac{R_t - R_o}{R_{100} - R_o}.$$

This assumes that the change of resistance with temperature follows a linear law. When greater accuracy is required it may be obtained by standardising at *three* known temperatures and assuming that

the resistance varies according to a parabolic formula. In this case we have

$$R_t = R_o(1 + \alpha t + \beta t^2), \qquad R_{100} = R_o(1 + 100\alpha + 100^2\beta),$$

and if P be written for the approximate value of the temperature given by the linear formula

$$P = 100\frac{R_t - R_o}{R_{100} - R_o} = 100\frac{\alpha t + \beta t^2}{100\alpha + 100^2\beta};$$

whence

$$t - P = t - \frac{\alpha t + \beta t^2}{\alpha + 100\beta} = \frac{100\beta t - \beta t^2}{\alpha + 100\beta}$$

$$= -\frac{\beta}{\alpha + 100\beta}t(t - 100)$$

or

$$t - P = -Dt(t - 100).$$

The approximate value P is called the *platinum-temperature*, and the above expression gives the correcting term which must be applied to it where accuracy is required. The value of D for any particular specimen of wire is ascertained by measuring its resistance at the temperature of boiling sulphur (444·53° C. on the air thermometer under normal pressure) as well as at 100° C. and 0° C.; the platinum-temperature (P_s) corresponding to the sulphur point is calculated, and D is then given by the formula

$$D = -\frac{444{\cdot}53 - P_s}{444{\cdot}53(444{\cdot}53 - 100)}.$$

The value of D for specimens of pure platinum is about $1{\cdot}5 \times 10^{-4}$ and this value may be assumed where great accuracy is not required. When the thermometer is specially required for low temperatures the boiling point of oxygen at atmospheric pressure is a convenient standard point; it corresponds to $-182{\cdot}5$°C.

CHAPTER XXXVII

ATMOSPHERIC ELECTRICITY

466. Potential at a Point in the Air.—Experiment shows that in an open space the potential at a point in the air is always different from that of the earth. Two methods may be used to determine the value of this potential.

Let a small insulated sphere, of radius r, be placed at the point in question and connected for a moment with the earth by means of a very fine wire. If V is the value of the potential at this point due to external charges, the potential of the earth being as usual taken as zero, the sphere will acquire a charge, Q, of electricity such that the potential of the sphere becomes zero, like that of the earth. The potential at the centre being $V+\frac{Q}{r}$, we have $V+\frac{Q}{r}=0$. If the sphere is then placed in a Faraday cylinder (80), we may measure its charge Q, and so obtain

$$V=-\frac{Q}{r}.$$

But the simplest method consists in putting at the place in question a point which forms part of an insulated conductor (19). Assuming that the point is *perfect*, equilibrium cannot exist so long as the point, and therewith the conductor of which it forms part, is at a different potential from that of the air near the point.

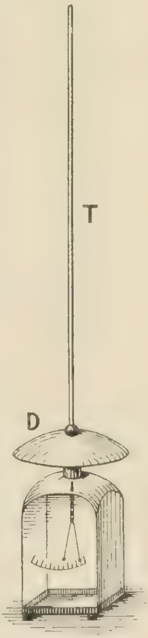


FIG. 360.

Saussure used a small electroscope provided with a point (Fig. 360). If the case is at the potential of the ground, the divergence of the leaves varies with the potential of the air at the end of the point; but the action of the point is too imperfect to allow us to consider that equilibrium is attained.

We have seen how a water-dropping apparatus (85) enables us to realise the effect of a perfect point, and how it assumes the

potential of the air at the point where the stream breaks into drops. The dropping apparatus (Fig. 361) is preferable to a lighted wick, owing to a small difference of potential produced by the combustion, which may amount to half a volt.

The insulated reservoir is connected with an electrometer, the deflection of which measures the potential. By using a mirror and allowing the image to fall on a uniformly moving band of photographic paper, a continuous registration of the indications of the instrument is obtained.

It is thus found that in *fine* weather the potential anywhere in the open air is always *positive;* that its value increases with the height of the point above the ground, and almost in direct proportion; but that at the same place rapid and often large variations occur.

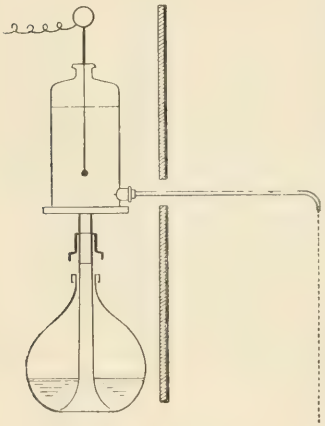


FIG. 361.

The results vary so much that it is difficult to give numerical statements. Above an open plain, for example, the change of potential with height is often between 10 and 1000 volts per metre, but it is sometimes far more.

If, instead of insulating the point, or the arrangement which acts as a point, we connect it with the earth, statical equilibrium cannot be established, and a continuous flow of electricity traverses the conductor. The flow is manifestly equal to the amount given out by the point. It increases with the difference of potential, but it cannot be used to measure this difference, the quantity given off being always so extremely small.

If there is a small break in the conducting wire, the difference of potential at the break may be great enough to produce a succession of sparks. Sparks are sometimes produced in this way between the needle and the quadrants of the electrometer.

467. Distribution of Potential.—At a given instant, the surfaces where the potential has constant and equidistant values above an open plain are equidistant horizontal planes. If the surface of the ground is irregular, the nearest equipotential surfaces follow its undulations, and approach each other over the elevated parts, and the more so the higher and more abrupt these are. Around a house, all parts of which may be regarded as

connected with the earth, and therefore at zero potential, the equipotential surfaces are vertical near the walls and follow the contours of the roof, being closer to each other over the ridge; on the other hand, they separate widely from each other in a court surrounded by high walls, or in a street. As we rise, the effects of the inequalities disappear, and it may be assumed that beyond a certain height the equipotential surfaces are horizontal planes.

As the equipotential surfaces tend to become more nearly parallel to the surface of the ground the nearer they are to it, the electric force at each point is perpendicular to the surface. And since, in fair weather, the value of the potential increases with the distance, it follows that the electric force is directed towards the earth; its intensity at each point is inversely proportional to the distance between two consecutive equipotential surfaces (**32**).

468. Negative Electrification of the Earth.—The phenomena observed near the ground are thus the same as they would be near a negatively electrified conductor in equilibrium. It follows from Coulomb's theorem (**44**), that if F is the value of the force in air near a conductor, the electric density σ at the corresponding point of the surface is given by the equation

$$\sigma = \frac{F}{4\pi}.$$

Let us assume that the equipotential surfaces are horizontal, and that the potential increases by 1 volt per centimetre—that is, in electrostatic C.G.S. units, by $\frac{10^8}{3 \times 10^{10}} = \frac{1}{300}$; we deduce from this for the value of F in dynes, $F = \frac{1}{300}$, and for the density, or charge per square centimetre, $\sigma = \frac{1}{1200\pi}$. Expressed in practical units, that is to say in coulombs, this is $\frac{10}{1200\pi \times 3 \times 10^{10}} = \frac{10}{36\pi 10^{12}}$ $= 10^{-13}$ nearly.

The electrostatic pressure—or, in other words, the force which would tend to lift a surface of 1 square centimetre placed on the ground—would be

$$2\pi\sigma^2 = \frac{10^{-4}}{72\pi} = 4{\cdot}4 \times 10^{-7},$$

that is to say, half a millionth of a dyne, a force wholly insufficient to raise the lightest body.

This charge, small as it is, might however be made evident by

a method like that of the proof plane: a disc like that of an electrophorus applied to the ground, and then removed by an insulating handle, would show a negative charge when tested by a delicate electroscope.

Let r be the radius of the disc; it will take a charge $q = \pi r^2 \sigma$, and since the capacity of a circular disc of radius r is $\frac{2r}{\pi}$, this charge will raise it to the potential $\frac{1}{2}\pi^2 r\sigma$ when it has been removed to a distance from the ground. If we assume $\sigma = \frac{1}{1200\pi}$, and remember that an electrostatic unit of potential is equal to 300 volts, the expression for the potential in volts will be $\frac{1}{8}\pi r$. If the radius were 25 centimetres, this would give a potential of nearly 10 volts.

The measurement of the charge of the disc would give the value of σ, and therefore of F. We deduce from it for the value of the potential at a height h from the ground,

$$V = Fh.$$

The potential of the air is not, however, always positive, and therefore that of the ground is not always negative; in cloudy weather, especially during rain, and sometimes, though very rarely, with cloudless sky, the potential of the air is negative, and therefore that of the soil positive. This, however, must be regarded as quite exceptional, and there is reason to believe that if at a given moment the sky were clear over the whole surface of the globe, the globe charge itself would be wholly negative.

469. Position of the Acting Charge.—The measurement of the potential at a point of the air near the ground teaches us nothing as to the situation of the acting electrical charge. An example will make this intelligible. Suppose that a water-dropping collector is placed in a closed room, and that a sphere charged with positive electricity is then introduced; the electrometer will at once show positive potential. Again, instead of bringing in an electrified body, let the positive charge of a Leyden jar, whose outer coating is connected with the inside of the room, escape by a point, the air of the room will be positively charged, and the electrometer will again indicate a positive electrification, equal, it may be, to that in the former case, although the acting charges are distributed in a totally different manner. As a matter of fact, any observations made near the ground in any one locality give nothing more than the electrical condition of the ground itself; they tell us, as we have seen, the density of the surface-layer, but

furnish no information as to whether this layer is the result of an independent charge, or is due to an external influence—that of the air, for example, which is positively electrified.

It may be observed, however, that if the electrification belongs exclusively to the earth, the potential above an extended plane ought to vary strictly as the distance—in other words, the electric force should be constant. On the other hand, if the mass of the air is itself electrified, the force must vary with the height, and should diminish if the air is positive, and increase if it is negative.

Experiments made in recent years by balloonists leave no doubt that though in the immediate neighbourhood of the ground the air may often be negatively charged, in the higher regions its charge is positive, but that up to a height of 4000 metres the sum of the charges of the earth and air is negative.

Since the air has an independent charge, the incessant variations of potential at a given point are explicable as due to the displacement of masses of air more or less electrified, and we might infer the distance of those masses from the extent of the surface of the ground over which the variations of potential at the same moment are proportional.

470. Origin of Atmospheric Electricity.—The question naturally suggests itself: What is the origin of the electricity of the air and the ground? A very seductive hypothesis is that which ascribes the electricity to the evaporation of water; the vapour being supposed to carry positive electricity with it, leaving negative electricity in the water and on the ground. Unfortunately none of the experiments made with a view of confirming this hypothesis have given it any conclusive support; on the contrary, the fact that rain is usually negatively electrified appears to contradict it.

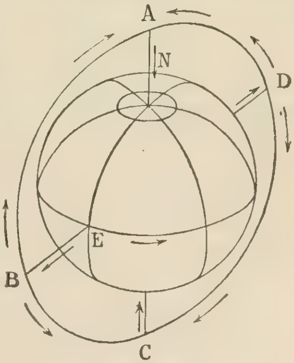


FIG. 362.

The origin of atmospheric electricity has also been sought for in the induction currents which the motion of the earth might develop in the upper regions of the atmosphere, supposing these to be conductors (**324**). Let us imagine a conducting arc, AB (Fig. 362), of any given shape, connecting the pole and the equator, to be either fixed, or at any rate to have a smaller angular velocity than the earth. The magnetic flux, cut by the arc as the earth turns from

west to east, develops therein an inductive electromotive force, the direction of which is from the equator to the pole. If the arc were in contact with the globe by sliding contacts at N and E, and the circuit closed, the induction would produce a continuous current; if the supposed conducting arc were insulated, there would be an accumulation of positive electricity at the pole and of negative at the equator, and to this accumulation might be ascribed the polar auroras on the one hand, and on the other the daily thunderstorms in the equatorial regions.

471. Polar Aurora.—In the polar regions the aurora borealis is visible, we may say, every night in winter. Its forms are excessively varied, and we shall not attempt to describe them. It is certain that the aurora is an electrical phenomenon. It is a discharge in rarefied air quite analogous to that produced in Geissler's tubes (**94**). The discharge, or, in other words, the flow of positive electricity, appears to be towards the surface from the upper regions of the air. The phenomenon is produced at very various distances; some have been observed at as small a height as a mile and a quarter, and others at more than ninety miles. The upper summits, moreover, are often seen to be bounded by a yellow glow. The light of the aurora is due, like that of the tubes, to gaseous substances made luminous by the discharge. In addition to rays ordinarily seen in tubes from which the air has been exhausted, spectrum analysis shows a special ray between the yellow and the green ($\lambda = 5570$) which sensibly coincides with one of the chief lines of Krypton. M. Lemström made experiments on a large scale on the production of the aurora; he constructed a network of copper wire in the polar regions extending over an elevated area of 900 square metres, and having a number of points directed towards the sky; he thus succeeded in producing the characteristic luminous phenomena, including the peculiar ray of the aurora.

472. Earth Currents.—All electric potentials are in practice referred to that of the earth taken as zero; if there were electric equilibrium, the potential of the earth would be the same at all points of the surface; but experiment shows that this is rarely the case. Telegraph lines, the two ends of which are connected to earth, are generally traversed by currents of very irregular strength and direction, which are sometimes so powerful as to stop the working of the lines. Such interruptions are remedied in certain cases by using condensers (**455**). Experiments show further that the currents are the same whether the wire is in the air or in the ground; that they have the same phases on two lines which are

in the same direction but of different lengths, and that the electromotive force is proportional to the distance of the extreme points. The simultaneous investigation of the currents on two lines in different directions, one parallel and the other perpendicular to the meridian, shows that the maximum is more or less north and south.

A comparison of the earth-current curves with the simultaneous curves of magnetic variations reveals an intimate relation between the two orders of phenomena. There is an obvious correspondence between the variations of current-strength parallel to the meridian and those of magnetic declination on the one hand, and on the other between the variations of the earth-currents perpendicular to the meridian and those of the horizontal component There is, however, this remarkable circumstance, that the phases do not coincide, but that the maxima of the one correspond to the minima of the other. This can only be explained by assuming that the same current which exerts an electromagnetic action on magnetic needles acts by induction on the lines. According to this, earth-currents must be looked upon as only secondary currents, and a comparison of the directions of these currents with the deflections of the needle shows that the primary currents, the direct cause of both sets of phenomena, have their seat in the upper regions of the atmosphere.

473. Thunderstorms.—To Franklin the credit is due of having demonstrated by the use of insulated pointed conductors that thunderstorms are purely electrical phenomena. Thunderclouds are electrified conductors, some of them charged positively and others negatively; lightning is a spark passing between two clouds, or between a cloud and the earth.

Passing from these general statements to details there is great uncertainty. Conjectures only can be made as to the constitution of thunderclouds and their origin, or as to the conditions and character of their discharge.

It is easy to understand that a cloud which is formed in contact with the earth may take with it negative electricity; also that the condensation of a mass of vapour in the midst of a mass of positively electrified air may produce a positive cloud; and lastly, that a cloud already charged may produce an influence-charge of the opposite kind upon another cloud that is for the moment in connection with the earth. We may even imagine actions like those of Holtz's machine, and conceive of electrification being produced by the relative displacement of two layers of clouds, one of which acts like the armature of such a machine. But can these causes

account for the duration of some storms, the multitude of lightning discharges by which they are accompanied, and for the enormous quantity of electricity they put in play?

Are thunderclouds themselves to be regarded as conductors charged with electricity only on the surface, or as agglomerations of isolated masses, each having its own charge? The latter opinion appears the more probable.

Up to a certain point we can form some idea of the charge. Let us assume that the surface of the cloud is parallel to the earth, the two forming a large parallel plate condenser; the density is the same on the two faces opposite each other. This density is 10^{-13} coulombs if the potential increases one volt per centimetre (**468**); if we assume that in a violent storm it increases 1000 volts per centimetre, which is certainly exaggerated, the charge will be 10^{-10} per square centimetre, and therefore one coulomb per square kilometre. This is certainly an outside estimate.

474. Various Kinds of Lightning.—In his celebrated *Notice sur le Tonnerre*, Arago discriminates between three classes of lightning, *zigzag* or *forked*, *sheet*, and *globe* lightning. The discharges of the first class consist of well-defined lines of fire, which, excepting as to dimensions, are perfectly similar to artificial electric sparks; they are accompanied by a more or less prolonged noise which is called *thunder*. Those of the second class are the flashes of light which suddenly illuminate clouds without being accompanied by any noise; they seem due to ordinary discharges concealed from the observer, or rather to partial discharges between the clouds. Globe lightning consists of balls of fire moving slowly through the air, and then exploding suddenly; if globe lightning is not a mere optical delusion, it is a phenomenon which in the present state of our knowledge is wholly inexplicable. The only known phenomena which at all resemble it are the luminous globules obtained by Planté on putting the poles of a battery of 3000 to 4000 volts in contact with the surface of a liquid.

Photographs of lightning flashes taken in recent years show that discharges of the first class have a much more complicated structure than appears at first sight. The principal line of light is always accompanied by numerous ramifications, which seems to prove that, from the electrical point of view, the cloud does not act as a continuous conductor.

475. Effects of the Discharge.—The effects are those of an electrical discharge, but on a larger scale. The lightning discharge heats conductors, sometimes to the melting, or even the volatilising point. It smashes and scatters badly conducting substances, kills

or paralyses living beings. Trees struck by lightning are generally torn to pieces as though by a sudden explosion of steam. In its path it frequently presents the most inexplicable appearances.

In order to get some idea of the energy at work, let us take the case of a cloud, such as that considered in (**473**), at a height of 1000 metres; its potential will exceed that of the earth by 10^8 volts. As the charge is one coulomb per square kilometre, the complete discharge will represent an amount of work equal to $\frac{1}{2}.10^8$ joules, or very nearly 5×10^6 kilogramme-metres (**39**); this represents the work done by a mass of 5000 kilogrammes falling from the height of the cloud.

476. Duration of the Lightning Discharge.—An interesting point to consider is the duration of the lightning discharge. Arago indicated the principle on which the observations may be made. Let the light from a discharge illuminate a wheel formed of a number of white radial lines on a black ground rotating with uniform velocity. Let the number of lines be 100, and the velocity 100 turns per second. The time required for one line to take the place of its neighbour will be the ten-thousandth part of a second. If the duration of the flash is equal or superior to the ten-thousandth of a second, the wheel, owing to the persistence of luminous impressions, will appear as a uniformly illuminated disc; but if the duration is one-half, one-third, or one-quarter of a ten-thousandth of a second, the wheel will appear formed of alternately bright and dark sections, the bright sections being equal to one-half, one-third, or one-quarter of the dark sections. If, as stated by Arago, the lines appear as sharp as if the wheel were stationary, the duration of the lightning is not an appreciable fraction of a ten-thousandth of a second. Everything seems to show that it is less than the hundred-thousandth of a second.

By photographing a lightning discharge upon a rapidly revolving plate, more definite results might be obtained than by means of Arago's wheel. Probably this process would show whether the discharge is continuous or oscillatory.

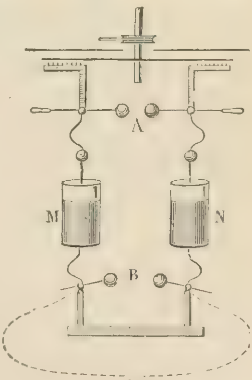


FIG. 363.

477. Two Kinds of Discharge.—We have already seen that discharges may be produced under two very different conditions. Let us return to the experiment of (**340**). At A (Fig. 363), the difference of potential increases gradually, and

when it has reached the value corresponding to the sparking distance, the discharge takes place; at B the difference of potentials is produced suddenly, and the spark passes at once without previous preparation. The effects of self-induction are more intense, the spark is longer and the lateral discharges more violent. We shall distinguish the two cases by the letters *a* and *b*. In order to show the application to the case of thunder-clouds, Sir Oliver Lodge arranged experiments in the following manner:—A and B (Figs. 364–366) are two insulated conducting plates, between which the discharge of a Leyden jar passes. The wires marked + and – connect the jar to the machine, and enable it to be charged. Fig. 364 corresponds to the case *a;* Figs. 365 and 366 to case *b*. Bodies of very various shapes are placed at varying heights between the two plates, a dome, a knob, a point, or a flame. In case *a*, the point and the

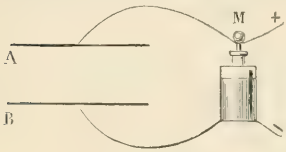


FIG. 364.

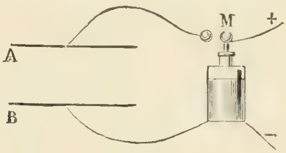


FIG. 365.

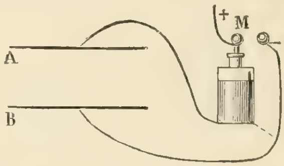


FIG. 366.

flame produce a silent discharge, and *protect* the other bodies; in case *b*, they are all struck indiscriminately, and by sparks of the same kind; it seems immaterial also whether the object is placed directly on the lower plate or is insulated.

Do these two cases occur in nature? There seems little doubt that they do, especially if we consider the clouds as made up of isolated masses.

In the experiments described, the discharge is oscillatory. Are the conditions which permit oscillation (**339**) realised in the case of a thunder-cloud? This is more doubtful; but it may be observed that the effects which we have in view are not so much connected with the oscillatory character of the discharge as with the extreme rapidity of the variations of potential.

478. Lightning Conductors.—Shortly after the discovery of the properties of points, Franklin conceived the idea of utilising them for protecting buildings against the effects of lightning discharges. Franklin's lightning conductor consists of a long metal rod pointed at the top, placed at the highest part of the building

to be protected, and connected with the earth by continuous good conductors. As now made, lightning conductors have a fine point of platinum or a stouter one of copper (Figs. 367 and 368). Franklin ascribes to the lightning conductor a twofold effect. Under the influence of the cloud the point allows the electricity of the opposite kind to escape, and this, carried by the particles of air, silently neutralises the electricity of the clouds. This is the *preventive effect.* But instead of supposing that the electricity which issues by the point neutralises that of the cloud, we may also attribute the preventive action to the electrification of the air by the point ; in this way there may be formed an electrified cloud, which tends to produce at all points below it a potential of opposite sign to that of the thundercloud, and therefore neutralises the influence of the latter. If such a cloud of electrified air remained floating above the conductor, equilibrium would be rapidly attained, and the point itself would cease to act.

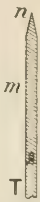


FIG. 368

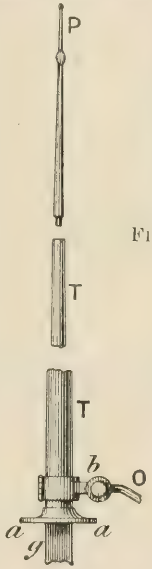


FIG. 367.

If, notwithstanding the point, a discharge takes place, it strikes the rod in preference to the other parts of the building which are within the radius of its protection, and the conductor leads the electricity to earth without any damage. This is the *protective effect.*

The conductor is usually an iron rod, 1·5 centimetres in diameter, experience having shown that lightning has never damaged a rod having this section ; its lower end should be connected with water or earth that is always damp. All large pieces of metal belonging to the building, whether inside or outside, should be connected with the conductor. It is sometimes assumed, but quite arbitrarily, that a conductor protects a circular area, the radius of which is equal to twice the height of the conductor.

In the erection of lightning conductors, and in instructions drawn up for this purpose, the case *a* has alone been taken into consideration, and even within these restricted conditions it does not appear that sufficient allowance has ever been made for the effects of self-induction. The duration of the discharge is so short that the effects of self-induction greatly preponderate over those of resistance. A flat band of metal is better than a solid circular

rod. The relative advantages of copper and iron have frequently been discussed. Iron appears preferable if magnetic permeability does not come into play. The experience of more than a century shows that Franklin's lightning conductor is an efficient protection in the great majority of cases, but it would be an exaggeration to say that it has never failed except by some defect in the construction.

479. Theoretical Conditions.—If we inquire what conditions theory prescribes for the protection of a structure and all that it contains against damage by lightning discharge, we arrive at the conclusion that the structure should be a closed conductor, such as a room, the walls, flooring, and ceiling of which are of iron. Whatever forces may be exerted from outside, the potential in the state of equilibrium would be constant for all points of the surface and the bodies which it encloses; there would be no trace of electricity on any of them. When equilibrium is disturbed, as by a lightning discharge, it is to be supposed that the sides would form a protecting screen for all bodies in the interior, and that sparks like those of Hertz would not be obtained. In any case, these sparks would not be dangerous, and, except perhaps in the case of a gunpowder magazine, there would be no need to consider them.

For an external body connected with the ground no sparks are possible in the case of equilibrium, but in the case of a sudden discharge, there might be dangerous sparks.

We have already seen that, in order to produce a conducting surface at constant potential, it is not necessary that the metal should be continuous; the conditions are realised by a network of even large meshes. The experiment in (**346**) shows that even when equilibrium is disturbed a network has the same effect as a continuous surface. The essential condition is that no conductor shall project into the interior, which might have a potential different from that of the envelope, and might therefore form a kind of electrode capable of giving sparks. Such, for instance, would be gas or water pipes; conductors of this kind should be connected with the conducting enclosure at the point where they enter.

480. Practical Conclusion.—The general result at which we ultimately arrive is that the most certain way to protect a building from the effects of lightning is to encase it in a metal network in perfect conducting communication with the earth. The network may be made of galvanised telegraph wire. The conductors should follow the ridge, the corners, the angles, the chimneys, &c., and

loops or sharp angles should be avoided. Gas and water pipes should be connected with the network on their entrance, and inside the building the pipes should be in metallic connection wherever they are near each other. The external network should be connected with the ground by the greatest possible number of points. All masses of metal on the outside—roofing, eaves, rain-troughs, &c.—should form part of the network. Connection should be made with the underground water-bearing stratum by means of large plates dipping in a pit dug in this stratum, and so deep that even when the water is at its lowest the plate is partially immersed. Large masses of metal inside the building should be connected with each other, but instead of joining them to the outer network, it is better that they should have an independent connection with the earth. The building thus protected might be struck by lightning, but there would be nothing to fear from the effects.

A few strokes of lightning, probably harmless in any case, might perhaps be avoided by providing the conductors with points. From this point of view, it should be attempted to surround the building with an electrified atmosphere, and this would be attained more easily the more numerous are the points and the more widely they are distributed. We are thus led to the plan of lightning conductors devised by Melsens.

The most important point, and that most frequently neglected, especially in the older installations, is that of connection with the earth. A system of lightning conductors badly connected with the earth is not only useless but dangerous.

CHAPTER XXXVIII

RECENT DEVELOPMENTS

In this chapter we shall briefly describe certain phenomena which have either been recently discovered or have recently come into prominence.

481. Cathode Stream.—We have seen that when an electric discharge takes place in a tube containing air at a pressure of less than a millimetre of mercury the appearance at the negative terminal consists of a blue glow close to the terminal, succeeded by a dark space, which in turn is succeeded by another luminous region (the negative glow). But besides these appearances, and crossing over them as though regardless of their presence, a shaft

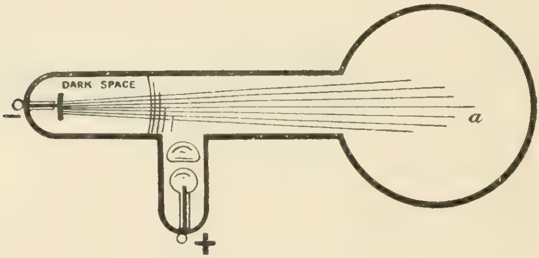


Fig. 369.

of bluish light *a* (Fig. 369) can often be seen proceeding in straight lines normally from the surface itself. If the terminal is a small disc or a short thick electrode surrounded, except at its end, by a closely fitting glass tube a nearly parallel beam is obtained; if, however, a concave terminal be employed, the beam converges approximately to a focal point a little beyond the centre of curvature; and after passing this point it once more diverges and spreads through the tube. These appearances are so similar to those presented by a beam of ordinary light that the term *Cathode Rays* was given to them; we shall adopt the name *Cathode Stream*, which is more in accordance with our present knowledge. Many of the properties of this stream have been known for a long time.

Heat sufficient to render platinum incandescent may be produced at the focal point. When the stream strikes the walls of the tube or a solid obstacle within it, it excites phosphorescence, the colour of which depends upon the substance struck. When this phosphorescent light is examined spectroscopically it is found to give rise usually to a continuous spectrum; in some cases this is traversed by well-defined lines characteristic of the substance struck. This is especially so in the case of the rare earths, and a very sensitive method for their detection has been founded by Crookes upon this property.

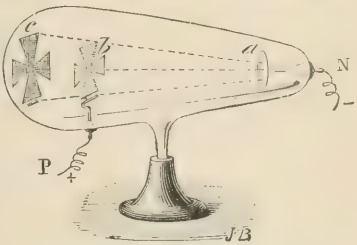


FIG. 370.

If objects be interposed in the paths of the stream, a sharp shadow is cast on the walls of the tube; this is well shown with a tube containing an aluminium cross mounted on a hinge so that it can be tilted into or out of the stream (Fig. 370). If it falls upon crystals of rock salt a beautiful purple tint is produced. This tint is much the same as that obtained by heating sodium chloride with sodium vapour. According to E. Wiedemann and Schmidt the coloration is due to the formation of a sub-chloride.

482. Nature of the Cathode Stream.—The following properties demonstrate that the stream does not consist in ordinary light:—

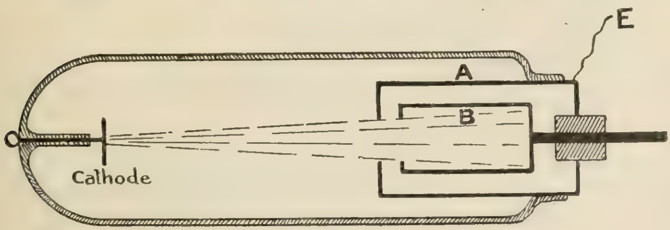


FIG. 371.

(*a*) Arrange a tube as shown in Fig. 371. A and B are hollow metal cylinders insulated from one another; A is earth-connected, and serves both as anode and as an electric screen to B, which is connected to an electroscope. A cathode stream is allowed to pass through the apertures in the ends of the cylinders; the electroscope then deflects as it would do if B received a negative charge.

(*b*) The stream is deflected by a magnetic field whose direction

is inclined to it. If care be taken by suitably shaping the terminal to obtain a cylindrical stream this can be deflected so as to form a circular ring or a spiral. When the field and stream are transverse to one another, the form is a ring whose axis is parallel to the field. When the field and stream are oblique to one another a spiral is obtained (Fig. 372). The direction of bending is that which would be expected if the stream consisted of negatively charged particles travelling away from the negative electrode, if we take for granted the validity of Rowland's experiments (355), which show that such charged particles in motion have the magnetic properties of an electric current. We can calculate the amount of bending in terms of the charge per unit mass of the stream.

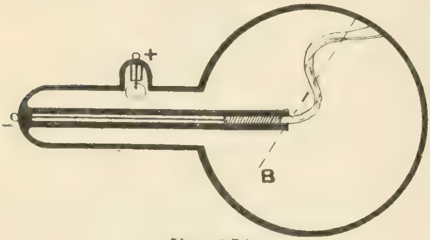


Fig. 372.

Let e be the charge on each particle, m its mass, and v its velocity. The electromagnetic force in a field of induction, B, at right angles to v is evB. Since this acts at right angles to the motion it is of the nature of a centripetal force; whence, if R is the radius of the deflected path,

$$\frac{mv^2}{R} = evB,$$

or

$$R = \frac{m}{e} \cdot \frac{v}{B}.$$

(*c*) The stream also undergoes deviation if placed in an electrostatic field. This again is in accordance with the above assumption, and important information can be obtained from it.

For example, let the stream pass between and parallel to two plates which form an air condenser. Let F be the electric force maintained (by a Voss machine, for example) between these plates. Then Fe is the mechanical force upon a particle carrying a charge e, and the direction of this force coincides with F, and is therefore transverse to the undisturbed motion. The transverse acceleration is Fe/m. Since the deflection is very small we may consider the acceleration as being at all points sensibly at right angles to the motion; the radius of curvature of the path may be taken as constant, and is given by the equation

$$\frac{v^2}{R} = \frac{Fe}{m}.$$

Let now the experiment be arranged so that either the magnetic or the electrostatic field can be excited at will; then, if their

strengths be adjusted until the same curvature of path is obtained for each,

$$\frac{v^2}{R}=\frac{evB}{m}=\frac{Fe}{m};$$

whence

$$v=\frac{F}{B},$$

and

$$\frac{e}{m}=\frac{v}{RB}=\frac{F}{RB^2}=\frac{F}{B^2}\,\frac{\theta}{l},$$

where θ is the angle between the deflected and original paths at a distance l from the origin. This experiment was performed by J. J. Thomson, and he obtained the following values:—

$$\frac{m}{e}=\text{from } 1{\cdot}1\times10^{-7} \text{ to } 1{\cdot}5\times10^{-7} \text{ electromagnetic C.G.S. units,}$$

$$v=\text{from } 2{\cdot}2\times10^{9} \text{ to } 3{\cdot}6\times10^{9} \text{ cms. per second.}$$

The values obtained with different gases, or by employing different electrodes, were the same, provided the difference of potential between the ends of the tube was maintained the same. The value now accepted (as the result of various methods of measurement) is

$$\frac{m}{e}=\cdot6\times10^{-7}$$

absolute units

or $$\frac{e}{m}=1{\cdot}7\times10^{7}.$$

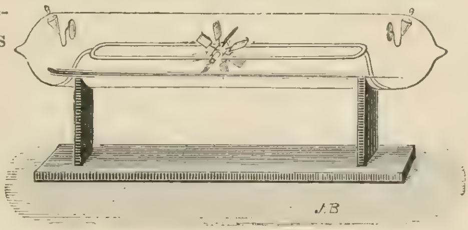


FIG. 373.

(*d*) The following experiment has also often been regarded as proving the same fact. A small wheel with vanes of mica (Fig. 373) rests with its axle on two glass rails; two small electrodes of aluminium are fused into the ends of the tube just above the middle. When the discharge is passed, it acts on the upper half of the wheel, and moves it from one end of the tube to the other away from the negative pole. The pressure upon delicately mounted vanes has been measured, and is very greatly in excess of the pressure which light exerts upon any surface upon which it falls. But it is noteworthy that it is also very much greater than that to be expected from the cathode stream supposed to consist of moving particles as above. Let N be the number of particles striking a surface normally in unit time, the mass of each being m, and its velocity v. Then, if we suppose

each to rebound with simple reversal of velocity the momentum transferred to the surface in unit time is $2Nmv$. If each particle carries a charge e the electric current is Ne. Now if we take the case for which the current has the exceptionally large value of 10^{-5} amperes, and for which $v = 10^{10}$ cms. per second, the force on the surface—equal to the rate at which momentum is transferred to it—is 2×10^{-3} dynes. This is equivalent to a difference in pressure on the two sides of a vane 1 sq. cm. in area of one five-hundred-millionth of an atmosphere; it is, therefore, much too small to produce the mechanical effects in the above experiment. These arise from a secondary cause which is probably the same as in an ordinary radiometer. Indeed, Starke has shown that if the ordinary radiometer effect is eliminated the observed mechanical effect is exceedingly small. The experiment therefore does not afford a criterion between the different theories of the nature of the stream.

483. Röntgen Rays.—We have seen that when the stream impinges upon the walls of the tube it excites phosphorescence; this phenomenon is not confined to the inside of the vessel. Lenard has shown that if the vessel walls are made very thin locally, or if a piece of thin aluminium foil is cemented over a hole in the tube, a cathode stream striking this spot *appears* to pass through: that is, it either does pass through or else a corresponding stream with similar properties issues from the outside. But it has now lost its simple character; for it was shown by Röntgen at the end of the year 1895 that other rays are now present which cannot be deflected by a magnet. These are called Röntgen-rays or X-rays.

They arise when the cathode stream strikes an obstacle; from the points struck they radiate very much as ordinary light would do. They are capable of affecting a photographic plate in the same way as light, *i.e.* so that the plate may when developed show a blackened image; and by this effect their properties may be investigated. They are capable of passing through solid substances, but the degree of transparency of a substance to ordinary light gives no clue to its transparency to these rays; *e.g.* aluminium is more transparent than glass, soda glass than lead glass, organic matter, such as muscle, than bone (which contains mineral matter). The degree of transparency also depends upon the tube in which the rays originate; when the vacuum is low their penetrating power is not nearly so great as when it is high.

The fact that they originate at the point struck by the cathode stream was first demonstrated by interposing, in a measured position between the tube and a photographic plate, a card on which were mounted a series of rings of tinfoil, T (Fig. 374). The foil,

which is relatively opaque compared with card, intercepted the rays, and a magnified shadow of the rings appeared upon the plate when developed. The position of the plate during exposure being also known, a diagram was constructed similar to the figure, showing the plate, card, and tube in their proper relative positions. When lines were drawn backwards from the boundaries of the shadows, D, on the plate through the corresponding edges of the tinfoil till they intersected, the point of intersection was at the metal *anode* which was at the focal point of the cathode stream. Experiments in

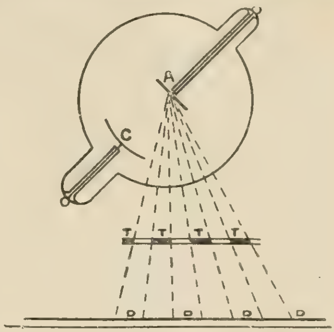


FIG. 374.

which the stream fell upon the walls of the vessel showed that these walls were then the source of the rays: and the shadows lost in sharpness the greater the area struck. The rays excite fluorescence, notably in the platinocyanides; screens of cardboard coated with platinocyanide of barium and other materials are employed for *seeing* the different transparency of bodies interposed. Thus, if the hand is interposed, rays pass with tolerable ease except through the bones. The bony skeleton is thus represented by a non-luminous region on the screen. The position of bullets embedded in the tissues can thus be detected, and this method is now very largely employed.

Although they resemble ultra-violet light in certain respects, yet these X-rays exhibit equally notable differences. They can neither be polarised nor diffracted. They are not deviated by prisms; *e.g.* passage through a centimetre thickness of a thirty-degree aluminium prism produces no perceptible shift of the edge of the shadow of a lead plate placed over it.

All the known facts can be best explained by a theory suggested by Stokes. According to this the rays consist in pulses in the ether (*i.e.* abrupt disturbances similar to explosion waves in air), arising from the sudden stoppage of the electrified particles of the cathode stream when they strike an object. It can be shown that such a pulse *must* originate whenever a moving electric charge suddenly stops; and, further, that its properties are precisely those which Röntgen rays possess.

484. Becquerel Rays.—Shortly after Röntgen's discovery Becquerel observed that uranium and its compounds are capable

of spontaneously acting upon a photographic plate and of discharging an electroscope. Further investigation has shown that it is constantly emitting (1) fast moving particles of small mass, which behave in every respect like those in a cathode stream, and in especial have the same value for the ratio e/m as determined by their magnetic and electrostatic deflection—these are known as β-particles; (2) slow moving and more massive particles, which when examined in the same way prove to have a positive charge and a mass comparable with the mass of a molecule of hydrogen, and (3) a group of rays which are indistinguishable from very penetrating Röntgen rays and are known as γ-rays. From a solution of a uranium salt a precipitate can be obtained which gives only β-particles and γ-rays, while the main body of stuff left behind emits only alpha-particles. Substances which give off these streams are now known as radioactive bodies. Schmidt showed in 1898 that thorium salts exhibit similar effects. These observations have since been extended by Rutherford. In Rutherford's earlier experiments the effects obtained were exceedingly capricious, depending to a large extent upon draughts. He traced this capriciousness to the emission of a *gas* which itself possesses radioactive properties. This *emanation*, as it is called, can be passed from vessel to vessel in every respect like a gas; it begins to condense at $-120°$ C. and deposits on the walls of the vessel; change of temperature does not affect its radioactivity. It is apparently formed not from thorium itself but from an intermediate product of thorium—Thorium X—which can be separated from its parent substance by chemical means. The emanation in turn, if removed from the thorium salt, gradually decays; the decay proceeds at the same rate before removal, but it is masked by the continuous production of fresh emanation; the rate of decay is such that half the initial amount disappears in about 54 seconds. The decomposition product (Thorium A) is not radioactive; but if it is allowed to form for a short while and the remaining emanation is blown out, the initially non-radioactive vessel gradually becomes again active. This is due to the gradual decay of Thorium A and formation of another body, Thorium B, which is radioactive. In each active stage α, β or γ products are given off; and the accepted theory is that each stage is formed from its parent substance by a process of disintegration in which these quick moving particles (α, β) and electromagnetic pulses (γ rays) are emitted, the new substance being the residue left behind. The genealogical table for the various stages, together with the different emissions and the times required for

each substance to fall to half value, when the supply is unreplenished by the decay of previous stages) is graphically shown in Figure 374A.

The last two taken together are known as "the active deposit," because they form upon the walls of the containing vessel and remain behind when the emanation is blown out.

The phenomena of radioactivity have led to the discovery of several new substances. The most interesting of these is *radium*,

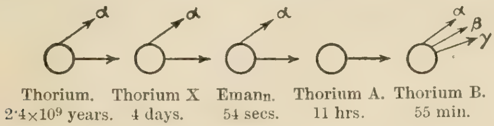


FIG. 374A.

discovered by P. Curie and Madame Curie in pitch-blende. This is an intensely radioactive substance giving off α and β particles and γ-rays, and also in one or more stages emitting much more slowly moving negatively charged particles (δ-particles). This substance also undergoes disintegration, forming in succession a series of products in the same manner as thorium. Their life-history is summarised in the adjoining figure (Fig. 374B).

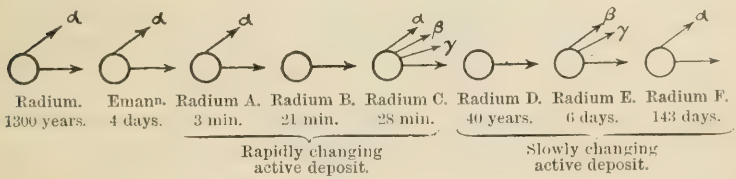


FIG. 374B.

It will be seen that Radium B and Radium D are non-radioactive; and that β-particles and γ-rays apparently are given off only by Radium C and E. The presence of the various particles is detected, and their amount measured, by means of the ionisation they produce (**485**). The active deposit is formed on the negative terminal if an electric field be produced in the containing vessel. The emission of positively charged alpha-particles in its formation would have led one to expect the deposit (Radium A) to be *negatively* charged and to move toward the *positive* terminal. It is possible, however, that negative particles are also emitted simultaneously, but are not detected because they move too slowly to produce ionisation.

Radium itself is a disintegration product of uranium; there are, however, one or more intermediate generations.

According to Rutherford each of these substances is formed from its immediate predecessor by a spontaneous process of disintegration, the rate of which is proportional to the amount of the predecessor present, but is otherwise a constant. Thus, if we take an amount E_0 of the emanation and separate it from the parent radium, its amount will cease to be replenished by a further decay of the radium, and it will itself decay; the rate of change of the amount E remaining at any instant is given by the equation

$$\frac{dE}{dt} = -\epsilon E,$$

where ϵ is a constant characteristic of the emanation known as its factor of decay. Hence by integration, the amount remaining at any time is

$$E = E_0 e^{-\epsilon t},$$

e being the base of natural logarithms. The time to face to *half* value is therefore $(\log_e 2)/\epsilon$; it is this number which is about 4 days.

As the amount of emanation diminishes, Radium A (which is initially absent) gradually forms at its expense; but it in turn decays by spontaneous disintegration. If a is its factor of decay, *and if one atom of* A *is formed from one atom of emanation*, the equation which determines the number of atoms of A at any moment is

$$\frac{dA}{dt} = \epsilon E - aA = \epsilon E_0 e^{-\epsilon t} - aA.$$

The general solution of this equation is

$$A = \frac{\epsilon E_0 e^{-\epsilon t}}{a - \epsilon} + \kappa e^{-at},$$

where κ is the constant of integration; or remembering that A is initially zero, and that therefore (putting $t = 0$)

$$0 = \frac{\epsilon E_0}{a - \epsilon} + \kappa,$$

the equation becomes

$$A = \frac{\epsilon E_0}{a - \epsilon}(e^{-\epsilon t} - e^{-at}).$$

This equation corresponds to the assumption that the amount of Radium A is zero at the first and that it again becomes zero after the lapse of an infinite time.

In a similar way all the successive changes might be considered. We will be content with writing down the differential equations of the next two changes. These are

$$\frac{dB}{dt} = \alpha A - \beta B,$$

$$\frac{dC}{dt} = \beta B - \gamma C,$$

where β and γ are the factors of decay of B and C, and B and C are the numbers of atoms of Radium A and B present at any moment, the assumption being again made that one atom is in each case formed from one atom of the parent.

Radium can be separated from its disintegration products. When a specimen of freshly prepared radium is left to itself the products of decay gradually accumulate. The emanation will cease to increase when $\rho R = \epsilon E$, where ρ is the rate of decay of the radium itself. Thus, neglecting the minute diminution in R, the amount of emanation will subsequently keep constant. Similarly, the amount of Radium A will cease to increase when $\epsilon E = \alpha A$; and so on for the other products. Thus, after sufficient time has elapsed, the various substances are present in amounts given by the continued equation

$$\rho R = \epsilon E = \alpha A = \beta B = \gamma C, \text{ \&c.}$$

These substances are then said to be in *radio-active equilibrium*. The approximate validity of this equation results from ρ being smaller than the other coefficients, so that the amount of R is sensibly constant. If a definite amount of emanation is isolated the growth of the products A, B, C, is shown on Fig. 375; the several curves being drawn to such scales as to display the fact that after about 3 hours the amounts of these substances vary proportionately to the emanation itself. The relative amounts then present are in the ratios

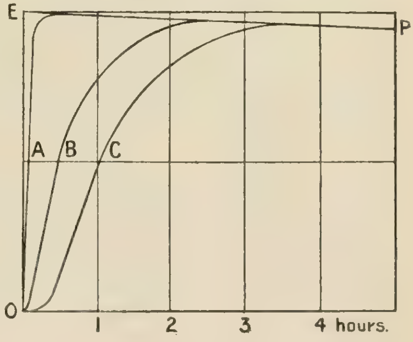


FIG. 375.

Curve EP, Emanation.
Curve OAP, Radium A.
Curve OBP, Radium B.
Curve OCP, Radium C.

$$1, \frac{\epsilon}{\alpha - \epsilon}, \frac{\alpha\epsilon}{(\alpha - \epsilon)(\beta - \epsilon)}, \frac{\alpha\beta\epsilon}{(\alpha - \epsilon)(\beta - \epsilon)(\gamma - \epsilon)};$$

or since ϵ is small compared with the other constants they are given by the equation

$$\epsilon E = \alpha A = \beta B = \gamma C;$$

that is, by the equation of radio-active equilibrium. These statements cannot be extended to the slow-changing products D, E, F, since their factors of decay are smaller than ϵ, so that ϵ never vanishes by comparison.

Comparatively few of these bodies can be obtained in sufficient quantities to enable either their weight or volume to be directly measured. About one gramme of radium has been isolated. Its atomic weight is about 225. The volume of the emanation from one gramme of radium in radioactive equilibrium with it is about ·57 cubic mm., when measured at atmospheric pressure, as first shown by Rutherford. When at a very low pressure its temperature of condensation is about $-150°$ C.; when at atmospheric pressure it is about $-60°$ C.

The temperature of radium is permanently higher than that of its surroundings. The discovery of this fact proved that the energy of the changes is not drawn by thermal conduction from the surrounding air. The possibility of an excess of temperature is easily understood on the hypothesis of atomic disintegration. The atomic system only passes spontaneously from any state to another in which the potential energy is less, the difference of energy being set free as heat.

We must refer to special treatises on the subject for further details in regard to the non-electrical properties of these interesting substances.

485. Electrical Action of Röntgen Rays, &c.—Air in its normal state at ordinary temperatures is either a perfect insulator or very nearly such. It does not appear, in fact, that an electric charge can be directly given to any gas or vapour. Water vapour arising from electrified water and mercury vapour arising from electrified mercury are absolutely unelectrified. But air through which any of these rays are passed becomes a conductor. Thus, the gold leaves of a charged electroscope collapse when a beam is incident upon them, and this takes place whether the leaves are charged positively or negatively. Metal plates, previously uncharged, when exposed to Röntgen rays acquire charges, positive in some cases and negative in others.

It is this electrical property chiefly which is made use of in detecting and measuring radioactive changes.

The active material is placed on the lower of two parallel plates, A and B, in a closed vessel (Fig. 376). This lower plate is connected with the insulated terminal of a battery; the upper plate is connected with an electrometer. The other terminals of the electrometer and the battery, as well as the case C, are earthed. The substance under test puts the air between A and B into a conducting state; hence the battery drives a current from A to B, and the electrometer is thereby gradually charged. The magnitude of this current tends to a certain limiting value for each substance as the difference of potential between the plates is gradually increased. About 300 volts, with the plates 3 or 4 cms. apart, produces this limiting current for substances whose activities are not too strong. The current depends upon the distance between the plates; it increases at first, other things being the same, as the distance increases. This fact, which at first appears anomalous, is explained by supposing that the current is a secondary effect, and is not the current carried by the alpha or beta particles themselves. These particles put the gas into a conducting state, their own energy being absorbed in the process. Now, if a particle can move several centimetres in the gas before it ceases to be efficient, it does not produce its full effect when the plates are less than this distance apart.

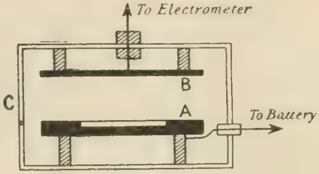


FIG. 376.

The particles are supposed to produce ions, which migrate in the electric field in the same manner as in the electrolysis of liquids: positive ions moving in the direction of the field and negative ions in the reverse direction. It is calculated that a single alpha particle is able to produce about 100,000 ions before it loses its efficiency.

With uranium salts the order of the current obtained with plates 8 cm. diameter and 3 cm. apart is 10^{-11} ampere—with pitch blende it is about eight times as great.

The electric method is thus an extraordinarily delicate means of detecting minute traces of radioactive matter. If a solution containing 10^{-7} grams of radium bromide is evaporated on a metal vessel, the activity which this minute quantity possesses is found to be sufficient to cause an extremely rapid discharge of a gold leaf electroscope when brought near it. In fact, the emission of a single alpha-particle can be detected; when the radioactive substance is taken in sufficiently small quantity, each particle as it is expelled

causes a separate small displacement of the electrometer needle, and therefore the number expelled can be counted.

Violet and ultra-violet light also produce a loss of charge if the body is charged *negatively;* they are without any action if the charge is positive. S. J. Allen has shown that the rate of loss of charge can be represented as the sum of two exponential terms, indicating that the decay is of a dual nature.

In the case of amalgams of the alkaline metals the action extends even to the red end of the spectrum. The azimuth of polarisation of the light is of importance. With an amalgam of potassium and sodium, on which the light was incident at an angle of 45°, Elster and Geitel found the effect to be a maximum when the planes of incidence and polarisation were perpendicular to one another.

Righi has shown that if an insulated uncharged body is illuminated by ultra-violet rays it becomes in general positively charged. The maximum difference of potential so produced depends on the metal of which the body is made; and in respect to this difference of potential the metals may be arranged in the same order as that of Volta—gold and zinc being at the extremities of the list.

Lenard has shown that a metal plate in a vacuum, illuminated by ultra-violet rays, emits rays which are charged negatively and can be deflected by a magnet.

Air also becomes a conductor when passed over phosphorus. The gases drawn from an electric arc, or from the neighbourhood of a glowing filament, or from flames, are conducting. In whatever way the conducting state is produced, it gradually vanishes on removal of the exciting cause. Its destruction may be accelerated by passing the gas through tightly packed glass wool or through water, or along a narrow metal tube. It is at once destroyed by passing the gas through a strong electric field; an electric current between the boundaries of the field accompanies the destruction. This last fact supports the suggestion that the conducting state is the result of the presence of charged particles within the mass of gas: these would be set in motion by the electric field, and their motion would constitute an electric current as in electrolysis. Since the gas is often uncharged on the whole, there must be both positive and negative particles present in such cases. The gas is said to be *ionised*, though there is no evidence that its state exactly corresponds to the state of an electrolytic liquid. The *permanence* of conducting power in the latter case is a notable difference.

486. Condensing Action.—The theory that particles are present which are different from the rest of the gas is confirmed by

the following facts. It was shown by Aitken that the condensation of water vapour, which takes place when air saturated with water vapour is cooled by being caused to expand adiabatically, occurs with freedom only when dust particles are present. It was shown by C. T. R. Wilson that in *ionised* air, saturated with water vapour and *freed from dust*, condensation takes place as the result of expansions which are too slight to produce condensation in dust-free normal air. It would thus appear that particles are present in ionised air which serve as nuclei encouraging condensation. J. J. Thomson had shown theoretically many years previously that electrifying a particle should tend to promote condensation upon it; and that for particles of less than a certain magnitude this tendency should more than outweigh the opposite tendency due to surface tension. The effect of electrically charged particles in promoting the condensation of vapour is strikingly shown by bringing a metal point connected with an electrical machine near an orifice from which a jet of steam is escaping: as soon as the machine is set in motion the cloud of steam becomes much denser and more opaque, but returns to its usual appearance directly the motion stops.

An approximation to the number of such nuclei present in any case has been made as follows. The mass of water that condenses as the result of a given expansion is known from thermodynamic data employed in the theory of the steam-engine. Assuming that each drop of water is a sphere formed round a single nucleus, the total number is known if the radius of each can be determined. The cloud that forms gradually settles down; now it has been shown by Stokes that a small sphere falling through a resisting gas of negligible density acquires a limiting velocity

$$v = \frac{2}{9}\frac{ga^2}{\mu},$$

where g = acceleration due to gravity alone,
a = radius of sphere,
μ = viscosity of the fluid.

Taking this as representing the velocity with which the cloud settles down, J. J. Thomson has calculated the average radius of a drop from the observed velocity, and thence the number of nuclei present in the particular case.

487. Charge on Nuclei.—We have said that the nuclei move in an electric field and that this motion constitutes an electric current. Let u be the mean velocity of the positive and negative nuclei, e the charge on each, and n their number per cubic centi-

metre. Then the current through each square centimetre of a surface at right angles to the flow is

$$C = neu$$

The value of n for a portion of the gas can be measured as in the previous section, C can be measured by means of an electrometer connected with the boundaries of the field (which for convenience may be two parallel metal plates), hence we need only u in order to find e. The determination of u for a given electric field is made by passing the stream of ionised air successively through two parallel sheets of metal gauze which are maintained at different potentials. If the gas were at rest the electric field between the plates would drive the positive and negative nuclei to opposite plates and the charges given to the plates would be carried away. Let the gas be now forced as a stream in the opposite direction to that in which the positive nuclei are driven by the field, and let its velocity be adjusted until the leak just ceases. Since the positive nuclei must now be at rest in space, the velocity of the stream of gas must be equal to the velocity which the nuclei previously possessed due to the electric field alone; that is to say, it equals u.

The results of experiments made by Sir J. J. Thomson and his pupils by this and other methods show that the charge on a nucleus is the same in whatever way the nucleus has been formed and to whatever gas it belongs; and further that it is of the same order as the charge calculated by dividing the charge which sets free a cubic centimetre of hydrogen in electrolysis by the number of atoms in a cubic centimetre; in other words, the charge on the nucleus is comparable with the charge on an atom of hydrogen. Its value may be taken as of the order of 10^{-20} absolute electromagnetic units. Since the ratio $\frac{e}{m}$ for the β-particles is about $1 \cdot 7 \times 10^7$ in the same system of units the mass of such a particle must be of the order of 6×10^{-28} grammes. Without making use of the very uncertain value for the number of atoms of hydrogen in a cubic centimetre, we may say that since $\frac{m}{e}$ is about $\cdot 6 \times 10^{-7}$ while for an atom of hydrogen $\frac{m}{e}$ (the electrochemical equivalent) is about 10^{-4} the mass of a β-particle must be about 1/1700 of the mass of an atom of hydrogen if it be assumed that their charges are the same.

488. Atmospheric Conductivity.—Although air can be obtained which is absolutely non-conducting, it was shown by Linss, and later by Elster and Geitel, that insulated conductors slowly lose their charge in atmospheric air; C. T. R. Wilson has shown that the rate of loss is nearly proportional to the pressure of the air. Both positive and negative charges diminish at the same rate near the sea-level. Neither wind nor mist has much influence, and the rate of loss is greatest when the air is clear and free from dust. Near the peaks and ridges of mountains a negative charge is dissipated much faster than a positive one; the opposite is true near waterfalls. Elster and Geitel attribute this phenomenon to the existence of charged nuclei in the atmosphere. The potential observed near waterfalls is negative. Lenard concludes from his observations that when water falls upon water the *air* round it becomes negatively electrified. The greater rate of dissipation of a *positive* charge near a fall would thus be easily accounted for.

489. Zeeman Effect.—When a source of light (flame, vacuum-tube, &c.) yielding a line-spectrum is placed between the poles of a powerful electromagnet, and examined spectroscopically *across* the magnetic field, the lines in the spectrum become broader; or, if the field is strong enough, each line separates into three or more components. The simplest case is where there are three such components; and in this case each of them is plane-polarised, the outer ones being oppositely polarised to the central one. On the other hand, if the line of sight be *along* the magnetic field, the simplest case is when the original line is split into two; and these are then circularly polarised in opposite directions. This effect, which had been unsuccessfully sought for by Faraday, was experimentally obtained by Zeeman in 1897.

Evidence that the frequency of vibration of the light is altered by a magnetic field is obtained by viewing the light from a hot sodium flame through a cooler sodium flame, the latter being between the poles of an electro-magnet. If the line of sight is along the magnetic lines the field of view brightens whenever the magnetic field is excited owing to the destruction of the complete coincidence of frequency of the two sources—a very small change being sufficient to prevent absorption taking place.

A full explanation (or working theory) has been given by Lorentz and Larmor. We shall be content with an outline of it in its simplest form.

Let us suppose that each atom of luminous gas consists of two charged bodies, one of which rotates round the other; let the

mass of the rotating one be m and its charge e. When the magnetic field is not present, its orbit will be determined by the mechanical force between it and its partner. If we take this force to be a quasi-elastic one, proportional to the distance from the central body and always directed toward it, the motion will either be simply periodic, or may be compounded of three such motions at right angles to one another, its projection on any plane being either circular or elliptical. In each case the frequency will be

$$n = \frac{\omega_o}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{f}{m}}$$

where f is the value of the force at unit distance, and ω_o is the frequency-constant.

When the magnetic field is introduced the force on the particle is modified: if the velocity of the particle is v and is at right angles to the field, there is an additional force upon it equal to evB at right angles to both v and B where B is the induction. If the plane of the motion is that of the paper, and the magnetic field acts downward through it, this additional force will be toward the centre in the case of a counter-clockwise motion of a positive charge and from the centre for a clockwise motion; in the case of a negative charge these statements must be interchanged. If the path is of radius r, the centripetal force may be written

$$m\omega^2 r = fr \pm evB$$

whence

$$\omega^2 = \frac{f}{m} \pm \frac{e\omega B}{m},$$

the positive or negative sign being chosen according to the above convention.

Now in any observed case the modification of the frequency by the field is exceedingly small; so that in the second term on the right the undisturbed value of ω may replace the actual value: hence

$$\omega^2 = \omega_o^2 \pm \frac{eB\omega_o}{m}$$

or

$$\omega = \omega_o \pm \frac{eB}{2m}.$$

Again, if there is any component of motion *parallel* to the magnetic lines, its frequency will undergo no modification. Thus, in general, we should expect three frequencies, viz. :—

$$\omega_o/2\pi, \qquad (\omega_o + eB/2m)/2\pi, \qquad \text{and } (\omega_o - eB/2m)/2\pi.$$

If the light is viewed *along* the magnetic lines, since the com-

ponent motion parallel to them will not give rise to a transverse wave in the ether travelling in this direction, the corresponding spectral line will be absent; the circular motions, however, will each give rise to circularly polarised light; and since in a mass of gas there will be charges which have a right-handed, and others which have a left-handed rotation, the original spectral line should resolve into *two*. If, however, the phenomena be viewed *across* the magnetic field, the circular orbits are now edge-ways on, and will give rise to plane-polarised waves in the ether travelling in the line of sight; the motion along the lines will also give rise to plane-polarised waves, but polarised in an azimuth at right angles to the former; the spectral line will in this case be resolved into *three*.

Thus, the assumptions we have made lead to an adequate account of the simplest form of the phenomenon. In numerous cases, however, four, six, or even nine lines appear; thus, the simple theory is not sufficient, and we may conclude that the events in the atom are more complicated than we have supposed.

From the experimental measurements of the frequency-differences between the components, the value of e/m can be calculated. The general character of the results is best exhibited by the following table, which gives the values obtained by Preston from measurements on lines of magnesium, zinc, and cadmium. The magnesium lines are the three b lines of the solar spectrum, while those of zinc and cadmium are analogous triplets, which for certain non-electrical reasons are considered to be allied to those of magnesium.

Wave Lengths.			$d\omega$.	Character of Components.
Magnesium.	Cadmium.	Zinc.		
5183·8	5086	4810·7	55 approx.	Diffuse triplets.
5172·8	4800	4722	87 ,,	Quartets.
5167·5	4678	4680	100 ,,	Pure triplets.

In the above table $d\omega$ is the magnetic separation between the components and is proportional to $\frac{e}{m}$. Thus corresponding lines in the spectra of different substances appear to arise from the motion of similarly charged particles; and this would appear to be the physical basis of the correspondence which from other considerations is known to exist between these lines.

These results have been extended by Runge and Paschen

(*Astrophysical Journal*, 1902, vol. i. p. 333), under conditions leading to a more complete resolution of the lines. Taking, for example, the case of the lines $\lambda 5184$ of magnesium and $\lambda 5086$ of cadmium they were able to measure nine components in a transverse magnetic field. The positions of these components with respect to the position of the undisturbed lines are given by the numbers in the following table, which represent values in the change of $^1/\lambda$ (in reciprocal tenth-metres) produced by a magnetic field of about 30,000 c.g.s. units.

Mg.	−2·76	−2·15	−1·39	−0·74	−0·01	+0·69	+1·43	+2·12	+2·80
Cd.	−2·83	−2·14	−1·43	−0·68	+0·04	+0·70	+1·43	+2·13	+2·78

The two spectra were formed at the same time, so as to avoid any uncertainty in the strength of the field in the two sets of measurements.

The absolute value of e/m obtained from these experiments is of the same order as that belonging to the carriers of which the cathode stream consists; it is a thousand times as great as that met with in the liberation of hydrogen in electrolysis.

The sign of the rotating charge is ascertained by observation of the sense of the rotation of each of the two circularly-polarised rays obtained when the phenomenon is viewed along the magnetic field. We have seen that when the magnetic field is pointed away from the observer the frequency would be decreased in the case of a clockwise motion of a positive charge; but increased for the same rotation of a negative charge. It should be noted that a clockwise rotation of either a positive or a negative charge produces light circularly-polarised in the *same* sense; for the direction of the electric field at points on the axis of the orbit is along the radius vector to the charge itself, and moves round with it whether the force is outwards or inwards along this radius. Now when the magnetic field tends away from the observer it is found that, of the two components, the one of greater frequency (*i.e.* the one lying on the violet side of the spectrum) is right-handedly polarised, and is therefore due to a particle rotating in this sense; similarly the component of lower frequency is left-handedly polarised and is due to a particle rotating left-handedly. Interpreting these results in the light of the equation determining the frequencies, we see that the moving particles must be negatively charged.

490. Canal-Rays.—When the negative electrode of a vacuum tube is perforated, streamers appear behind it. These have been

shown to be *positively* electrified, to travel much more slowly than the cathode stream, and to be coloured differently for different gases. Their velocity and the value of e/m can be obtained from the deflection which is produced by magnetic and electric fields. The value of e/m which is thus obtained is *comparable with the value met with in electrolysis.* It would thus appear that what we speak of as a *positive* charge is much more closely associated with the material molecule than the negative charge is. Although many phenomena are now known in which the motion of the positive charge can be followed, no case is known in which it can be shown to be separated from matter.

Thus the difference between positive and negative charges seems to be of a fundamental character, and is not adequately represented by a mere difference in sign.

491. Electrons.—To explain the various facts recorded in this chapter it is necessary to regard any substance as built up in the main of material atoms possessing charges of positive electricity, or else to regard the atoms themselves as being in one of their aspects electrical bodies. Besides this main framework much smaller particles exist, each having a mass very small compared with an atom, and possessing all the properties of a negatively electrified body. Even the small mass that each of these possesses seems to arise merely from its motion in the way described in (**355****). For, as we have mentioned, the mass arising from the motion of a charge increases rapidly when the motion becomes comparable with that of light. Now the beta-particles from radium move with different speeds (from $\frac{1}{5}$ to $\frac{9}{10}$ that of light), and Kaufmann has shown that their effective masses vary with their velocities in the mode that we would expect if their mass was wholly of electromagnetic origin. Hence we might speak of these as particles of negative electricity. The electricity in each is comparable with and is supposed to be identical to that set free from an atom of hydrogen in electrolysis. The name *electron* was given many years ago by Johnstone Stoney to this unitary quantity of electricity, and the name has been retained for the carrier with which it is always associated.

In ordinary electrolysis it travels with the migrating atoms—to which in electrolytes it must be considered as firmly attached—and its motion constitutes the electric current; in metals, the attachment to matter becomes loosened, and it travels amongst the metallic molecules under the influence of the electric force, giving up part of its energy to these molecules, and this consequently appears as heat (RC^2t). In the cathode stream the electrons have

2 Q

become completely detached from their material attendants; and thus is explained the fact that the stream has the same properties whatever the gas in which it originates, or the electrode from which it pours.

492. Metallic Conduction.—If conduction in metals is due to an actual transfer of electric charges or electrons, these must be fairly free to move amongst the molecules of the metal itself. A theory of conduction has been put forward which starts by supposing that some of the electrons in a metal are free to move, and in fact are in rapid motion in much the same way as if they were the particles of a gas dissolved in the metal. Although this theory does not lead to results in exact agreement with experiment, it is at least suggestive, and a short account of it will give an idea of the direction in which scientific thought is tending.

Applying to the electrons the theorem which holds good for gases, that the mean-square velocities of the molecules of any two gases vary inversely as their masses, the electrons must have an average velocity of about 58·3 times that for hydrogen, or about $1{\cdot}07 \times 10^7$ centimetres per second at 0° C. When these particles are moving at random in all directions no electric current flows although the particles are charged. But when an electromotive force is applied each particle gains a component velocity which, on account of its negative charge, will be in the opposite direction to the e.m.f. applied. Thus a drift of the particles will be set up, and this will be equivalent to a current. Let us suppose that there are n electrons per unit volume, having a root-mean-square velocity, u; and that when in an electric field X they acquire in addition a small velocity opposite to the field. This velocity will be gained by any one molecule between two collisions, but this gain will be held in check by a perpetual transfer of momentum between each electron and the molecules with which it comes into collision; for we may assume that when collision occurs with a second molecule the shock is so violent that the particle afterwards moves on with much the same velocity as if it were not in an electric field, and then again gains speed and so on repeatedly. Being of mass, m, and acted upon by a force eX, the mean drift velocity, v, during the time τ between two collisions is $\frac{1}{2}\frac{Xe}{m}\tau$. The electric current per unit area corresponding to the motion of n such particles per unit volume is $nev = \frac{1}{2}\frac{Xe}{m}ne\tau$. Assuming that v is so small that the average value of τ is unchanged by its existence we have $\tau = \lambda/u$ where λ is the mean free path. In this connection it should be

noted that the time between two collisions is on the average the same whether we consider electrons moving in the direction of the drift or in any other azimuth; and therefore the drift velocity opposed to X is the same on the average for all directions of the random motion. Thus the current per unit area is $\frac{1}{2}\frac{Xe}{m} . ne . \frac{\lambda}{u}$.

Let us suppose that there are ν free electrons for each atom of the body; then if N is the number of atoms per cubic centimetre the expression for the conductivity (which is the above expression divided by X) may be written

$$\sigma = \frac{1}{2}\frac{e}{m} . \nu . Ne . \frac{\lambda}{u},$$

where the terms are intentionally grouped in such a way as to indicate which quantities are most readily ascertainable by experiment. Thus, e/m is known with much more certainty than either e or m separately. Similarly with respect to Ne. For let N_H be the number of *atoms* of hydrogen per unit volume under standard conditions, then

$$\frac{N}{N_H} = \frac{\text{density of metal}}{\text{density of hydrogen}} \times \frac{\text{atomic wt. of hydrogen}}{\text{atomic wt. of metal}};$$

hence Ne can be calculated if we know $N_H e$. But $N_H e$ is the electrolytic charge corresponding to the atoms in unit volume of hydrogen; and since 1 c.g.s. unit of electricity sets free 1·15 c.c. of hydrogen under standard conditions we have

$$1{\cdot}15 N_H e = 1.$$

Hence Ne is ascertainable without any necessity for a knowledge of either N or e separately. The uncertain data in the equation for σ are ν, λ, and u. Now we do not know the value of the mean free path (λ) for an electron in a metal; but as the value for air at atmospheric pressure is only 10^{-5} centimetres, and the molecules in a solid are more than 1000 times as crowded as in a gas, the mean free path in a solid (which may be taken roughly proportional to the *square* of the linear dimensions) cannot be greater than 10^{-7} centimetres. The value of u has been stated to be about $1{\cdot}07 \times 10^7$ centimetres per second. Whence for silver

$$\sigma = \frac{1}{2} \times (1{\cdot}7 \times 10^7) \times \left(\frac{1}{1{\cdot}15} \times \frac{10{\cdot}5}{{\cdot}00009} \times \frac{1}{108}\right) \times \frac{10^{-7}}{1{\cdot}07 \times 10^7}\,\nu$$
$$= 7{\cdot}5 \times 10^{-5}\,\nu.$$

Taking the value of σ as 1/1600 this gives $\nu = 8$ nearly; or the number of free electrons concerned in electric conduction in even

so highly conducting a metal as silver is comparable only with the number of atoms of the metal.

493. Thermal Conductivity.—The expression for the electrical conductivity can be written

$$\sigma = \tfrac{1}{2}\,\frac{e}{m}\cdot\frac{e}{mu^2}\cdot nmu\lambda,$$

where the last group contains the only terms (n, λ) which change from material to material. Now the thermal conductivity of a *gas* is shown in treatises on the kinetic theory to be given by the expression $\kappa = \frac{1}{3}\,nmu\lambda c$ when c is the specific heat of the gas at constant volume, and the other data have the same signification as in the preceding section. Experimentally it is found that the thermal and electrical conductivities of many metals are in nearly the same ratio to one another. We can account for this proportionality if we *not only liken the electrons in a metal to a gas, but, at the same time, assume them to be the responsible carriers of heat*, for on these assumptions the expression $nmu\lambda$ stands for precisely the same quantity in the expressions for σ and κ; and the ratio of κ to σ becomes

$$\frac{\kappa}{\sigma} = \frac{2}{3}\cdot\frac{m}{e}\cdot\frac{mu^2}{e}\cdot c\,;$$

or remembering that c for a monatomic gas is equal to $\frac{1}{2}\,\frac{u^2}{T}$, where T is the absolute temperature,

$$\begin{aligned}\frac{\kappa}{\sigma} &= \frac{1}{3T}\left(\frac{mu^2}{e}\right)^2,\\ &= \frac{1}{3\times 273}\left(\frac{1}{1{\cdot}7\times 10^7}\cdot(1{\cdot}07\times 10^7)^2\right) \text{ at } 0^\circ \text{ C.},\\ &= 5{\cdot}8\times 10^{10}.\end{aligned}$$

The following are values found experimentally for the ratio :—

Metal.	$\frac{\kappa}{c}$
Copper	$6{\cdot}71\times 10^{10}$
Silver	$6{\cdot}86\times 10^{10}$
Gold	$7{\cdot}09\times 10^{10}$
Nickel	$6{\cdot}99\times 10^{10}$
Zinc	$6{\cdot}72\times 10^{10}$
Aluminium	$6{\cdot}36\times 10^{10}$
Platinum	$7{\cdot}53\times 10^{10}$
Iron	$8{\cdot}38\times 10^{10}$
Bismuth	$9{\cdot}64\times 10^{10}$

The approximate identity of this number with the theoretical one lends support to the conclusion that the theory is correctly based, in spite of the somewhat startling assumption which excludes the atoms of the metal themselves from taking any sensible part in thermal conduction.

The difficulty of the theory is that the assumption that the electrons behave as a gas attributes to them more energy than the thermal data for the metal show to be possible. The specific heat of a monatomic gas is equal to $\frac{1}{2}u^2/T$, or its thermal capacity per unit volume is $\frac{1}{2}nmu^2/T$. Now in a cubic centimetre of silver there are about $1{\cdot}6 \times 10^{23}$ atoms of silver, and we have seen that the electrical conduction requires about eight times as many electrons, for each of which $\frac{1}{2}mu^2$ is about $\frac{1}{2} \times 6 \times 10^{-28} \times (1{\cdot}07 \times 10^7)^2$. Hence the thermal capacity of the electrons alone per unit volume would be more than 10^8 ergs per degree, or (in thermal measure) 2·4 calories per degree. But the heat required to raise unit volume of silver one degree is only about ·6 calories. No satisfactory way has yet been found of removing this discrepancy. Since, however, it is difficult to think of the electrons behaving as a perfect gas when they are eight times as crowded as the atoms of the solid metal itself, it is probable that the explanation will depend upon a modification of this fundamental assumption.

494. Thermo-Electric Effects.—Any isolated piece of metal may be conceived at any temperature as possessing per unit volume a definite number of electrons characteristic of the particular metal. When pieces of two different metals at the same temperature are brought into contact, the electrons have freedom to flow across the boundary between them, and this flow will take place from the metal in which the pressure of the electrons is greater to the one in which it is less. Since the electrons are charged this flow is, however, soon checked; for the metal from which they escape will be left with an excess of positive electricity, while the other will become negatively charged. A field of force is thus set up in the boundary layers equal to X (say), and the flow will cease as soon as this balances the force due to the difference of the pressure on the two sides. Let the slope of pressure at a point in the thin layer near the boundary be $\frac{dp}{dx}$; and let n be the number of electrons per unit volume at this point. Then when equilibrium is set up we must have

$$\frac{dp}{dx} = Xne.$$

Let V be the volume containing one electron $\left(\text{so that } v = \frac{1}{n}\right)$; then this becomes

$$X = \frac{1}{e} v \frac{dp}{dx}.$$

Integrating across the layer we obtain for the difference of potential between the two sides

$$V_2 - V_1 = \int X dx = \frac{1}{e}\int v dp,$$

or retaining the conception of the gaseous character of the electrons

$$V_2 - V_1 = \frac{mRT}{e} \log \frac{p_2}{p_1} = \frac{mRT}{e} \log \frac{n_2}{n_1}$$
$$= \frac{T}{40 \times 273} \log \frac{n_2}{n_1}.$$

This difference of potential would correspond to the Peltier effect. For the two metals, antimony and bismuth, for which the Peltier effect is exceptionally large, the ratio of n_2 to n_1 would be 3·8; for other pairs of substances it would be less. But it is doubtful how much dependence may safely be placed upon this number. Similar considerations show that there will be a difference of potential set up in a single conductor whose ends are maintained at different temperatures, provided that the number of electrons per unit volume depends upon the temperature. This would explain the existence of the Thomson thermo-electric effect.

495. Magnetic Effects.—If a magnetic field be set up at right angles to the flow of a current in a thin metal plate we have seen (**293**) that a *transverse* difference of potential is excited, which is known as the Hall effect. On the convection theory of electric conduction this is explained by taking into account the deflecting action of the magnetic field upon the charged particles in motion. In order to explain all the effects observed it is necessary to assume the presence of carriers of electricity of both signs. Let us consider first the positive carriers; if their drift velocity is U in the direction of the current the deflecting force due to the magnetic induction B is eBU, e being the charge. Thus these charged particles will acquire a transverse drift which will, however, be ultimately checked, in part by the difference of potential set up and in part by the extra pressure near that margin of the plate toward which they crowd. These opposing forces are of the same kind as those considered in connection with the diffusion of ions in a concentration cell (**165**), and are calculable in a similar manner. If z denote

the transverse direction, $\frac{dV}{dz}$ the rate of decrease of potential in this direction, p_1 the pressure of the positive carriers, and n_1 their number per unit volume, we have, in the state of equilibrium, the relation

$$eBU - e\frac{dV}{dz} - \frac{1}{n_1}\frac{dp_1}{dz} = 0.$$

Similarly, for the *negative* carriers moving in the opposite direction to the main current we have

$$eBY + e\frac{dV}{dz} - \frac{1}{n_2}\frac{dp_2}{dz} = 0,$$

if, as we suppose for simplicity, their charges are equal. We shall call these the two fundamental equations. Absence of charge on the whole necessitates in this case that $n_1 = n_2$, and hence $p_1 = p_2 = p$ (say). The main current is given by the equation

$$C = n(U + Y)e.$$

Eliminating $\frac{1}{n}\frac{dp}{dz}$ from the first two equations:

$$eB(U - Y) = 2e\frac{dV}{dz};$$

whence

$$\frac{dV}{dz} = \frac{BC(U - Y)}{2(U + Y)ne} = \frac{BC(u - y)}{2(u + y)ne},$$

where, as when considering electrolysis, we write u and y for the velocities of the two kinds of carrier under unit potential gradient.

The sign of this transverse potential gradient, $\frac{dV}{dz}$, will depend upon the sign of $u - y$; that is to say, upon the sign of the carrier which moves more easily in an electric field. In the case of those substances for which the Hall coefficient is positive the positive carrier would, on this theory, have the greater mobility $(u > y)$; in those substances that have a negative coefficient the negative carrier would move the more easily. The coefficient G of the Hall effect would be $\frac{u - y}{(u + y)2ne}$; in the case of silver its value is experimentally found to be about -9×10^{-4} C.G.S. units.

If we suppose that the current in silver is carried wholly by negative carriers, each with a charge 10^{-20} C.G.S. units, then

$$n = \frac{1}{2 \times 10^{-20} \times 9 \times 10^{-4}} = \frac{10^{24}}{18}.$$

This is less than but of about the same order as the supposed number of atoms contained in one cubic centimetre of silver.

Eliminating $\frac{dV}{dz}$ by adding the two fundamental equations we obtain

$$\frac{dp}{dz}=\frac{BC}{2},$$

an equation which gives the slope of pressure of each kind of carrier. This gradient of pressure corresponds to a transverse gradient of temperature which was experimentally discovered by von Ettingshausen and Nernst, the pressure increasing with the temperature as it does in a gas. It is easily seen that such a gradient should accompany the Hall effect. Positive carriers moving with the current and negative carriers moving against it are deflected in the *same* direction by the magnetic field; each in moving across the field gains energy in the interval between its collisions with the molecules of the metal; and at each collision it gives up to a molecule the energy it has gained. Thus there will be a transfer of kinetic energy across the field in the direction in which the deflection takes place; and the side of the conductor toward which this drift occurs will be maintained in this way at a higher temperature than the opposite side. Experiment shows that this is the case in every metal examined except iron; for this metal the sign is reversed. The gradient of temperature is found experimentally to satisfy the equation

$$\frac{dT}{dz}=P\frac{CB}{\text{Thickness}},$$

where P is a characteristic coefficient depending upon the material of the plate. The effect is known as the von Ettingshausen and Nernst effect or as the galvano-magnetic transverse temperature effect.

Similar transverse effects are observed when a flow of heat is substituted for the electric current. The ends of the plate being maintained at different temperatures there exists, while the magnetic field normal to the flow is present, a transverse difference of potential (the *thermo-magnetic transverse gradient of electric potential*) and a difference of temperature (the *thermo-magnetic transverse temperature difference*). In this case we must put $C=0$; whence $U=-Y$; that is the two kinds of carriers diffuse in company with one another as in electrolytic diffusion after a small initial separation has taken place, producing a longitudinal gradient of potential, $\frac{dV}{dx}$. Since the pressure of the carriers is greater at

the hot end of the plate there is also a gradient of pressure along the plate, $\frac{dp}{dx}$ say.

Then
$$U = u\left(e\frac{dV}{dx} + \frac{1}{n}\frac{dp}{dx}\right)$$
$$-Y = y\left(-e\frac{dV}{dx} + \frac{1}{n}\frac{dp}{dx}\right) = U$$

whence
$$U\frac{u+y}{uy} = \frac{2}{n}\frac{dp}{dx}.$$

Subtracting one of the two fundamental equations from the other in this case we obtain
$$2eBU = 2e\frac{dV}{dz};$$

whence
$$\frac{dV}{dz} = \frac{uy}{u+y}\frac{2}{n}B\frac{dp}{dx}.$$

Assuming merely that p increases with the temperature T, so that $\frac{dp}{dx}$ is of the same sign as $\frac{dT}{dx}$, this equation shows that the transverse gradient of potential would also have the same sign when B is positive. Experimentally it is found to be sometimes positive and sometimes negative. Moreover, if we add the two fundamental equations in the case when $U = -Y$ we find that $\frac{dp}{dz}$ must be zero; that is, there would be *no* transverse temperature gradient set up by a longitudinal gradient of temperature in a normal magnetic field. Experimentally a small one is obtained which can be represented by the equation $\frac{dT}{dz} = SB\frac{dT}{dx}$ where S is a characteristic constant. The associated potential slope can be represented by the formula
$$\frac{dV}{dz} = QB\frac{dT}{dx}.$$

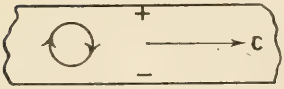
Drude has further developed the theory, allowing for the possibility of the presence of unequal numbers of the two kinds of carrier with unequal charges, and the discrepancies from experiment which we have mentioned are thereby removed.

These several effects have been investigated by von Ettingshausen, Nernst, von Everdingen, Zahn, and others. The following table embodies some of the results of Zahn, expressed in absolute measure, the signs of the coefficients being given as positive when the data are related as indicated in the annexed schemes, in which

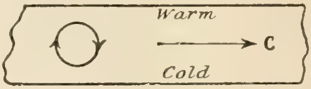
the longitudinal current of electricity or heat is denoted by a straight arrow, the circular arrow indicates the direction of the current in the electro-magnet, and the transverse effects are indicated by +, −, and warm, cold respectively.

GALVANO-MAGNETIC TRANSVERSE EFFECTS.

Electric (G).



Thermal (P).



THERMO-MAGNETIC TRANSVERSE EFFECTS.

Electric (Q)

Thermal (S).

	G.	P.	Q.	S.
Bismuth .	$-6{\cdot}33$	$+3{\cdot}53\times10^{5}$	$-1{\cdot}78\times10^{-1}$	$-2{\cdot}05\times10^{6}$
Nickel . .	$-4{\cdot}69\times10^{3}$	$+2{\cdot}8\times10^{8}$	$-1{\cdot}3\times10^{-3}$	$-2{\cdot}0\times10^{7}$
Silver . .	$-8{\cdot}97\times10^{4}$	......	$+4{\cdot}3\times10^{4}$	$-4{\cdot}04\times10^{7}$
Copper .	$-4{\cdot}28\times10^{-4}$	......	$+2{\cdot}7\times10^{4}$	$-2{\cdot}32\times10^{-7}$
Platinum .	$-1{\cdot}27\times10^{-4}$	......	......	$-2{\cdot}1\times10^{-8}$
Zinc . .	$+1{\cdot}04\times10^{-3}$	......	$+2{\cdot}4\times10^{-4}$	$+1{\cdot}28\times10^{-7}$
Cobalt . .	$+1{\cdot}61\times10^{3}$	$+9\times10^{7}$	$-2{\cdot}0\times10^{3}$	$+1{\cdot}3\times10^{-7}$
Iron . .	$+1{\cdot}08\times10^{2}$	$5{\cdot}7\times10^{8}$	$+1{\cdot}05\times10^{-3}$	$+3{\cdot}9\times10^{7}$
Antimony	$+0{\cdot}219$	$+1{\cdot}94\times10^{-6}$	$-1{\cdot}76\times10^{-2}$	$+2{\cdot}01\times10^{-6}$

These measurements were all made on the same samples of material. The values of G differ from those given in **293** by amounts in some cases so large as to indicate that traces of impurity have a large influence. The table may be taken as indicating at least the order and sign of the several effects upon which all observers are in agreement. Comparisons with the table in **140** will show that, with the exception of platinum and cobalt, the metals stand in relation to the Hall effect in the order of their thermo-electric power.

Besides these four transverse effects there exist also longitudinal effects which depend upon them. For the *transverse* difference in temperature must in general excite longitudinal

gradients in potential and temperature. The longitudinal thermomagnetic electric effect (otherwise known as the longitudinal Hall effect) simulates an increase in resistance to the main current. It is detected, in any very marked degree, only in bismuth, which shows (**293**) an apparent increase in resistance in a magnetic field. This effect, according to the theory, would be proportional to the magnetic induction and the transverse temperature gradient conjointly. Since the latter is itself proportional to the magnetic induction, the apparent change in resistance should depend upon the square of the induction. The increase found experimentally is not as great as this law implies.

TABLES

DIELECTRIC CONSTANTS (Specific Inductive Capacities).

Quartz, parallel to axis	4·64	Plate glass	6·1
Quartz, perpendicular to axis	4·54	Paraffin wax	2·32
Fused quartz	3·78	Beeswax	4·75
Flint glass:—		Shellac	2·5 to 3·0
Density 4·65	10·64	Ebonite	2·5 to 2·8
,, 4·12	8·52	Sulphur	4·03
,, 3·30	6·90	Gutta percha	3·6 to 4·2
,, 2·87	6·61		

Air	1·0000	Carbonic acid gas	1·0008
Vacuum	·9985	Sulphurous acid gas	1·0037
Hydrogen	·9998		

DIELECTRIC STRENGTHS

According to Steinmetz, the thickness d, in hundredths of a millimetre, which can be pierced by the difference of potentials given by the following formulæ, in which E is expressed in kilovolts, is for

Mica	$d=0{\cdot}24E+0{\cdot}014E^2$.
Paraffined paper	$d=3E$.
Boiled linseed oil	$d=12{\cdot}4E$.

M'Farlane and Pierce give the following dielectric strengths in volts per centimetre:—

Oil of turpentine	94,000	Paraffin wax	130,000
Paraffin oil	87,000	Paraffined paper	360,000
Olive oil	82,000	Bees-waxed paper	540,000

Air (thickness, 5 cms.)	23,800	Hydrogen (5 cms.)	15,100
Carbonic acid (5 cms.)	22,700	Coal gas	22,300
Oxygen (5 cms.)	22,200		

The following are in kilovolts per millimetre:—

Manilla paper	2·2 to 2·9	Caoutchouc	10 to 21
Varnished cotton	10 to 21	Ebonite	28·5 to 31·5
Gutta percha	7·7 to 19		

RESISTANCE OF METALS AND ALLOYS.

	Resistance in ohms of a length of 10 metres.		Coefficient of variation for 1°.
	1 sq. mm. section.[1]	1 mm. diameter.	a.[2]
Silver, annealed . .	0·1492	0·1900	0·00380
„ hard-drawn . .	0·1620	0·2062	...
Copper, annealed . .	0·1584	0·2017	0·00388
„ hard-drawn . .	0·1620	0·2063	...
Gold, annealed . .	0·2041	0·2599	0·00365
„ hard-drawn . .	0·2077	0·2645	...
Aluminium . . .	0·2889	0·3679	0·00390
Platinum	0·8982	1·1435	0·0039
Iron	0·9638	1·227	0·00463
Nickel	1·236	1·573	...
Mercury	9·407[3]	11·978	0·000887
2 gold + 1 silver . .	1·078	1·372	0·00065
9 platinum + 1 iridium .	2·163	2·754	0·000133
2 platinum + 1 silver .	2·419	3·080	0·00025
German silver . . .	2·076	2·643	0·00040

RESISTANCE OF AQUEOUS SOLUTIONS.

1 part Crystallised Salt dissolved in	Specific Resistance in C.G.S. Units.			
	At 10° C.		Sulphuric Acid at 16°.	
	Copper Sulphate.	Zinc Sulphate.	Density.	Resistance.
40 parts water .	$16{\cdot}44 \times 10^{10}$	$18{\cdot}29 \times 10^{10}$	1·10	$0{\cdot}845 \times 10^{10}$
20 „ „ .	9·87 „	11·11 „	1·20	0·666 „
10 „ „ .	5·90 „	6·38 „	1·25	0·624 „
5 „ „ .	3·81 „	4·21 „	1·30	0·662 „
2·6 „ „	2·93 „ (saturated)	...	1·50	1·72 „
0·752 „ „ .	...	3·37 „ (saturated)	1·70	4·23 „

[1] By shifting the decimal point in this column four places to the right, the values of the *specific resistances* in C.G.S. units are obtained.

[2] $R_t = R_o\ (1 + at)$.

[3] Length of 1 ohm = 106·3 cm.

ELECTROMOTIVE FORCES OF GALVANIC CELLS.

Volta	Zinc Water Copper	about 1 volt
Leclanché	Amalgamated zinc Solution of sal-ammoniac Binoxide of manganese and carbon	1·46 „
Poggendorff	Amalgamated zinc 12 potassium bichromate, 25 sulphuric acid, 100 water Carbon	2·01 „
Daniell	Amalgamated zinc 1 part sulphuric acid, 10 parts water Saturated solution of copper sulphate Copper	1·07 „
Grove	Amalgamated zinc 1 sulphuric acid, 12 water Nitric acid Platinum	1·88 „
Latimer Clark (standard)	Amalgamated zinc Saturated solution of zinc sulphate Paste of mercurous sulphate Mercury	1·432 „ at 15° C.
Weston (standard)	Amalgamated cadmium Paste of cadmium sulphate Paste of mercurous sulphate Mercury	1·0183 at 17° C.

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References are to sections, not to pages.

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